ANNALES
POLONICI MATHEMATICI
L (1990)

On the solvability of singular BVPs for second-order ordinary differential equations

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Abstract. In the paper, the existence of solutions of the equation $\varphi(t)x'' = f(t, x, x')$ satisfying $x(0) = \alpha$, $x(\infty) = 0$ is established by means of the degree theory and compactness argument. The function φ vanishes only at 0.

Introduction. The general aim of the present work is to get the solvability of the boundary value problem BVP, $\varphi(t)x'' = f(t, x, x')$, $x(0) = \alpha$, $\lim_{t\to\infty} x(t) = 0$, where φ is a nonnegative continuous function on $\langle 0, \infty)$ vanishing only at 0 (in particular ∞ is not a singular point). When φ is, in addition, bounded, the existence of solutions can be established by using the method from [4]. We shall consider the case of unbounded φ; then it is necessary to assume a some kind of the asymptotic regularity of φ.

First, we shall examine the nonsingular case, $\varphi = 1$. It has already been studied in [4] for f independent of the first derivative x'. Here, we shall generalize it assuming that f satisfies a growth condition of Bernstein type (cf. [1], [3]).

Next, we shall study the fundamental system u_1, u_2 of solutions of the linear equation $\varphi(t)x'' = \mu^2 x$. The behaviour of these functions is similar to $\exp \mu t$, $\exp(-\mu t)$ being the fundamental system for nonsingular equations. We shall be interested in asymptotic behaviour of these solutions and the functions v_1, v_2 forming the fundamental system for $\varphi(t)x'' = \nu^2 x$, where $\nu < \mu$. The exact formulas, we shall obtain, enable us to consider the singular case similarly as the nonsingular one.

The method, we use in the paper, is different from the one of [4]. It has proved that the theory of DC-mappings is not necessary to solve the problem. Here, we find solutions of the sequence of BVPs on finite intervals $\langle 0, n\rangle$ by using the Leray–Schauder theory and then we show that the sequence of these solutions is relatively compact in a certain topology. Passing to a convergent subsequence, we obtain a solution of our problem on the half-line. The main result not only generalizes that of [4] but weakens the assumptions of [4] as well.

In the last section, we shall show some applications of the results to the equations of: Boltzmann–Poisson, Thomas–Fermi and Emden–Fowler.

1. The nonsingular boundary value problem. Let us consider the BVP:

$$x'' = f(t, x, x'), \quad x(0) = \alpha \in \mathbf{R}, \quad \lim_{t\to\infty} x(t) = 0. \tag{1.1}$$

We shall assume that f satisfies the following conditions:

$$f\colon \langle 0, \infty) \times \mathbf{R}^2 \to \mathbf{R} \quad \text{is continuous,} \tag{1.2}$$

$$x[f(t, x, -\mu x) - \mu^2 x] > 0 \quad \text{for } |x| \geqslant K\exp(-\mu t), \tag{1.3}$$

where μ and K are positive constants,

$$|f(t, x, p)| \leqslant a(t, x)p^2 + b(t, x), \tag{1.4}$$

where a and b are such functions that, for any $M > 0$,

$$\sup_{t\geqslant 0} \sup_{|x|\leqslant M\exp(-\mu t)} a(t, x)\exp(-\mu t) =: a_M < \infty, \tag{1.5}$$

$$\sup_{t\geqslant 0} \sup_{|x|\leqslant M\exp(-\mu t)} b(t, x)\exp \mu t =: b_M < \infty. \tag{1.6}$$

THEOREM 1. *Under the above assumptions, the* BVP (1.1) *has a solution* $x \in C^2(\langle 0, \infty))$ *such that*

$$\lim_{t\to\infty} x(t)\exp \nu t = 0 = \lim_{t\to\infty} x'(t)\exp \nu t$$

for any $\nu \in (0, \mu)$.

Proof. Let us consider the sequence of BVPs

$$x_n'' = f(t, x_n, x_n'), \quad x_n(0) = \alpha, \quad x_n(n) = 0, \tag{1.7}$$

for $n \in \mathbf{N}$. Each of them has a solution, as we shall show below by using the Leray–Schauder degree.

Fix $n \in \mathbf{N}$ and consider the homotopical family of BVPs

$$x_n'' - \mu^2 x_n = \lambda\big(f(t, x_n, x_n') - \mu^2 x_n\big), \quad x_n(0) = \alpha, \quad x_n(n) = 0, \tag{1.8}$$

where $\lambda \in \langle 0, 1\rangle$. This is equivalent to the equation

$$x_n(t) = w_n(t) + \lambda G\big(f(\cdot, x_n, x_n) - \mu^2 x_n\big)(t), \tag{1.9}$$

where w_n is the unique solution of the problem

$$w_n'' - \mu^2 w_n = 0, \quad w_n(0) = \alpha, \quad w_n(n) = 0,$$

and G is the Green operator of the problem

$$x_n'' - \mu^2 x_n = y, \quad x_n(0) = 0 = x_n(n).$$

The Green operator is linear and completely continuous on the space of C^1-functions defined on $\langle 0, n\rangle$. It suffices to find a common a priori bound for solutions of (1.9). We shall use the equivalent norm in $C^1(\langle 0, n\rangle)$:

$$\|x\| = \max\{\sup_t |x(t)|e^{\mu t}, \ \sup_t |x'(t)|e^{\mu t}\}. \tag{1.10}$$

Let x_n be a solution of (1.9), hence also of (1.8), and let $y_n(t) = x_n(t)\exp\mu t$. If $|y_n|$ takes values greater than $\max\{|\alpha|, K\}$, then there exists t_0 such that either y_n has a maximum at t_0, $y_n(t_0) > K$ or y_n has a minimum at t_0, $y_n(t_0) < -K$. In the first case, $y_n'(t_0) = 0$ and

$$\begin{aligned} 0 \geqslant y_n''(t_0) &= [x_n''(t_0) + 2\mu x_n'(t_0) + \mu^2 x_n(t_0)]\, e^{\mu t_0} \\ &= \lambda[f(t_0, x_n(t_0), -\mu x_n(t_0)) - \mu^2 x_n(t_0)]\, e^{\mu t_0} > 0 \end{aligned}$$

for $\lambda > 0$. The case of minimum leads to the contradiction in the same way. When $\lambda = 0$, $y_n(t) = w_n(t)\exp\mu t$ and the estimate can be obtained explicitly $|y_n(t)| \leqslant |\alpha|$. Thus

$$|y_n(t)| \leqslant \max\{|\alpha|, K\} =: M.$$

Now, we shall estimate y_n',

$$\begin{aligned} y_n'' &= (x_n'' + 2\mu x_n' + \mu^2 x_n)\, e^{\mu t} = (x_n'' - \mu^2 x_n)\, e^{\mu t} + 2\mu y_n' \\ &= \lambda[f(t, x_n, x_n') - \mu^2 x_n]\, e^{\mu t} + 2\mu y_n', \\ |y_n''| &\leqslant \lambda a(t, x_n)\, x_n'^2 \cdot e^{\mu t} + \lambda b(t, x_n)\, e^{\mu t} + \lambda\mu^2 |y_n| + 2\mu|y_n'| \\ &\leqslant \lambda a_M (x_n' e^{\mu t})^2 + \lambda b_M + \lambda\mu^2 M + 2\mu|y_n'| \\ &\leqslant a_M |y_n'|^2 + 2\mu(1 + Ma_M)|y_n'| + b_M + \mu^2 M(1 + Ma_M). \end{aligned}$$

Consider $\psi(z) = a_M z^2 + 2\mu(1 + Ma_M)z + b_M + \mu^2 M(1 + Ma_M)$. Then

$$\int_0^\infty z\psi(z)^{-1}\, dz = \infty. \tag{1.11}$$

Let us divide $\langle 0, n\rangle$ into intervals where y_n' has a constant sign. We should examine cases: (i) $y_n'(t) > 0$ on (t_1, t_2), (ii) $y_n'(t) > 0$ on $\langle 0, t_1)$, (iii) $y_n'(t) > 0$ on $(t_2, n\rangle$, (iv) $y_n'(t) > 0$ on $\langle 0, n\rangle$, and the same for the negative derivative. Let us study (i). Since $y_n'(t_1) = 0$ then

$$\frac{y_n''(t)\, y_n'(t)}{\psi(y_n'(t))} \leqslant y_n'(t), \qquad t \in (t_1, t_2).$$

Integrating both sides over $\langle t_1, t\rangle$, we get

$$\int_0^{y_n'(t)} z\psi(z)^{-1}\, dz \leqslant y_n(t) - y_n(t_1) \leqslant 2M$$

and, by (1.11), $y_n'(t) \leqslant M'$ for some M'. Similar calculations give a priori bounds in the remaining cases (cf. [3]). Thus $|y_n'(t)| \leqslant M'$ for $t \in \langle 0, n\rangle$ and hence

$$|x_n'(t)|\, e^{\mu t} \leqslant M_1 =: M' + \mu M.$$

Therefore, we have obtained a priori estimates for solutions of (1.9) in the space $C^1(\langle 0, n\rangle)$ with the norm given by (1.10). It follows that the Leray–Schauder degrees of $I - G$ and I on the ball centered at 0 with radius $R = \max(M, M_1) + 1$ are the same.

Hence, each of the BVPs (1.7) has a solution x_n, $n \in N$, and their norms (1.10) are uniformly bounded (however, functions x_n belong to different spaces). We extend x_n to the whole half-line, by putting $x_n(t) = 0$ for $t > n$. Let $\nu \in (0, \mu)$. Consider the space of all continuous functions x: $\langle 0, \infty) \to R$ such that the norm

$$\|x\| = \sup_t |x(t)|\, e^{\nu t}$$

is finite. The sequence x_n, $n \in N$, is relatively compact in this space. In fact, it suffices to notice that this sequence satisfies the assumptions of the Ascoli–Arzela Theorem on an arbitrary interval $\langle 0, a\rangle$ and that the limit

$$\lim_{t \to \infty} x_n(t)\, e^{\nu t} = 0$$

is uniform with respect to $n \in N$.

Take a subsequence x_{n_k}, $k \in N$, such that

$$x_{n_k}(t)\, e^{\nu t} \to x(t)\, e^{\nu t}$$

uniformly on the half-line, and fix an interval $\langle 0, a\rangle$. The sequences x_{n_k}' and x_{n_k}'', $k \in N$, are uniformly bounded on $\langle 0, a\rangle$ so, by the Ascoli–Arzela Theorem, there exists a uniformly convergent subsequence, x' is its limit. Applying the diagonal procedure of subsequence choice, we get $x_{n_k}' \to x'$ uniformly on compact subsets of $\langle 0, \infty)$ (we do not change the symbol of the subsequence for the simplicity). At last $x_{n_k}'' = f(t, x_{n_k}, x_{n_k}') \to x''$ uniformly on compact subsets and, therefore, x is a C^2-function satisfying the differential equation and the boundary conditions.

It can be easily shown that the function $t \mapsto x(t)\exp \mu t$ and its derivative are bounded (use the method of getting a priori bounds for y_n, y_n'). It follows that, for $\nu \in (0, \mu)$, x and x' tend to 0 faster than $\exp \nu t$. □

2. Asymptotic behaviour of solutions of linear equations. In this section, we shall deal with the linear equation

$$\varphi(t)x'' = \mu^2 x. \tag{2.1}$$

We shall assume that φ: $\langle 0, \infty) \to \langle 0, \infty)$ is continuous,

$$\varphi(0) = 0, \qquad \varphi(t) > 0 \quad \text{for } t > 0, \tag{2.2}$$

(2.3) $$\varphi(t)^{-1} \quad \text{is integrable at } 0,$$

(2.4) $$\int_0^\infty \varphi(t)^{-1}\,dt = \infty,$$

(2.5) $$\varphi \text{ is increasing for } t \geqslant t_0 \text{ and } \lim_{t\to\infty} \varphi(t) = \infty,$$

(2.6) $$|\varphi(t')-\varphi(t)| \leqslant L|t'-t| \quad \text{for } t',\ t \geqslant t_0.$$

Condition (2.3) gives that all solutions of (2.1) belong to $C^1(\langle 0, \infty)) \cap C^2((0, \infty))$. After [5], we can find the fundamental system of solutions u_1, u_2 such that $u_1(0) = u_2(0) = 1$, u_1 is strictly increasing,

$$\lim_{t\to\infty} u_1(t) = \lim_{t\to\infty} u_1'(t) = \infty,$$

$$u_2(t) = u_1(t)\Big[1-\Big(\int_0^\infty u_1(s)^{-2}\,ds\Big)^{-1}\int_0^t u_1(s)^{-2}\,ds\Big],$$

u_2 is strictly decreasing,

$$\lim_{t\to\infty} u_2(t) = \lim_{t\to\infty} u_2'(t) = 0$$

((2.4) is needed for the last formulas).

PROPOSITION 1. *Under the above assumptions,*

$$\lim_{t\to\infty} \frac{\mu u_1(t)}{\sqrt{\varphi(t)}\,u_1'(t)} = 1.$$

Proof. We omit subindex 1. Let us notice that

$$u'(t)^2-u'(t_0)^2 = 2\mu^2\int_{t_0}^t \varphi(s)^{-1}u(s)u'(s)\,ds = \mu^2\int_{t_0}^t \varphi(s)^{-1}\,du(s)^2$$

$$= \mu^2\Big[\varphi(t)^{-1}u(t)^2-\varphi(t_0)^{-1}u(t_0)^2-\int_{t_0}^t u(s)^2\,d\varphi(s)^{-1}\Big]$$

(the integrals in the sense of Riemann–Stieltjes). Dividing by $u'(t)^2$ and taking into account that $u'(t)\to\infty$ and the last integral is negative ($\varphi(s)^{-1}$ is decreasing), we get the boundedness of

$$\frac{\mu^2 u(t)^2}{\varphi(t)u'(t)^2} \leqslant \gamma.$$

But the above calculations show that

$$\lim_{t\to\infty}\Big\{\frac{\mu^2 u(t)^2}{\varphi(t)u'(t)^2}-\frac{\mu^2}{u'(t)^2}\int_{t_0}^t u(s)^2\,d\varphi(s)^{-1}\Big\} = 1.$$

We shall prove that the second summand tends to 0,

$$-\frac{\mu^2}{u'(t)^2}\int_{t_0}^{t} u(s)^2\, d\varphi(s)^{-1}$$

$$=\frac{1}{u'(t)^2}\int_{t_0}^{t}\frac{\mu^2 u(s)^2}{\varphi(s)\,u'(s)^2}\,u'(s)^2\, d\ln\varphi(s) \leqslant \frac{\gamma}{u'(t)^2}\int_{t_0}^{t} u'(s)^2\, d\ln\varphi(s).$$

Take a division T: $t_0 < t_1 < \ldots < t_n = t$ of the interval $\langle t_0, t\rangle$ such that $L\cdot\operatorname{diam} T \leqslant \varphi(t_0)$. Then we have an estimate for Riemann sums:

$$\sum_{k=1}^{n} u'(t_k)^2 \left(\ln\varphi(t_k) - \ln\varphi(t_{k-1})\right) \leqslant \sum_{k=1}^{n} u'(t_k)^2 \ln[1 + L\varphi(t_{k-1})^{-1}(t_k - t_{k-1})]$$

$$\leqslant 2L\sum_{k=1}^{n} u'(t_k)^2\, \varphi(t_k)^{-1}(t_k - t_{k-1}).$$

Hence

$$\frac{\gamma}{u'(t)^2}\int_{t_0}^{t} u'(s)^2\, d\ln\varphi(s) \leqslant \frac{2L\gamma}{u'(t)^2}\int_{t_0}^{t}\frac{u'(s)^2}{\varphi(s)}\,ds.$$

Applying the l'Hôspital Theorem to the right-hand side, we get the limit of $u'(t)/u(t)$. But

$$\frac{u'(t)^2}{u(t)^2} = \frac{2u'(t_0)^2}{u(t)^2} + \frac{2}{u(t)^2}\int_{t_0}^{t}\mu^2\varphi(s)^{-1}u(s)\,u'(s)\,ds$$

$$\leqslant \frac{2\mu^2}{u(t)}\int_{t_0}^{t}\varphi(s)^{-1}u'(s)\,ds + o(t)$$

and

$$\lim_{t\to\infty} u(t)^{-1}\int_{t_0}^{t}\varphi(s)^{-1}u'(s)\,ds = \lim_{t\to\infty}\varphi(t)^{-1} = 0.$$

Thus

$$\lim_{t\to\infty} u'(t)\,u(t)^{-1} = 0 \tag{2.7}$$

and the proof is complete. □

Proposition 2.

$$\lim_{t\to\infty}\frac{\mu u_2(t)}{\sqrt{\varphi(t)}\,u_2'(t)} = -1.$$

Proof. Let

$$W = \Big(\int_{0}^{\infty} u_1(s)^{-2}\,ds\Big)^{-1}.$$

Then

$$u_2(t) = Wu_1(t) \int_t^{\infty} u_1(s)^{-2}\, ds,$$

$$u_2'(t) = Wu_1'(t) \int_t^{\infty} u_1(s)^{-2}\, ds - Wu_1(t)^{-1}.$$

By Proposition 1,

$$\begin{aligned}\lim_{t\to\infty} \frac{\sqrt{\varphi(t)}\, u_2'(t)}{\mu u_2(t)} &= 1 - \lim_{t\to\infty} \frac{\sqrt{\varphi(t)}}{\mu u_1(t)^2} \Big(\int_t^{\infty} u_1(s)^{-2}\, ds\Big)^{-1} \\ &= 1 - [\lim u_1'(t) u_1(t) \int_t^{\infty} u_1(s)^{-2}\, ds]^{-1} \\ &= 1 - \lim u_1'(t)^{-2} [u_1'(t)^2 + \mu^2 \varphi(t)^{-1} u_1(t)^2] \\ &= 1 - 2 = -1. \ \square\end{aligned}$$

Now, it is easy to get

$$\lim_{t\to\infty} \frac{2\mu u_1(t) u_2(t)}{W\sqrt{\varphi(t)}} = 1, \tag{2.8}$$

where W is from the last proof. The Wrońskian of u_1, u_2 equals $-W$.

Remark. Condition (2.6) can be weakened by

$$|\varphi(t') - \varphi(t)| \leqslant L(t)\, |t' - t| \quad \text{for } |t' - t| < \delta,$$

where $L(t)/\sqrt{\varphi(t)} \to 0$. The proof is quite similar.

Now, let $\nu \in (0, \mu)$ and let v_1, v_2 be the fundamental system of solutions of the equation $\varphi(t) v'' = \nu^2 v$.

PROPOSITION 3.

$$\lim_{t\to\infty} v_1(t) u_1(t)^{-1} = \lim_{t\to\infty} u_2(t) v_2(t)^{-1} = 0.$$

Proof. Due to Proposition 1,

$$\frac{\sqrt{\varphi(t)}\, v_1'(t)}{\nu v_1(t)} < \frac{\sqrt{\varphi(t)}\, u_1'(t)}{\mu u_1(t)} \cdot \frac{\mu}{\nu}$$

for sufficiently large t. It follows that $v_1 \cdot u_1^{-1}$ is decreasing for such t's. On the other hand, $v_1 \leqslant u_1$ by the Comparison Theorem (see [5]). Thus the limit of $v_1 \cdot u_1^{-1}$ when $t \to \infty$ exists and

$$\lim_{t\to\infty} v_1(t) u_1(t)^{-1} = \lim_{t\to\infty} v_1''(t) u_1''(t)^{-1} = \nu^2 \mu^{-2} \lim_{t\to\infty} v_1 \cdot u_1^{-1},$$

Therefore, this limit vanishes. The second part is proved similarly. $\square$

3. The singular boundary value problem. We shall examine the general BVP:

$$\varphi(t)x'' = f(t, x, x'), \quad x(0) = \alpha, \quad \lim_{t\to\infty} x(t) = 0. \tag{3.1}$$

The assumptions on φ are from Section 2: (2.2)–(2.6) but we claim that the Lipschitz condition is satisfied for all t. The assumptions on f are similar to those from Section 1 with slight modifications:

$$f\colon \langle 0, \infty) \times \mathbf{R}^2 \to \mathbf{R} \quad \text{is continuous,} \tag{3.2}$$

$$x\left[f\left(t, x, xu_2'(t)u_2(t)^{-1}\right) - \mu^2 x\right] > 0 \quad \text{for } |x| \geqslant Ku_2(t), \tag{3.3}$$

where u_2 is the solution of the linear problem

$$\varphi(t)u'' = \mu^2 u, \quad u(0) = 1, \quad \lim_{t\to\infty} u(t) = 0, \tag{3.4}$$

$$|f(t, x, p)| \leqslant a(t, x)p^2 + b(t, x), \tag{3.5}$$

where a and b are such functions that, for any $M > 0$,

$$\sup_{t\geqslant 0}\ \sup_{|x|\leqslant Mu_2(t)} a(t, x)u_2(t)\varphi(t)^{-1} =: a_M < \infty, \tag{3.6}$$

$$\sup_{t\geqslant 0}\ \sup_{|x|\leqslant Mu_2(t)} b(t, x)u_2(t)^{-1} =: b_M < \infty, \tag{3.7}$$

Moreover, for any $M > 0$, there exists $\delta > 0$ such that

$$\sup_{t\in\langle 0,\delta\rangle}\ \sup_{|x|\leqslant M}\ \sup_{p\in\mathbf{R}} |f(t, x, p)| =: C_M < \infty. \tag{3.8}$$

THEOREM 2. *Under the above assumptions, the* BVP (3.1) *has a solution x such that, for any $\nu < \mu$ and v_2 being the unique solution of problem* (3.4) *with ν instead of μ,*

$$\lim_{t\to\infty} x(t)v_2(t)^{-1} = 0 = \lim_{t\to\infty} x'(t)v_2(t)^{-1}\sqrt{\varphi(t)}.$$

Proof. Let us consider the BVPs

$$\varphi(t)x_n'' = f(t, x_n, x_n'), \quad x_n(0) = \alpha, \quad x_n(0) = 0 \tag{3.9}$$

for $n\in\mathbf{N}$. Fix $n\in\mathbf{N}$ and take a sequence of BVPs

$$\varphi(t)x_m'' = f(t, x_m, x_m'), \quad x_m(1/m) = \alpha, \quad x_m(n) = 0 \tag{3.10}$$

for $m\in\mathbf{N}$. These problems have solutions belonging to $C^2(\langle m^{-1}, n\rangle)$ (see [3]). We shall establish the solvability of (3.9) within $C^2((0, n\rangle)\cap C^1(\langle 0, n\rangle)$ if we find common bounds for sup-norms of solutions of (3.10) and its derivatives.

Put $y_m(t) = x_m(t)u_2(t)^{-1}$ and suppose that $|y_m(t)| > \max(|\alpha|, K)$ for a certain t. Then there exists t_0 such that $y_m(t_0) > K$, $y_m'(t_0) = 0$, $y_m''(t_0) \leqslant 0$ (or $y_m(t_0) < -K$, $y_m'(t_0) = 0$, $y_m''(t_0) \geqslant 0$). Hence $x_m'(t_0)u_2(t_0) = x_m(t_0)u_2'(t_0)$

and

$$0 \geqslant y_m''(t_0) = [\varphi(t_0)x_m''(t_0) - \mu^2 x_m(t_0)]/\varphi(t_0)u_2(t_0) > 0$$

by (3.3). It follows that the sup-norm $\|y_m\| \leqslant \max(|\alpha|, K)$ independently of $m \in N$ and, therefore, $\|x_m\| \leqslant M$ for a certain M.

Let δ in (3.8) be chosen for this M. Due to the Mean Value Theorem

$$\inf_{t \in \langle m^{-1}, \delta \rangle} |x_m'(t)| \leqslant 4M/\delta$$

for sufficiently large m ($m \geqslant 2/\delta$). Thus, for $t \leqslant \delta$,

$$|x_m'(t)| \leqslant \frac{4M}{\delta} + C_M \int_0^\delta \varphi(s)^{-1}\, ds.$$

For $t > \delta$, we use the Bernstein method with (3.5) (see [3]). It is possible, since $\langle \delta, n \rangle \ni t \mapsto \varphi(t)^{-1}$ is bounded. Therefore, x_m' are uniformly bounded and we can apply the Ascoli–Arzela Theorem in any interval $\langle d, n \rangle$, $d > 0$, to get a solution of (3.9).

Now, we apply the compactness arguments to the sequence x_n, $n \in N$, of solutions of (3.9). Let $y_n(t) = x_n(t)u_2(t)^{-1}$ as previously. Repeating the above consideration, we get $|y_n(t)| \leqslant M = \max(|\alpha|, K)$. Similarly, one can obtain a common bound for y_n' on $\langle 0, \delta \rangle$. This estimation is more difficult for $t > \delta$. By Proposition 1,

$$\sup_{t > \delta} \frac{\sqrt{\varphi(t)}\,|u_2'(t)|}{u_2(t)} =: C < \infty.$$

Below, we omit the arguments $t > \delta$ and $|x| \leqslant M$ in the following calculations:

$$\begin{aligned}\varphi|y_n''| &\leqslant \frac{a x_n'^2 + b + \mu^2 |x_n|}{u_2} + 2\sqrt{\varphi}\,|y_n'|\,\frac{\sqrt{\varphi}\,|u_2'|}{u_2} \\ &\leqslant a y_n'^2 u_2 + a x_n^2 u_2'^2 u_2^{-3} + 2a|y_n'|\,|x_n|\,|u_2'| + b_M + \mu^2 M + 2C\sqrt{\varphi}\,|y_n'| \\ &\leqslant a_M(\sqrt{\varphi}\,|y_n'|)^2 + 2C(a_M + 1)(\sqrt{\varphi}\,|y_n'|) + a_M M^2 C^2 + b_M + \mu^2 M \\ &=: c_1(\sqrt{\varphi}\,|y_n'|)^2 + c_2(\sqrt{\varphi}\,|y_n'|) + c_3.\end{aligned}$$

For $n \geqslant n_0$, we have

$$\inf |y_n'(t)| \leqslant \frac{2M}{n_0 - \delta} =: \gamma.$$

Divide $\langle \delta, n \rangle$ into intervals on which y_n' has a constant sign. We shall find a bound for $\sqrt{\varphi}\, y_n'$ on the interval $\langle t_1, t_2 \rangle$ such that

$$y_n'(t) > 0, \quad t \in (t_1, t_2), \quad |y_n'(t_1)| \leqslant \gamma.$$

The remaining cases are examined similarly.

Write

$$\psi(z) = c_1 z^2 + c_2 z + c_3.$$

Then $\varphi(t) y_n''(t) \leqslant \psi(\sqrt{\varphi(t)}\, y_n'(t))$ and

$$\int_{\gamma}^{\sqrt{\varphi(t)}y_n'(t)} z\psi(z)^{-1}\,dz = \int_{t_1}^{t} \sqrt{\varphi}\, y_n' \psi(\sqrt{\varphi}\, y_n')^{-1}\, d(\sqrt{\varphi}\, y_n')$$
$$\leqslant \int_{t_1}^{t} y_n'' y_n' \psi(\sqrt{\varphi}\, y_n')^{-1}\, ds + \tfrac{1}{2} L \int_{t_1}^{t} y_n'^2 \psi(\sqrt{\varphi}\, y_n')^{-1}\, ds,$$

where we have applied the Riemann sums and the Lipschitz condition for φ. The function under the first integral is not greater than y_n', so this integral is bounded by $2M$. In order to estimate the second summand, we choose $\bar{t}$ such that $\varphi(t) \geqslant 1$ for $t \geqslant \bar{t}$ and divide $\langle t_1, t\rangle$ into $\langle t_1, \bar{t}\rangle$ and $\langle \bar{t}, t\rangle$ if $\bar{t} \in (t_1, t)$. The function under the integral is not greater than $c_4 \varphi^{-1}$ on $\langle t_1, \bar{t}\rangle$, where c_4 is a bound of $z^2\psi(z)^{-1}$. On $\langle \bar{t}, t\rangle$, we estimate the integrated function by $c_2^{-1} y_n'$. Therefore, we have

$$\int_{\gamma}^{\sqrt{\varphi(t)}y_n'(t)} z\psi(z)^{-1}\,dz \leqslant 2M + \tfrac{1}{2} L c_4 \int_0^{\bar{t}} \varphi^{-1}\,ds + L c_2^{-1} M.$$

But the function $z\psi(z)^{-1}$ is not integrable on unbounded intervals, thus $\sqrt{\varphi}\, y_n' \leqslant \tilde{M}$.

Now, we know that functions y_n, $\sqrt{\hat{\varphi}}\, y_n'$ are bounded by constants independent of $n \in N$, where $\hat{\varphi}(t) = \varphi(t)$ for $t > \delta$, $\hat{\varphi}(t) = \varphi(\delta)$ for $t \in \langle 0, \delta\rangle$. Hence the sequence $\varphi y_n''$, $n \in N$, is also bounded. It follows that the same is true for sequences $x_n u_2^{-1}$, $\sqrt{\hat{\varphi}}\, x_n' u_2^{-1}$, $\varphi x_n'' u_2^{-1}$, $n \in N$. Applying arguments similar to those from Section 1 and Proposition 3, we get a subsequence x_{n_k}, $k \in N$, uniformly convergent and such that x_{n_k}', $k \in N$, and x_{n_k}'', $k \in N$, are uniformly convergent on compact subsets of $\langle 0, \infty)$ and $(0, \infty)$, respectively. Thus $x = \lim x_{n_k}$ is twice differentiable on $(0, \infty)$ and once on $\langle 0, \infty)$, and it satisfies the differential equation and the first boundary condition $x(0) = \alpha$.

The above a priori estimates technique can be applied to x and it gives that xu_2^{-1} and $\sqrt{\hat{\varphi}}\, x' u_2^{-1}$ are bounded. Hence

$$\lim_{t\to\infty} x(t) v_2(t)^{-1} = 0 = \lim_{t\to\infty} x'(t) v_2(t)^{-1} \sqrt{\varphi(t)}$$

for any $v < \mu$ and v_2 being the decreasing solution of the problem $\varphi(t) v'' = v^2 v$, $v(0) = 1$, $v(\infty) = 0$. □

4. Applications and remarks. Theorems 1 and 2 can be applied to several important equations.

1. $x'' = \operatorname{sh} x$ — the Boltzmann–Poisson equation from the theory of electrolits,

2. $\sqrt{t}\,x'' = \sqrt{x^3}$ — the Thomas–Fermi equation of the ionized atom. Here $\alpha > 0$. It is easy to see that a solution (if it exists) is a positive function, hence we can apply Theorem 2, though f is defined only for $x \geqslant 0$.

3. $\varphi(t)x'' = x^{\beta}$ — the Emden–Fowler equation describing polytropic gas. This equation generalizes 2. Assumptions on φ are from Section 3 and $\beta > 0$.

If, in Theorem 2, the function f does not depend on the first derivative x', it is not necessary to assume that φ satisfies the Lipschitz condition on the whole domain.

The question of uniqueness can be solved by giving some classical assumptions. The most important is the monotonicity of f with respect to the variable x (cf. [3]).

One can consider different BVPs by using the described method:

$$\alpha x(0)+\beta x'(0) = \gamma, \qquad \lim_{t\to\infty} x(t) = 0.$$

The second boundary condition is, however, essential.

Recently, Furi and Pera [2] presented the general continuation method of the solvability of BVPs on noncompact intervals. It seems that this method works also in our case but the estimation of asymptotic behaviour of solutions can be obtained rather in the above framework. On the other hand, the calculations leading to a priori bounds are always necessary and similar.

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Reçu par la Rédaction le 29.02.1988