ANNALES
POLONICI MATHEMATICI
LIII.1 (1991)

Abstract differential inclusions with some applications to partial differential ones

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Abstract. Let V, W, and Z be linear normed spaces and consider an operator A: $\operatorname{dom} A \subset W \to V$, a mapping j: $\operatorname{dom} j \subset W \to Z$, and a multifunction F: $\operatorname{dom} F \subset Z \to \operatorname{cl}(V)$. The paper deals with the problem of existence of solutions for an abstract differential inclusion:

Find $w \in W$ such that $Aw \in F(jw)$.

Such a generally stated problem admits a solution under suitable regularity assumptions on j, A, and F, and the compactness and growth conditions with the exponents α_j, α_R, α_F, where $\alpha_j \alpha_R \alpha_F < 1$ and R is a coretraction to A. The idea of the proof is based on an existence theorem for continuous selections.

The main result is applied to the following problems for partial differential inclusions:

(a) of elliptic type: Find u from the Sobolev space $H^2(T, \mathbf{R}^k)$ such that $\Delta u \in G(t, u(t), \nabla u(t))$ a.e. in an open bounded set $T \subset \mathbf{R}^m$ and $u(t) = 0$ on the boundary ∂T, where G: $T \times \mathbf{R}^k \times \mathbf{R}^{mk} \to \operatorname{cl}(\mathbf{R}^k)$;

(b) of parabolic type: solve $du/dt + Lu = g(t)$ with an initial condition $u(0) \in F(ju)$, where $t \in T = (0, 1)$, $I = [0, a]$, $L = -\partial^2/\partial x^2$ is an operator defined on the Sobolev space $H_0^1(T, \mathbf{R}^k) \cap H^2(T, \mathbf{R}^k)$ and $g \in L^1(I, L^2(T, \mathbf{R}^k))$.

Introduction. Consider an abstract differential inclusion

$$Du \in F(u), \tag{1}$$

where U and V are function spaces and D: $U \to V$ is a differential operator. In the case when the right-hand side $F(u)$ is a lower semicontinuous multifunction from U into closed convex subsets of V, the existence of solutions to (1) can be concluded from the well-known Michael Theorem [11], [13]. Namely, under suitable growth and compactness conditions on $F(u)$, every differential equation $Du = f(u)$, with a continuous selection $f(u)$ of $F(u)$, $f(u) \in F(u)$, admits a solution $u \in \operatorname{dom} F := \{u \in U:\ F(u) \neq \emptyset\}$.

If the values $F(u)$ are not necessarily convex, a continuous selection need not exist. However, in some situations, the convexity assumption in the Michael Theorem can be replaced by another one. This is the case when V is the Lebesgue space of s-integrable functions, $s \in [1, \infty)$, defined on a measure space $(T, \mathscr{L}, \mu)$. If we assume that $\operatorname{dom} F$ is compact and all sets $F(u)$ are decomposable, i.e. for all v, $v' \in F(u)$ and every $A \in \mathscr{L}$,

$$\chi_A v + (1-\chi_A)v' \in F(u),$$

then the mapping $F(\cdot)$ admits a continuous selection. This is proved in [7] for $V = L^1(T, X)$, where X is a Banach space (in [7] X was required to be separable, but the proof does not actually use this condition).

In the present paper we extend this result in a simple way to an arbitrary space $L^s(T, X)$, $s \in [1, \infty)$ (see also [3], [4]). We give a precise formulation in Section 2, where we also provide some necessary definitions and results. They allow us to show the scope of the class of decomposable lower semicontinuous multifunctions. Section 3 is devoted to the proof of our main result on the existence of solutions to (1), which we apply in Section 4 to elliptic and parabolic differential inclusions. Our theorem generalizes the existence results to a wide range of differential inclusions such as partial differential inclusions [1] and a new nonstationary diffusion problem. In this case a qualitatively new phenomenon appears: we may consider a constraint on the initial conditions which depends on the future values of the trajectory.

2. Preliminaries. Let W and Z be topological spaces and let $\mathrm{cl}(Z)$ stand for the class of all nonempty closed subsets of Z. By a *multifunction* (multivalued mapping or simply mapping) we mean a mapping $F\colon W \to \mathrm{cl}(Z)$.

DEFINITION 1. A mapping $F\colon W \to \mathrm{cl}(Z)$ is called *lower semicontinuous* (*l.s.c.*) if for any closed $B \subset Z$ the set

$$F^+(B) = \{w \in W\colon\ F(w) \subset B\} \tag{2}$$

is closed in W.

Remark 1. If Z is a metric space with metric d, then F being l.s.c. is equivalent to the upper semicontinuity (u.s.c.) of the function $(w, z) \mapsto \mathrm{dist}(z, F(w))$. ∎

Since now let T be a locally compact Hausdorff topological space with a σ-field $\mathscr{L}$ given by a nonnegative σ-finite regular Borel measure μ. For an arbitrary Banach space X with norm $|\cdot|$ we shall denote by $\mathscr{M}(T, X)$ the space (of equivalence classes) of $\mathscr{L}$-measurable functions from T into X and by $L^s(T, X)$ the subspace of $\mathscr{M}(T, X)$ equipped with the norm

$$\|z\|_{s,T} = \{\int_T |z(t)|^s \mu(dt)\}^{1/s}, \quad s \in [1, \infty). \tag{3}$$

Let Z be a normed space contained in $\mathscr{M}(T, X)$.

DEFINITION 2. (i) A set $K \subset Z$ is *decomposable* if for all z, $z' \in K$ and every $A \in \mathscr{L}$

$$\chi_A z + (1-\chi_A)z' \in K \quad (\text{in } \mathscr{M}(T, X)). \tag{4}$$

(ii) A mapping $F: W \to \mathrm{cl}(Z)$ is *decomposable* if for every $w \in W$ the set $F(w)$ is decomposable.

Consider a continuous mapping $f: X \to X$ and denote by $T_f: Z \to \mathcal{M}(T, X)$ the mapping given by

$$(T_f z)(t) = f(z(t)). \tag{5}$$

Such a T_f can be a source of new decomposable mappings:

PROPOSITION 1. *Let Z and Z_1 be normed spaces in $\mathcal{M}(T, X)$ and let W and W_1 be topological spaces. Assume that the mapping T_f given by (5) is a homeomorphism between Z and Z_1. Let $F_1: W_1 \to \mathrm{cl}(Z_1)$ be decomposable and l.s.c., and $J: W \to W_1$ a continuous mapping. Then $F = T_f^{-1} \circ F_1 \circ J: W \to \mathrm{cl}(Z)$ is also decomposable and l.s.c. where $T_f^{-1}: \mathrm{cl}(Z_1) \to \mathrm{cl}(Z)$ is induced by $T_f^{-1}: Z_1 \to Z$.*

Proof. That F is l.s.c. is obvious. To prove that F is decomposable, fix $A \in \mathcal{L}$ and $w \in W$. For $z_1, z_2 \in F(w)$ set $z = \chi_A z_1 + (1-\chi_A) z_2$. One can check that $T_f z = \chi_A T_f z_1 + (1-\chi_A) T_f z_2$. But $T_f z_i \in F_1(Jw)$ for $i = 1, 2$ and thus $T_f z \in \chi_A F_1(Jw) + (1-\chi_A) F_1(Jw) \subset F_1(Jw)$, by decomposability of F_1. Therefore $z \in (T_f^{-1} \circ F_1 \circ J)(\omega) = F(w)$, which completes the proof. ∎

COROLLARY 1. *For any $p, s \in [1, \infty)$, the classes of all decomposable l.s.c. mappings from W into $\mathrm{cl}(L^s(T, X))$ and $\mathrm{cl}(L^p(T, X))$ respectively are in a bijective correspondence.*

Proof. Let $f_{s,p}: X \to X$ be the function given by

$$f_{s,p}(x) = \begin{cases} |x|^{s/p-1} x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Consider the mapping $T_{s,p}$ given by (5) with $f = f_{s,p}$. Then $T_{s,p}$ is the famous Mazur homeomorphism ([2], Ch. IV) between $L^s(T, X)$ and $L^p(T, X)$ and now our claim follows from Proposition 1. ∎

Consider a space $V \subset \mathcal{M}(T, X)$. We shall say that V has the *continuous selection property* iff for any compact topological space W, every l.s.c. decomposable mapping $F: W \to \mathrm{cl}(V)$ admits a continuous selection $f(\cdot)$, $f(w) \in F(w)$ for all $w \in W$. First the continuous selection property for the spaces $L^1(T, X)$ has been shown in [7] with T compact and X separable and next extended in [3] to the general case (for W a separable space). The same remains true also for $L^s(T, X)$, where $s \in [1, \infty)$:

PROPOSITION 2. *Assume that W is a compact topological space and let $F: W \to \mathrm{cl}(L^s(T, X))$ be l.s.c. and decomposable. Then F admits a continuous selection.*

Proof. Let $T_{1,s}$ denote the Mazur homeomorphism from $L^1(T, X)$ onto $L^s(T, X)$. From Proposition 1 it follows that $F_1 = (T_{1,s})^{-1} \circ F$ is l.s.c. and decomposable as well. Therefore F_1 admits a continuous selection f_1. One can check that $f = T_{1,s} \circ f_1$ is the required selection of F. ∎

Needless to say, the family of l.s.c. decomposable mappings is very wide. Now we are going to present a class of such mappings; first, we recall the notion of measurability of multivalued mappings.

Let $(S, \mathscr{S})$ be an arbitrary measure space and Y a separable metric space.

DEFINITION 3. A mapping $G\colon S \to \mathrm{cl}(Y)$ is called *$\mathscr{S}$-measurable* (or simply *measurable*) if $G^+(B) \in \mathscr{S}$ for every $B \in \mathrm{cl}(Y)$.

Now we present a class of l.s.c. decomposable mappings which appears in the theory of differential inclusions. Let X and Y be two finite-dimensional Banach spaces and consider a mapping $G\colon T \times X \to \mathrm{cl}(Y)$ satisfying the following conditions:

(i) G is $\mathscr{L} \otimes \mathscr{B}$-measurable, where $\mathscr{B}$ stands for the Borel σ-field of X,

(ii) $G(t, \cdot)$ is l.s.c. for every $t \in T$,

(iii) G satisfies the following growth condition: there exist $\gamma > 0$, $c > 0$ and $m \in L^s(T, \boldsymbol{R})$ such that for each $t \in T$ and $x \in X$

$$\sup\{|r|\colon r \in G(t, x)\} \leqslant m(t) + \epsilon(1+|x|)^\gamma, \qquad s \in [1, \infty).$$

For such G and $w \in L^p(T, X)$ we define a new multifunction by

$$F(w) = \{z \in \mathscr{M}(T, Y)\colon z(t) \in G(t, w(t)) \text{ a.e. in } T\}. \tag{6}$$

If $\gamma \in (0, p/s]$ then $F(w)$ is a nonempty closed subset of $L^s(T, Y)$. Moreover, we have the following

PROPOSITION 3. *Assume that $G\colon T \times X \to \mathrm{cl}(Y)$ satisfies* (i)–(iii) *and consider the mapping F given by* (6). *If $\gamma \in (0, p/s]$ and $\mu(T) < +\infty$ then F: $L^p(T, X) \to \mathrm{cl}(L^s(T, X))$ is l.s.c. and decomposable and satisfies the following growth condition: There is $C > 0$ such that*

$$\sup\{\|z\|_{s,T}\colon z \in F(w)\} \leqslant C(1 + \|w\|_{p,T})^\gamma \quad \textit{for every } w \in L^p(T, X). \tag{7}$$

Proof. Fix $w \in L^p(T, X)$. One can check that the set $F(w)$ is decomposable. By a standard argument it is also closed. We shall prove that it is nonempty, condition (7) holds and hence the mapping $F(\cdot)$ is l.s.c.

(a) *Nonemptiness.* From (i) it follows that $G(\cdot, w(\cdot))$ is measurable, and hence admits measurable selections, by the Kuratowski and Ryll-Nardzewski Theorem [10], [5]. From (iii) we can conclude that every measurable selection $z(\cdot)$ of $G(\cdot, w(\cdot))$ satisfies

$$|z(t)| \leqslant m(t) + c(1 + |w(t)|)^\gamma \quad \text{a.e. in } T. \tag{8}$$

But $(1 + |w(\cdot)|)^\gamma \in L^s(T, \boldsymbol{R})$ and therefore (8) means that $F(w)$ is a nonempty subset of $L^s(T, X)$ and for every $z \in F(w)$

$$\|z\|_{s,T} \leqslant \|m\|_{s,T} + cc_0(\|1 + |w|\,\|_{p,T})^\gamma,$$

where c_0 is the norm of the embedding of $L^{p/\gamma}(T, \mathbf{R})$ into $L^s(T, \mathbf{R})$. The latter inequality provides the required estimate with $C = \|m\|_{s,T} + cc_0(1+\mu(T)^{1/p})^{\gamma}$ since $\|m\|_{s,T} \leqslant \|m\|_{s,T}(1+\|w\|_{p,T})^{\gamma}$.

(b) *l.s.c.* Fix $B \in \mathrm{cl}(L^s(T, Y))$ and let $w_n \in F^+(B)$ be a sequence converging to w_0. We shall show that $w_0 \in F^+(B)$, i.e. $F(w_0) \subset B$. For every $z_0 \in F(w_0)$ and $n = 1, 2, \ldots$ there exist $z_n \in F(w_n)$ such that

$$|z_n(t) - z_0(t)| = \mathrm{dist}\big(z_0(t), G(t, w_n(t))\big) \quad \text{a.e. in } T.$$

Since $\{w_n\}$ converges in $L^s(T, X)$ to w_0, by taking a subsequence, if necessary, we may also assume that it converges a.e. in T. Let $T_0 \subset T$ be a set of full measure such that for each $t \in T_0$

$$\lim w_n(t) = w_0(t), \qquad z_n(t) \in F\big(t, w_n(t)\big).$$

Then (ii) shows that for all $t \in T_0$

$$\begin{aligned}\limsup |z_n(t) - z_0(t)| &= \limsup \mathrm{dist}\big(z_0(t), G(t, w_n(t))\big)\\ &\leqslant \mathrm{dist}\big(z_0(t), G(t, w_0(t))\big) = 0,\end{aligned}$$

which means that $z_n(t)$ tends to $z_0(t)$ a.e. in T. Moreover, from (8) it follows that for any $A \in \mathscr{L}$

$$\int_A |z_n(t)|^s \mu(dt) \leqslant 2^{s-1} \int_A |m(t)|^s \mu(dt) + 2^{s-1} c^s \int_A \big(1+|w_n(t)|\big)^{\gamma s} \mu(dt).$$

But since $\gamma s \leqslant p$, using the Mazur homeomorphism we have

$$\lim \int_A \big(1+|w_n(t)|\big)^{\gamma s} \mu(dt) = \int_A \big(1+|w_0(t)|\big)^{\gamma s} \mu(dt),$$

which means in particular that the functions $|z_n(\cdot)|^s$ are equiintegrable. By the Vitali–Hahn–Saks Theorem [6], $\{z_n\}$ converges in $L^s(T, Y)$ to z_0. But $z_n \in F(w_n) \subset B$ and B is closed, thus $z_0 \in B$. Since $z_0 \in F(w_0)$ has been arbitrary, we have $F(w_0) \subset B$, which completes the proof. ∎

3. Abstract differential inclusions. Let V and W be arbitrary normed spaces. A mapping $A\colon \mathrm{dom}\, A \to V$, where $\mathrm{dom} A \subset W$, is called a *retraction* on V if A admits a "right inverse", i.e. there exists a continuous mapping $R\colon V \to W$ such that $A \circ R = \mathrm{id}_V$. Of course we must have $\mathrm{im}\, R \subset \mathrm{dom}\, A \subset W$. The map R is said to be a *coretraction* of A.

Let Z be a normed space and consider a multifunction F: $\mathrm{dom}\, F \subset Z \to \mathrm{cl}(V)$. We shall say that F satisfies the *growth condition with exponent* $\alpha_F \geqslant 0$ iff there exists a constant $C_F > 0$ such that for every $z \in \mathrm{dom}\, F$

$$\sup\{\|v\|\colon v \in F(z)\} \leqslant C_F(1+\|z\|)^{\alpha_F}. \tag{9}$$

In particular, if F is a function then the growth condition reads

$$\|F(z)\| \leqslant C_F(1+\|z\|)^{\alpha_F}. \tag{10}$$

Assume now that Z is a Banach space and $V \subset \mathcal{M}(T, X)$ has the continuous selection property and F is a l.s.c. decomposable mapping. Consider a completely continuous mapping j: $\operatorname{dom} j \subset W \to Z$. The main result of this work is the following:

THEOREM 1. *Suppose that F, j and a coretraction R of A satisfy the growth conditions with nonnegative exponents α_F, α_j and α_R (and constants C_F, C_j and C_R) respectively and additionally*

(i) $\operatorname{im} R \subset \operatorname{dom} j$, $\operatorname{im} j \subset \operatorname{dom} F$ *and* $\operatorname{dom} F$ *is a closed convex subset of* Z;

(ii) $\alpha_F \alpha_j \alpha_R < 1$.

Then the problem

$$\text{(11)} \qquad \text{Find } w \in W \text{ such that } Aw \in F(jw)$$

has a solution $w \in \operatorname{im} R \subset \operatorname{dom} j \cap \operatorname{dom} A$.

Proof. Fix $r > 0$ and set $B_r = \{w \in \operatorname{dom} j: \|w\|_W \leqslant r\}$. Then $B_r \neq \emptyset$ for any $r \geqslant \|R(0)\|_W$ since then $R(0) \in B_r$. Let $S_r = \overline{\operatorname{co}}\, j(B_r)$, the closed convex hull of $j(B_r)$. Since j is completely continuous and B_r is bounded, $j(B_r)$ is relatively compact and thus S_r is compact. Moreover, for each $z \in S_r$

$$\text{(12)} \qquad \|z\|_Z \leqslant \sup\{\|j(w)\|_Z: w \in B_r\} \leqslant C_j(1+r)^{\alpha_j}.$$

Clearly $S_r \subset \operatorname{dom} F$ and the mapping F restricted to S_r is l.s.c. and decomposable as well. Thus F admits a continuous selection on S_r. The growth condition yields that for every such selection f: $S_r \to V$ and for each $z \in S_r$

$$\text{(13)} \qquad \|f(z)\|_V \leqslant C_F(1+\|z\|_Z)^{\alpha_F}.$$

The last inequality together with (12) implies that there exists a constant C_f such that for each $z \in S_r$

$$\text{(14)} \qquad \|f(z)\|_V \leqslant C_f(1+r)^{\alpha_j \alpha_F}.$$

Since $\|(R \circ f)(z)\|_W \leqslant C_R(1+\|f(z)\|)^{\alpha_R}$, from (14) we obtain for each $z \in S_r$

$$\text{(15)} \qquad \begin{aligned} \|(R \circ f)(z)\|_W &\leqslant C_R\big(1 + C_f(1+r)^{\alpha_j \alpha_F}\big)^{\alpha_R} \\ &\leqslant C_R(1+C_f)^{\alpha_R}(1+r)^{\alpha_j \alpha_F \alpha_R}. \end{aligned}$$

But $\alpha_j \alpha_F \alpha_R < 1$, so there exists $r > 0$ such that $[2C_R(1+C_f)^{\alpha_R}]^{1/(1-\alpha_j\alpha_F\alpha_R)} \leqslant r$. For such an r any selection f of F restricted to S_r produces the mapping $R \circ f$: $S_r \to B_r$, since im $R \subset \operatorname{dom} j$. Then for a fixed continuous selection f of F on S_r the mapping $j \circ R \circ f$: $S_r \to j(B_r) \subset S_r$ is continuous as well and by the Schauder theorem it has a fixed point $z_0 \in j(B_r)$. One can check that $w = (R \circ f)(z_0)$ is then a solution of (11). ■

Since every Hölder mapping satisfies the growth condition with the same exponent we obtain

COROLLARY 2. *Theorem* 1 *remains true if instead of the growth conditions on* j *and* R *we assume that they are Hölder continuous with exponents* α_j *and* α_R.

Remark 2. Note that if $F\colon Z \to \mathrm{cl}(V)$ satisfies the condition

(ii)′ there exists a bounded set V_0 such that $F(z) \subset V_0$ for any $z \in Z$,

then F satisfies the growth condition with exponent $\alpha_F = 0$. ∎

COROLLARY 3. *Theorem* 1 *is also true if we replace* (ii) *by* (ii)′.

4. Applications. The existence theorem for abstract inclusions (11) will be applied to some examples concerning partial differential inclusions of parabolic and elliptic type.

Let T be an open set in the euclidean space $\boldsymbol{R}^m$ with smooth boundary ∂T and elements $t = (t_1, \ldots, t_m)$. For $\alpha = (\alpha_1, \ldots, \alpha_m)$ let $D^\alpha = D_1^{\alpha_1} \ldots D_m^{\alpha_m}$, where $D_j = \partial/\partial t_j$ stands for the distributional derivative and let $D = (D_1, \ldots, D_m)$ be the gradient. Recall that

$$H^1(T, \boldsymbol{R}^k) = \{v \in L^2(T, \boldsymbol{R}^k)\colon Dv \in L^2(T, \boldsymbol{R}^{mk})\},$$

$$H^2(T, \boldsymbol{R}^k) = \{v \in L^2(T, \boldsymbol{R}^k)\colon D^\alpha v \in L^2(T, \boldsymbol{R}^k) \text{ for } |\alpha| \leqslant 2\},$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_m$, are Sobolev spaces with the respective norms

$$\|v\|_1 = \Big\{\int_T (|v(t)|^2 + |Dv(t)|^2) dt\Big\}^{1/2}, \quad \|v\|_2 = \Big\{\int_T \Big(\sum_{|\alpha| \leqslant 2} |D^\alpha v(t)|^2\Big) dt\Big\}^{1/2}.$$

By $H_0^1(T, \boldsymbol{R}^k)$ we mean the subspace of $H^1(T, \boldsymbol{R}^k)$ consisting of those elements $v \in H^1(T, \boldsymbol{R}^k)$ for which $v|_{\partial T} = 0$.

Now we are ready to give some examples of the abstract inclusion (11).

4.1. Elliptic inclusions. Consider the following problem:

(16) Find $u \in H^2(T, \boldsymbol{R}^k)$ such that $\Delta u(t) \in G(t, u(t), Du(t))$ a.e. in T, $u|_{\partial T} = 0$,

where $\Delta\colon H^2(T, \boldsymbol{R}^k) \to L^2(T, \boldsymbol{R}^k)$ is the Laplace operator and $G\colon T \times \boldsymbol{R}^k \times \boldsymbol{R}^{mk} \to \mathrm{cl}(\boldsymbol{R}^k)$.

THEOREM 2. *Suppose that*

(a) *G is $\mathscr{L} \otimes \mathscr{B}(\boldsymbol{R}^{k+mk})$-measurable, where $\mathscr{L}$ is the σ-algebra of Lebesgue measurable sets in T and $\mathscr{B}(\cdot)$ stands for the Borel sets,*

(b) *the mapping $G(t, \cdot, \cdot)$ is l.s.c. for every $t \in T$,*

(c) *there exist $\gamma < 1$, $M \geqslant 0$ and $m \in L^2(T, \boldsymbol{R})$ such that for any $(x, y) \in \boldsymbol{R}^k \times \boldsymbol{R}^{mk}$*

$$\sup\{|r|\colon r \in G(t, x, y)\} \leqslant m(t) + M(1 + |x| + |y|)^\gamma \quad \text{a.e. in } T.$$

Then the problem (16) *has a solution $u \in H^2(T, \boldsymbol{R}^k) \cap H_0^1(T, \boldsymbol{R}^k)$.*

Proof. Let G be the mapping from $L^2(T, \boldsymbol{R}^{k+mk})$ into $\mathrm{cl}(L^2(T, \boldsymbol{R}^k))$ given by

$$G(u) = \{z \in L^2(T, \boldsymbol{R}^k)\colon z(t) \in G(t, u_0(t), \ldots, u_m(t)) \text{ a.e. in } T\},$$

where $u(\cdot) = (u_0(\cdot), u_1(\cdot), \ldots, u_m(\cdot))$. This mapping is decomposable and l.s.c. by Proposition 3.

Since the operator (id, D) is continuous from $H^1(T, \boldsymbol{R}^k)$ into $L^2(T, \boldsymbol{R}^k) \times L^2(T, \boldsymbol{R}^{mk})$, $F(\cdot)$ defined on $H^1(T, \boldsymbol{R}^k)$ by

$$F(v) = \{z \in L^2(T, \boldsymbol{R}^k)\colon\ z(t) \in G(t, v(t), Dv(t)) \text{ a.e. in } T\}$$

is decomposable and l.s.c. as well. By Proposition 3, the mapping $F(\cdot)$ satisfies the growth condition with exponent γ (and constant C_F).

By Rz denote the unique solution in $H^2(T, \boldsymbol{R}^k)$ of the Dirichlet problem

$$\Delta u = z, \qquad u|_{\partial T} = 0$$

for any $z \in L^2(T, \boldsymbol{R}^k)$. The mapping R is a continuous linear operator from $L^2(T, \boldsymbol{R}^k)$ into $H^2(T, \boldsymbol{R}^k) \cap H_0^1(T, \boldsymbol{R}^k)$ (see [12]). Therefore in order to obtain a solution of (16) we can apply Theorem 1 for $W = H^2(T, \boldsymbol{R}^k)$, $V = L^2(T, \boldsymbol{R}^k)$, $Z = H^1(T, \boldsymbol{R}^k)$, $A = \Delta$ and the embedding $j\colon H^2(T, \boldsymbol{R}^k) \to H^1(T, \boldsymbol{R}^k)$, which is compact by the Rellich Theorem. ∎

EXAMPLE 1. A particular case of (16) is the problem

(17) $$\Delta u \in G(u), \qquad u|_{\partial T} = 0,$$

where $T \subset \boldsymbol{R}^2$ and $G\colon \boldsymbol{R}^2 \to \mathrm{cl}(\boldsymbol{R}^2)$ is the standard continuous multifunction given by

$$G(x) = \begin{cases} \{-x/|x|\} & \text{for } |x| \geqslant 1, \\ \{(\cos\alpha, \sin\alpha)\colon |\pi\alpha - \varphi| \leqslant \pi(1-|x|)\} & \text{if } x = |x|(\cos\alpha, \sin\alpha) \\ & \text{for } 1 > |x| > 0, \\ \{(\cos\alpha, \sin\alpha)\colon \alpha \in \boldsymbol{R}\} & \text{for } x = 0, \end{cases}$$

which does not admit a continuous selection. Thus the problem (17) has a solution which is a Lipschitz function on $\bar{T}$ (by the Sobolev embedding theorems). ∎

4.2. Parabolic inclusions. Let $T = (0, 1)$ and $I = [0, a]$ and set $H = L^2(T, \boldsymbol{R}^k)$ with the norm $|\cdot|$.

Consider the class of parabolic problems

(18) Find a continuous function $u\colon [0, a] \to H$ such that

$$\frac{du}{dt} + Lu = g(t), \qquad u(0) \in F(ju),$$

where the operator L is defined on $H_0^1(T, \boldsymbol{R}^k) \cap H^2(T, \boldsymbol{R}^k)$ by $Lu = (-\partial^2 u_1/\partial x^2, \ldots, -\partial^2 u_k/\partial x^2)$ with $u = (u_1, \ldots, u_k)$ and $g \in L^1(I, H)$. We impose the following assumptions on the data j, Z, and F:

(19) Z is a Banach space;

(20) $j\colon \mathscr{C}(I, H) \to Z$ is completely continuous and satisfies the growth condition with exponent α (and constant C_j);

(21) F is l.s.c. and decomposable and satisfies the growth condition with exponent β (and constant C_F).

We now explain the notions of "strong" and "weak" solution of the equation $du/dt+Lu = g$ and of the problem (18).

By a *strong solution of the equation* above we mean a mapping $u \in \mathscr{C}(I, H)$ such that for every $\varepsilon > 0$, u: $[\varepsilon, a-\varepsilon] \to H$ is absolutely continuous, $u(t) \in \operatorname{dom} L$ a.e. in I and

$$du/dt+Lu = g \quad \text{a.e. in } I. \tag{22}$$

A *weak solution* is a limit u of strong solutions u_n satisfying (22) with right-hand sides g_n tending to g in $L^1(I, H)$.

A *weak solution of the problem* (18) is a weak solution of the equation satisfying the initial condition $u(0) \in F(ju)$.

Now we are in a position to formulate our result.

THEOREM 3. *Assume that* (19)–(21) *hold and* $\alpha\beta < 1$. *Then the problem* (18) *admits a weak solution* u: $I \to H$.

This is a generalization of Theorem 38 in [8] which asserts that for every $g \in L^1(I, H)$ and $f \in H$ there exists a weak solution of the problem (18) with $Z = \{0\}$ and $F(0) = \{f\}$.

A unique weak solution of

$$du/dt+Lu = g, \qquad u(0) = f \tag{23}$$

corresponding to given f and g is denoted by $S(g, f)$.

For a fixed $g \in L^1(I, H)$ we define an operator R: $H \to \mathscr{C}(I, H)$ by the formula

$$Rf = S(g, f). \tag{24}$$

Before we present a proof of Theorem 3 we need two technical lemmas.

LEMMA 1. *The operator R given by* (24) *maps H isometrically into* $\mathscr{C}(I, H)$.

Proof. First we show that $S(g, f)$ is a Lipschitz contraction on the product $L^1(I, H) \times H$, i.e.

$$\|S(g_1, f_1)-S(g_2, f_2)\|_{\mathscr{C}(I,H)} \leqslant |f_1-f_2|+\|g_1-g_2\|_{L^1(I,H)} \tag{25}$$

for all g_1, $g_2 \in L^1(I, H)$ and f_1, $f_2 \in H$.

Indeed, Theorem 34 in [8] asserts that two weak solutions u_1 and u_2 of (22) with right-hand sides g_1 and g_2 respectively satisfy

$$|u_1(t)-u_2(t)| \leqslant |u_1(s)-u_2(s)|+\int_s^t |g_1(\tau)-g_2(\tau)|\,d\tau$$

for all $0 \leqslant s \leqslant t \leqslant a$. Moreover, for each $f \in \overline{\operatorname{dom} L}$ there exists exactly one weak solution $S(g, f)$ of the initial problem. If we take $u_1 = S(g_1, f_1)$ and

$u_2 = S(g_2, f_2)$, the weak solutions of (18) with initial values f_1 and $f_2 \in \overline{\operatorname{dom} L} = H$ respectively and $s = 0$, we obtain

$$\|S(g_1, f_1) - S(g_2, f_2)\|_{\mathscr{C}(I,H)} = \sup_{t \in I} |u_1(t) - u_2(t)|$$

$$\leqslant \sup_{t \in [0,a]} \Big(|u_1(0) - u_2(0)| + \int_0^t |g_1(\tau) - g_2(\tau)|\, d\tau\Big)$$

$$\leqslant |f_1 - f_2| + \int_0^a |g_1(\tau) - g_2(\tau)|\, d\tau = |f_1 - f_2| + \|g_1 - g_2\|_{L^1(I,H)}.$$

This completes the proof of (25).

From (25) applied to $g_1 = g_2 = g$ we conclude that the operator R given by (24) satisfies

$$\|Rf_1 - Rf_2\|_{\mathscr{C}(I,H)} \leqslant |f_1 - f_2|.$$

On the other hand, $\sup |Rf_1(t) - Rf_2(t)| \geqslant Rf_1(0) - Rf_2(0)| = |f_1 - f_2|$, which proves that R is an isometry. ∎

Remark 3. The above lemma implies the existence of the inverse operator to R with domain $\operatorname{im} R$ closed in $\mathscr{C}(I, H)$. ∎

LEMMA 2. *The operator A from $\mathscr{C}(I, H)$ into H given by $Au = u(0)$ is a retraction on H. A coretraction for it is the map R given by* (24). *Moreover, R satisfies the growth condition with exponent $\alpha_R = 1$.*

Proof. Observe that $(A \circ R)f = Au = u(0)$ for every weak solution of the problem (23). Thus $(A \circ R)f = f$ for each $f \in H$, because $\operatorname{dom} L = H^2(T, \mathbf{R}^k) \cap H_0^1(T, \mathbf{R}^k)$ is dense in H and a weak solution of the above problem exists for each initial value f in $H = \overline{\operatorname{dom} L}$.

Since R is an isometry by Lemma 1, it is a Lipschitz mapping and, as in Corollary 2, it satisfies the inequality

$$\|Rf\|_{\mathscr{C}(I,H)} \leqslant C(1 + |f|) \quad \text{for some constant } C. \ ∎$$

Now having proved lemmas we can proceed to the proof of Theorem 3.

Proof of Theorem 3. By Lemma 2, the coretraction R satisfies the growth condition with exponent $\alpha_R = 1$. Because of (20) and (21), the same holds for j and F with $\alpha_j = \alpha$ and $\alpha_F = \beta$. Thus $\alpha_R \alpha_j \alpha_F < 1$ and $\operatorname{dom} j \supset \operatorname{im} R$ and $\operatorname{dom} F = Z$. This means that all assumptions of Theorem 1 are fulfilled. Therefore we conclude that there exists $u \in \mathscr{C}(I, H)$ such that $u(0) \in F(ju)$ and $u \in \operatorname{im} R$; in other words, u is a weak solution of (22). ∎

EXAMPLE 2. A special case of (18) is the class of problems

(26) Find a function u such that

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=g(t,x),$$

$$u(0,x)\in F\Big[\Big|\int_I\int_T \chi(\tau,z,x)u|u|^{q_1-1}(\tau,z)d\tau dz\Big|^{1/q_2}\Big],$$

$$u(t,0)=0=u(t,1),$$

where $I=[0,a]$, $T=(0,1)$, $\chi\in L^1(I\times T\times T, \boldsymbol{R}^{k\times k})$ and the mapping $F(\cdot)$ from $L^2(T,\boldsymbol{R})$ into the closed subsets of $H=L^2(T,\boldsymbol{R}^k)$ need not admit a continuous selection. The physical model of such a situation can be the problem of seeking the distribution of concentrations and/or temperatures $u(t,x)$ in a pipe of length 1 for a given arbitrary emission-absorption field $g(t,x)$. Initial constraints depend on a nonlinear mean-value observed from a measuring device with characteristic χ which can depend on time. If we fix $q_2=2$, $q_1=1$ and $\chi=\chi_{[a-\varepsilon,a]\times T\times T}\mathrm{id}_{\boldsymbol{R}^k}$, we deduce that for an arbitrary time a there exists a real $b\geqslant 0$ such that

$$b^2=\Big|\int_{a-\varepsilon}^{a}\int_T u(t,z)dt dz\Big|$$

and the trajectories $u(t,x)$ are generated by the set of initial values $F(b)$.

COROLLARY 4. *The problem* (26) *admits a weak solution* $u\in\mathscr{C}(I,H)$ *if we assume that* $0<q_1<2$, $\max(1,q_1)<q_2<+\infty$, $\chi(\cdot,\cdot,x)\in L^r(I\times T,\boldsymbol{R}^{k\times k})$, $\chi(\tau,z,\cdot)\in L^\sigma(T,\boldsymbol{R}^{k\times k})$, $r\sigma>1$ *and* $r,\sigma\in[1,+\infty]$.

It is sufficient to show that the mapping $j\colon \mathscr{C}(I,H)\to L^2(T,\boldsymbol{R})$ defined by

$$(ju)(x)=\Big|\int_I\int_T \chi(\tau,z,x)u|u|^{q_1-1}(\tau,z)d\tau dz\Big|^{1/q_2} \tag{27}$$

is completely continuous and fulfils the growth condition with $Z=L^2(T,\boldsymbol{R})$. ∎

LEMMA 3. *The mapping* $j\colon\mathscr{C}(I,H)\to Z$ *given by* (27) *is completely continuous and satisfies the growth condition with exponent* $\alpha=q_1/q_2<1$.

Proof. *Step* 1. The mapping T_{q_1} given by $T_{q_1}(u)=u|u|^{q_1-1}$ is continuous from $\mathscr{C}(I,H)$ into $L^s(I\times T,\boldsymbol{R}^k)$ as in the proof of Corollary 1 and, moreover,

$$\|T_{q_1}u\|_{s,I\times T}\leqslant C_1\|u\|^{q_1}_{\mathscr{C}(I,H)},$$

provided only $s=2/q_1\geqslant 1$.

Step 2. The operator U from $L^s(I\times T,\boldsymbol{R}^k)$ into $L^s(T,\boldsymbol{R}^k)$ given by

$$U(v)=\int_I\int_T v(\tau,z)\chi(\tau,z,x)d\tau dz$$

is linear and compact if we assume that $s>1$ and $r\sigma>1$. This is an immediate consequence of Theorem XI.3.3 from [9].

Step 3. The mapping $|T_{1/q_1}|(w) = |w|^{1/q_1}$ is continuous from $L^s(T, \mathbf{R})$ into $L^{2q_2/q_1}(T, \mathbf{R})$ if $q_2 > 1$ and into $Z = L^2(T, \mathbf{R})$ for $q_1/q_2 \leqslant 1$. We also have

$$|T_{1/q_2} w|_H \leqslant C_2 \|w\|_{s,T}^{1/q_2}.$$

Step 4. The mapping $j = |T_{1/q_2}| \circ U \circ T_{q_1}$ is actually into W and since U is compact, j is completely continuous. Moreover,

$$|ju| \leqslant C_j(1 + \|u\|_{\mathscr{C}(I,H)})^{q_1/q_2}.$$

Indeed,

$$|ju| \leqslant C_2 \|(U \circ T_{q_1})(u)\|_{s,T}^{1/q_2} \leqslant C_3 \|T_{q_1} u\|_{s,I\times T}^{1/q_2} \leqslant C \|u\|_{C(I,H)}^{q_1/q_2}. \ \blacksquare$$

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Reçu par la Rédaction le 19.12.1988
Révisé le 20.09.1989