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TIGHT EXTENSIONS OF GROUP-VALUED QUASI-MEASURES

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The main result of the paper (Theorem 1) is concerned with extension of an additive and exhaustive set function μ on an algebra $\mathfrak{M}$, further called a quasi-measure, with values in an Abelian complete Hausdorff topological group. The quasi-measure μ is assumed to be $\mathfrak{K}$-tight, where $\mathfrak{K}$ is a lattice of sets. (In Marczewski's terminology [13], $\mathfrak{K}$ approximates $\mathfrak{M}$ with respect to μ.) The domain of the extension ν is the algebra generated by $\mathfrak{M} \cup \mathfrak{K}$ and ν is a $\mathfrak{K}$-tight quasi-measure. The result is an improvement of an earlier theorem of the author [8]. Its two major consequences (Theorems 2 and 3) imply the corresponding results of Henry [4], Lembcke [7], Bachman and Sultan [1], and Dalgas [2], which are, in turn, generalizations of several previous results. It should be pointed out, however, that, in contrast with the effective method of Dalgas, the proof of Theorem 1 is necessarily based on an uncountable form of the axiom of choice.

The terminology and notation to be used below mostly follow those of [8]. Throughout G denotes an Abelian complete Hausdorff topological group and X stands for an arbitrary (nonempty) set. The family of all subsets of X is denoted by 2^X.

Let $\mathfrak{M}$ be an algebra of subsets of X. An additive set function μ: $\mathfrak{M} \to G$ generates a group topology on $\mathfrak{M}$ (equipped with the symmetric difference of sets as the group operation), the *μ-topology*, which is determined by the neighbourhood base at $\emptyset$

$$\{M \in \mathfrak{M}: \ \mu(S) \in V \text{ for all } M \supset S \in \mathfrak{M}\},$$

where V runs through a neighbourhood base at 0 in G (see, e.g., [8], p. 24). The denseness and closure with respect to this topology will be referred to as *μ-denseness* and *μ-closure*, respectively.

We say that μ is *$\mathfrak{K}$-tight*, where $\mathfrak{K} \subset 2^X$, if for every $M \in \mathfrak{M}$ and a neighbourhood V of 0 in G there exist $K \in \mathfrak{K}$ and $\tilde{M} \in \mathfrak{M}$ such that $\tilde{M} \subset K \subset M$ and

$$\mu(S) \in V \text{ whenever } S \in \mathfrak{M} \text{ and } S \subset M \setminus \tilde{M} \ (^1).$$

(1) For positive real-valued set functions this notion was introduced by Marczewski [13], p. 116; see also [4], p. 237.

In case $\mathfrak{K} \subset \mathfrak{M}$, we can, clearly, take $\tilde{M} = K$. In that generality the notion of $\mathfrak{K}$-tightness appears in [8], Definition 2, and, under the name of $\mathfrak{K}$-regularity, in [2], 2.6, and [6], p. 188.

We say that an additive set function μ: $\mathfrak{M} \to G$ is a *quasi-measure* if it is *exhaustive*, i.e., $\mu(M_i) \to 0$ for every sequence (M_i) of pairwise disjoint sets in $\mathfrak{M}$ ([8], Definition 1). The function μ generates its inner and outer extensions μ_* and μ^* to 2^X (see [8], p. 22, for definitions).

A set function μ: $\mathfrak{M} \to G$ is called a *measure* if $\mathfrak{M}$ is a σ-algebra and μ is σ-additive.

LEMMA. *Let $\mathfrak{M}$ be an algebra of subsets of X, let $\mathfrak{K} \subset 2^X$ and let $Z \subset X$. Every $\mathfrak{K}$-tight quasi-measure μ: $\mathfrak{M} \to G$ extends to a quasi-measure ν: $\mathfrak{N} \to G$, where $\mathfrak{N}$ is the algebra generated by $\mathfrak{M} \cup \{Z\}$, such that ν is tight with respect to the family*

$$\{(K_1 \cap Z) \cup K_2 \colon K_1, K_2 \in \mathfrak{K} \text{ and } K_2 \subset Z^c\}$$

and $\mathfrak{M}$ is ν-dense in $\mathfrak{N}$.

Proof. Put $\nu(N) = \mu^*(N \cap Z) + \mu_*(N \cap Z^c)$ for $N \in \mathfrak{N}$. In view of [8], Lemma 5 and its proof, we only have to show the tightness assertion. Since

$$\mathfrak{N} = \{(M_1 \cap Z) \cup (M_2 \cap Z^c) \colon M_1, M_2 \in \mathfrak{M}\}$$

and $\emptyset \in \mathfrak{K}$, it is enough to establish the approximation condition separately for the disjoint sets $M_1 \cap Z$ and $M_2 \cap Z^c$. Fix a closed neighbourhood V of 0 in G.

Choose $K_1 \in \mathfrak{K}$ and $\tilde{M}_1 \in \mathfrak{M}$ such that $\tilde{M}_1 \subset K_1 \subset M_1$ and $\mu(S) \in V$ whenever $S \in \mathfrak{M}$ and $S \subset M_1 \setminus \tilde{M}_1$. We have $\tilde{M}_1 \cap Z \subset K_1 \cap Z \subset M_1 \cap Z$. Since $\mathfrak{M}$ is ν-dense in $\mathfrak{N}$, it follows from [10], Lemma, that $\nu(N) \in V$ whenever $N \in \mathfrak{N}$ and $N \subset M_1 \setminus \tilde{M}_1$. In particular, the same holds whenever $N \in \mathfrak{N}$ and

$$N \subset (M_1 \cap Z) \setminus (\tilde{M}_1 \cap Z).$$

Let W be a neighbourhood of 0 in G with $W + W \subset V$. According to [8], Lemma 1, there exists $M \in \mathfrak{M}$ such that $M \subset M_2 \cap Z^c$ and $\mu(S) \in W$ whenever $S \in \mathfrak{M}$ and $S \subset (M_2 \cap Z^c) \setminus M$. Since μ is $\mathfrak{K}$-tight, there exist $K_2 \in \mathfrak{K}$ and $\tilde{M}_2 \in \mathfrak{M}$ such that $\tilde{M}_2 \subset K_2 \subset M$ and $\mu(S) \in W$ whenever $S \in \mathfrak{M}$ and $S \subset M \setminus \tilde{M}_2$. We have $\tilde{M}_2 \subset K_2 \subset M_2 \cap Z^c$. Assume that $N \in \mathfrak{N}$ and $N \subset (M_2 \cap Z^c) \setminus \tilde{M}_2$. For every $S \in \mathfrak{M}$ with $S \subset N$ we have

$$S = (S \cap M) \cup (S \setminus M) \quad \text{and} \quad S \cap M \subset M \setminus \tilde{M}_2 \quad \text{and} \quad S \setminus M \subset (M_2 \cap Z^c) \setminus M.$$

Hence $\mu(S) \in V$. It follows that $\nu(N) = \mu_*(N) \in V$.

The following result generalizes [8], Theorem 3 (see also [14], Corollary), and, partially, [7], Satz 3.1. The proof is based on a method due to J. Łoś and E. Marczewski. In fact, we use a combination of two improvements of this method due to Henry [4] (see also [1]) and the author [8] (see also [11], Remark 5).

THEOREM 1. *Let $\mathfrak{M}$ be an algebra of subsets of X and let $\mathfrak{K}$ be a lattice of subsets of X. Every $\mathfrak{K}$-tight quasi-measure μ: $\mathfrak{M} \to G$ extends to a $\mathfrak{K}$-tight quasi-measure φ: $\mathfrak{F} \to G$, where $\mathfrak{F}$ is the algebra generated by $\mathfrak{M} \cup \mathfrak{K}$, such that $\mathfrak{M}$ is φ-dense in $\mathfrak{F}$.*

Proof. Consider the class M of all pairs $(\mathfrak{N}, \nu)$ with the following properties:

(a) $\mathfrak{N}$ is an algebra of subsets of X with $\mathfrak{M} \subset \mathfrak{N} \subset \mathfrak{F}$.

(b) ν: $\mathfrak{N} \to G$ is a quasi-measure.

(c) $\nu | \mathfrak{M} = \mu$.

(d) $\mathfrak{M}$ is ν-dense in $\mathfrak{N}$.

(e) ν is $\mathfrak{K}$-tight.

We define a (partial) ordering $\leqslant$ in M by putting $(\mathfrak{N}_1, \nu_1) \leqslant (\mathfrak{N}_2, \nu_2)$ provided that $\mathfrak{N}_1 \subset \mathfrak{N}_2$ and $\nu_2 | \mathfrak{N}_1 = \nu_1$. Let $\{(N_t, \nu_t)\colon t \in T\}$ be a chain in M. Put $\mathfrak{N} = \bigcup_{t \in T} \mathfrak{N}_t$ and $\nu(N) = \nu_t(N)$ if $N \in \mathfrak{N}_t$. We claim that $(\mathfrak{N}, \nu) \in M$ and $(\mathfrak{N}_t, \nu_t) \leqslant (\mathfrak{N}, \nu)$. Clearly, (a) holds and ν is well defined and additive on $\mathfrak{N}$. Moreover, (c) holds. Property (d) follows from [11], Lemma 1. Hence, in view of [8], Lemma 4, ν is exhaustive, and so (b) holds. To prove (e), fix $N \in \mathfrak{N}$. Since $N \in \mathfrak{N}_t$ for some $t \in T$, given a closed neighbourhood V of 0 in G, there exist $K \in \mathfrak{K}$ and $\tilde{N} \in \mathfrak{N}_t$ such that $\tilde{N} \subset K \subset N$ and $\nu_t(S) \in V$ whenever $S \in \mathfrak{N}_t$ and $S \subset N \setminus \tilde{N}$. Since $N \setminus \tilde{N} \in \mathfrak{N}_t$ and $\mathfrak{N}_t$ is ν-dense in $\mathfrak{N}$, it follows from [10], Lemma, that $\nu(R) \in V$ whenever $R \in \mathfrak{N}$ and $R \subset N \setminus \tilde{N}$.

Let $(\mathfrak{N}_0, \nu_0)$ be a maximal element of M with respect to the ordering $\leqslant$ whose existence follows from the Kuratowski–Zorn Lemma. We claim that $\mathfrak{N}_0 = \mathfrak{F}$, which proves the theorem. Otherwise, take $Z \in \mathfrak{K} \setminus \mathfrak{N}_0$ and apply the lemma above to $\mathfrak{N}_0$, $\mathfrak{K}$, Z and ν_0. The resulting pair $(\mathfrak{N}_1, \nu_1)$ obviously satisfies conditions (a)–(c) and (e), while (d) follows from [10], Lemma. Moreover, $(\mathfrak{N}_0, \nu_0) \leqslant (\mathfrak{N}_1, \nu_1)$. Thus $(\mathfrak{N}_0, \nu_0)$ is not maximal in M, a contradiction.

Remark. In the situation of Theorem 1, if μ is $\mathfrak{L}$-tight, where $\mathfrak{L} \subset \mathfrak{K}$, then $\mathfrak{L}$ is φ-dense in $\mathfrak{F}$. Indeed, by the definition of $\mathfrak{L}$-tightness and [10], Lemma, the φ-closure of $\mathfrak{L}$ in $\mathfrak{N}$ contains $\mathfrak{M}$.

Our next result partially generalizes a theorem of Lembcke ([7], Satz 4.5). The case where $\mathfrak{M}$ is a σ-algebra and μ is $(\mathfrak{K} \cap \mathfrak{M})$-tight and takes values in $[0, \infty)$ is due to Plebanek ([15], Theorem 2.6.2).

THEOREM 2. *Let $\mathfrak{M}$ be an algebra of subsets of X and let $\mathfrak{K}$ be a lattice of subsets of X such that for every sequence (K_n) in $\mathfrak{K}$ with $K_n \downarrow \emptyset$, we have $K_{n_0} = \emptyset$ for some n_0* (2). *Then every $\mathfrak{K}$-tight quasi-measure μ: $\mathfrak{M} \to G$ extends*

(2) That is, $\mathfrak{K}$ is a compact class of sets in the sense of Marczewski ([13], Section 2; cf. also [12], p. 22).

to a $\mathfrak{K}_\delta$-tight measure ν: $\mathfrak{N} \to G$, where $\mathfrak{N}$ is the σ-algebra generated by $\mathfrak{M} \cup \mathfrak{K}$, such that $\mathfrak{M}$ is ν-dense in $\mathfrak{N}$.

Proof. By Theorem 1, μ extends to a $\mathfrak{K}$-tight quasi-measure φ: $\mathfrak{F} \to G$, where $\mathfrak{F}$ is the algebra generated by $\mathfrak{M} \cup \mathfrak{K}$, such that $\mathfrak{M}$ is φ-dense in $\mathfrak{F}$. In view of [9], Theorem 1, φ is σ-additive. Hence φ extends uniquely to a measure ν: $\mathfrak{N} \to G$ (see, e.g., [3], Theorem 9.2). In view of [9], Lemma 4, ν is $\mathfrak{K}_\delta$-tight. Moreover, $\mathfrak{F}$ is ν-dense in $\mathfrak{N}$ (cf. [3], Theorem 8.2). It follows that $\mathfrak{M}$ is ν-dense in $\mathfrak{N}$ ([10], Lemma).

The following theorem generalizes results of Henry ([4], Théorème 1; cf. also Lemme 1 thereof), Bachman and Sultan ([1], Theorem 2.1), and Dalgas ([2], Theorem 3.6 without property (*)).

THEOREM 3. *Let $\mathfrak{M}$ be an algebra of subsets of X and let $\mathfrak{K}$ and $\mathfrak{L}$ be lattices of subsets of X such that $\mathfrak{L} \subset \mathfrak{K}$ and for every sequence (K_n) in $\mathfrak{K}$ with $K_n \downarrow \emptyset$ and $L \in \mathfrak{L}$ there exists a sequence (M_n) in $\mathfrak{M}$ with $M_n \downarrow \emptyset$ and $K_n \cap L \subset M_n$ for $n = 1, 2, \ldots$* [3]. *Then every σ-additive $\mathfrak{L}$-tight quasi-measure μ: $\mathfrak{M} \to G$ extends to a $\mathfrak{K}_\delta$-tight measure ν: $\mathfrak{N} \to G$, where $\mathfrak{N}$ is the σ-algebra generated by $\mathfrak{M} \cup \mathfrak{K}$, such that $\mathfrak{M}$ is ν-dense in $\mathfrak{N}$. If, additionally, $K \cap L \in \mathfrak{L}$ whenever $K \in \mathfrak{K}_\delta$ and $L \in \mathfrak{L}$, then ν is $\mathfrak{L}$-tight.*

Proof. Let φ and $\mathfrak{F}$ have the same meaning as in the proof of Theorem 2. Once we know that φ is σ-additive, we can proceed exactly as in the proof of that theorem.

Since φ is $\mathfrak{K}$-tight, to prove the σ-additivity of φ, it is enough to show that for every sequence (K_n) in $\mathfrak{K}$ with $K_n \downarrow \emptyset$ we have $\varphi(K_n) \to 0$ ([9], Theorem 1). Fix a closed neighbourhood V of 0 in G and choose $L \in \mathfrak{L}$ such that $\varphi(F) \in V$ whenever $F \in \mathfrak{F}$ and $F \cap L = \emptyset$ (see the remark above). By assumption, there exists a sequence (M_n) in $\mathfrak{M}$ with $M_n \downarrow \emptyset$ and $K_n \cap L \subset M_n$ for $n = 1, 2, \ldots$ Let n_0 be such that $\mu(M) \in V$ whenever $M \in \mathfrak{M}$ and $M \subset M_{n_0}$ (see, e.g., [6], Lemma 13). Then $\varphi(F) \in V$ whenever $F \in \mathfrak{F}$ and $F \subset M_{n_0}$ ([10], Lemma). In particular, $\varphi(K_n \cap L) \in V$ for $n \geqslant n_0$. It follows that

$$\varphi(K_n) = \varphi(K_n \cap L) + \varphi(K_n \setminus L) \in V + V$$

for $n \geqslant n_0$.

To establish the additional assertion, take V and L as in the first part of the proof. Fix $N \in \mathfrak{N}$ and choose $K \in \mathfrak{K}_\delta$ such that $K \subset N$ and $\nu(R) \in V$ whenever $R \in \mathfrak{N}$ and $R \subset N \setminus K$. Then $K \cap L \in \mathfrak{L}$ and $K \cap L \subset N$. Moreover, if $R \in \mathfrak{N}$ and $R \subset N \setminus (K \cap L)$, then

$$R = [R \cap (N \setminus K)] \cup [R \cap (K \setminus L)].$$

Hence, in view of [10], Lemma, $\nu(R) \in V + V$.

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[3] That is, $\mathfrak{M}$ dominates $\mathfrak{K}$ on $\mathfrak{L}$ in the terminology of [2], Definition 3.2. In case $X \in \mathfrak{L}$, this is identical with $\mathfrak{K}$ being $\mathfrak{M}$-countably paracompact in the terminology of [1].

The next simple result yields a sufficient condition for the uniqueness of an extension in the situation of Theorems 1–3.

PROPOSITION (cf. [1], Theorem 2.1, and [2], Lemma 5.18(1)). *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be algebras of subsets of* X *with* $\mathfrak{M} \subset \mathfrak{N}$ *and let* v: $\mathfrak{N} \to G$ *be a* $\mathfrak{K}$*-tight additive set function, where* $\mathfrak{K} \subset \mathfrak{N}$. *If every pair of disjoint sets in* $\mathfrak{K}$ *can be separated by disjoint sets in* $\mathfrak{M}$, *then for every* $K \in \mathfrak{K}$ *we have*

$$v(K) = \lim \{v(M)\colon\ K \subset M \in \mathfrak{M}\},$$

where the index set $\{M \in \mathfrak{M}\colon\ K \subset M\}$ *is directed downwards by inclusion.*

Proof. Fix $K \in \mathfrak{K}$ and a neighbourhood V of 0 in G. Then we can find $K_1 \in \mathfrak{K}$ with the properties: $K_1 \subset X \setminus K$ and $v(N) \in V$ whenever $N \in \mathfrak{N}$ and $N \subset X \setminus (K \cup K_1)$. By assumption, there exists $M \in \mathfrak{M}$ with $K \subset M$ and $M \cap K_1 = \emptyset$. Let $\tilde{M} \in \mathfrak{M}$ and $K \subset \tilde{M} \subset M$. Then

$$\tilde{M} \setminus K \subset X \setminus (K \cup K_1), \quad \text{whence } v(\tilde{M}) - v(K) = v(\tilde{M} \setminus K) \in V.$$

The following corollary to Theorem 2 improves a result of Khurana ([5], Theorem 1), which is, in turn, a generalization of the corresponding result for positive real-valued measures due to H. Bauer, J. Hardy and H. E. Lacey, and Henry [4] (see [5] for other references). Before formulating the corollary, we recall that a G-valued measure μ on the Borel σ-algebra $\mathfrak{B}(Y)$ of a compact (Hausdorff) space Y is termed *(inner) regular* if it is $\mathfrak{K}(Y)$-tight, where $\mathfrak{K}(Y)$ is the family of all compact subsets of Y.

COROLLARY. *Let* X *and* Y *be compact topological spaces and let* $f\colon X \to Y$ *be continuous and surjective. Then for every regular measure* $\mu\colon \mathfrak{B}(Y) \to G$ *there exists a regular measure* $v\colon \mathfrak{B}(X) \to G$ *such that* $\mu(B) = v(f^{-1}(B))$ *for all* $B \in \mathfrak{B}(Y)$ *and the* σ*-algebra*

$$\mathfrak{M} = \{f^{-1}(B)\colon\ B \in \mathfrak{B}(Y)\}$$

is v*-dense in* $\mathfrak{B}(X)$.

Proof (cf. [4], proof of Théorème 2). Put $\mu_0(f^{-1}(B)) = \mu(B)$ for all $B \in \mathfrak{B}(Y)$. Since f is surjective, μ_0 is well defined and σ-additive on $\mathfrak{M}$. Since f is continuous, $\mathfrak{M} \subset \mathfrak{B}(X)$ and $\{f^{-1}(K)\colon\ K \in \mathfrak{K}(Y)\} \subset \mathfrak{K}(X)$. It follows that μ_0 is $\mathfrak{K}(X)$-tight. The assertion is now a consequence of Theorem 2.

Postscript. A recent paper by Adamski [0], which appeared after the submission of our paper for publication, deals with related problems concerning tight set functions with values in $[0, \infty]$. We shall compare some results of [0] with those obtained above. The notation used in the sequel without explanation follows [0].

1. In the case where λ is bounded, Lemma 2.1 and Theorem 2.2 of [0] follow from the lemma and Theorem 1 above with the help of [9], Theorems 1 and 3(d).

2. For finite μ, Theorem 3.1 of [0] follows from Theorem 3 above with $\mathfrak{M} = \mathscr{A}$, $\mathfrak{K} = \mathscr{K}_2$ and $\mathfrak{L} = \mathscr{K}_1$ with the help of [13], 4(i).

3. The existence part of [0], Theorem 3.4(a), follows from Theorem 1 above, while the uniqueness part is a consequence of the proposition above.

4. The existence part of [0], Theorem 3.4(b), follows from Theorem 3 above with $\mathfrak{M} = \sigma(\mathscr{K}_1)$, $\mathfrak{K} = \mathscr{K}_2$ and $\mathfrak{L} = \mathscr{K}_1$, while the uniqueness part is a consequence of the proposition above.

5. For finite μ, the existence parts of Theorems 3.2 and 3.3(a) of [0] follow from Theorems 1 and 3 above, respectively, with the help of [8], Theorem 2. The details will appear elsewhere.

Added in proof. Related results are contained in the author's paper *On unique extensions of positive additive set functions. II*, Archiv der Mathematik, to appear.

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