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On the Chaplighin method for partial differential equations of the first order

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In the present paper we discuss several topics concerning the Chaplighin method (see [1] and [2], pp. 145-208) for first order partial differential equations. The basic tool in our investigations are theorems of Szarski [6]-[9] and some of their limit forms.

In sections 1 and 2 we describe some general features of the Chaplighin method in case the unknown function as well the members of the Chaplighin sequences are defined in a suitable Haar pyramid. The rest of the paper deals with more general domains than Haar pyramids but for hyperbolic systems. There is discussed in details the Chaplighin method for hyperbolic equations and the results are more complete than in case of general systems investigated in sections 1 and 3.

1. We write $y = (y_1, y_2, \ldots, y_n)$, $z = (z_1, z_2, \ldots, z_m)$ and $q = (q_1, \ldots, q_n)$. The function $f(x, y_1, \ldots, y_n, z_1, \ldots, z_m, q_1, \ldots, q_n)$ can be shortly written as $f(x, y, z, q)$; x stands for a single real variable. If the partial derivatives $\partial u/\partial y_i$ exist, we write $\partial u/\partial y = (\partial u/\partial y_1, \ldots, \partial u/\partial y_n)$. We write $(\bar{z}_1, \ldots, \bar{z}_m) = \bar{z} \leqslant \bar{\bar{z}} = (\bar{\bar{z}}_1, \ldots, \bar{\bar{z}}_m)$ if $\bar{z}_k \leqslant \bar{\bar{z}}_k$ for $k = 1, \ldots, m$.

For the sake of simplicity we introduce the following conditions:

Condition (A). The functions $f^i(x, y, z, q)$, $i = 1, \ldots, m$, are defined in the set

$$(1.1) \qquad 0 \leqslant x < a, \quad |y_i| \leqslant b_i - Mx \quad (i = 1, \ldots, n)$$

$(a \leqslant \min\limits_i b_i/M)$ and z, q arbitrary, and satisfy

$$(1.2) \qquad |f^i(x, y, z, \bar{q}) - f^i(x, y, z, \bar{\bar{q}})| \leqslant M \sum_{k|1}^{n} |\bar{q}_k - \bar{\bar{q}}_k|$$

for $i = 1, \ldots, m$ and $x > 0$.

Condition (W). For every index k, if $\bar{z}_i \leqslant \bar{\bar{z}}_i$ for $i \neq k$ and $\bar{z}_k = \bar{\bar{z}}_k$, then $f^k(x, y, \bar{z}, q) \leqslant f^k(x, y, \bar{\bar{z}}, q)$, $x > 0$.

We are now able to formulate the following lemma (see [6]):

LEMMA 1.1. *Let Conditions* (A) *and* (W) *be satisfied. Assume that the vector functions* $u = \big(u^1(x,y), \ldots, u^m(x,y)\big)$ *and* $v = \big(v^1(x,y), \ldots, v^m(x,y)\big)$ *are continuous in* (1.1), *and* $u^i(0,y) < v^i(0,y)$ *for* $i = 1, \ldots, m$, $|y_j| \leqslant b_j$ $(j = 1, \ldots, n)$.

Suppose that all u^i *and* v^i *have the total differentials in* (1.1) *for* $x > 0$ *and satisfy there the inequalities*

$$u_x^i < f^i\big(x, y, u(x,y), u_y^i(x,y)\big),$$

$$v_x^i \geqslant f^i\big(x, y, v(x,y), v_y^i(x,y)\big)$$

for $i = 1, \ldots, m$. *Then* $u^i(x,y) < v^i(x,y)$ *in* (1.1) *for* $i = 1, \ldots, m$.

Suppose now that the functions $g^i(x, y, z, q)$ have partial derivatives $g^i_{z_k}, g^i_{q_j}$ in (1.1).

We will prove the following theorem:

THEOREM 1.1. *Let the functions* $u^i(x,y)$, $i = 1, \ldots, m$, *be continuous in* (1.1) *and suppose that they have total differentials for* $x > 0$.

Assume that

$$u_x^i < g^i(x, y, u, u_y^i) \text{ in } (1.1) \text{ for } x > 0 \text{ and } i = 1, \ldots, m. \tag{1.3}$$

Suppose that

$$g^i_{z_k}\big(x, y, u(x,y), u_y^i(x,y)\big) \geqslant 0 \text{ for } i \neq k \text{ and } x > 0 \tag{1.4}$$

and

$$\big|g^i_{q_j}\big(x, y, u(x,y), u_y^i(x,y)\big)\big| \leqslant M \text{ for } i = 1, \ldots, m,\ j = 1, \ldots, n \tag{1.5}$$

in (1.1) *for* $x > 0$.

Let the functions $v^i(x,y)$, $i = 1, \ldots, m$, *be continuous in* (1.1) *and assume that* v^i *have total differentials in* (1.1) *for* $x > 0$. *We assume that* v^i *satisfy in* (1.1), *for* $x > 0$, *the system of equations*

$$\begin{aligned} v_x^i = g^i(x, y, u, u_y^i) + \sum_{k|1}^{m} g^i_{z_k}(x, y, u, u_y^i)(v^k - u^k) + \\ + \sum_{j|1}^{n} g^i_{q_j}(x, y, u, u_y^i)(v^i_{y_j} - u^i_{y_j}). \end{aligned} \tag{1.6}$$

Under the above assumptions, if $u^i(0,y) < v^i(0,y)$ *for* $i = 1, \ldots, m$ *and* $|y_j| \leqslant b_j$ $(j = 1, \ldots, n)$, *then* $u^i(x,y) < v^i(x,y)$ *in* (1.1) *for* $i = 1, \ldots, m$.

Proof. We define, for $x > 0$,

$$f^i(x, y, z, q) = g^i(x, y, u, u^i_y) + \sum_{k|1}^{m} g^i_{z_k}(x, y, u, u^i_y)(z_k - u^k) + \\ + \sum_{j|1}^{n} g^i_{q_j}(x, y, u, u^i_y)(q_j - u^i_{y_j}) .$$

It follows from (1.4) that f^i satisfy Condition (W). By (1.5) f^i satisfy (1.2). On the other hand,

$$f^i(x, y, u, u^i_y) = g^i(x, y, u, u^i_y) \qquad (x > 0)$$

and consequently, by (1.3),

$$u^i_x < f(x, y, u, u^i_y) \qquad (i = 1, \ldots, m)$$

in (1.1) for $x > 0$. Applying Lemma 1.1 we get the desired conclusion.

Let us introduce the following definition:

We say that the functions u^i $(i = 1, \ldots, m)$ are *regular* if they are continuous in (1.1) and have total differentials for $x > 0$.

Next we introduce the following condition:

Condition (B). The functions $g^i(x, y, z, q)$, $i = 1, \ldots, m$, are defined in (1.1). The derivatives $g^i_{z_k z_j}$, $g^i_{q_k q_j}$ and $g^i_{q_k z_j}$ exist in (1.1) and are continuous in (z, q). For p defined as $p = (z_1, \ldots, z_m, q_1, \ldots, q_n)$ and $\tilde{g}^i(x, y, p) = g^i(x, y, z, q)$ the forms $\varphi_i(\lambda) = \sum \tilde{g}^i_{p_k p_j} \lambda_k \lambda_j$ are strictly positive, i.e. $\sum \tilde{g}^i_{p_k p_j} \lambda_k \lambda_j \geqslant m_i \sum \lambda_s^2$ with positive constants m_i, in all of (1.1).

The next theorem is the following one:

Theorem 1.2. *Let* u^i, v^i $(i = 1, \ldots, m)$ *be two systems of regular functions. Suppose that*

$$(1.7) \qquad u^i(0, y) < v^i(0, y), \quad i = 1, \ldots, m, \ |y_j| \leqslant b_j, \ j = 1, \ldots, n,$$

$$(1.8) \qquad u^i_x < g^i(x, y, u, u^i_y), \quad i = 1, \ldots, m, \ \textit{in } (1.1) \textit{ for } x > 0 .$$

Assume that v^i *satisfy* (1.6) *in* (1.1) *for* $x > 0$. *Let* g^i *satisfy* (B) *and*

$$(1.9) \qquad |g^i_{q_j}| \leqslant M \quad \textit{in } (1.1) \textit{ for } x > 0,$$

$$(1.10) \qquad g^i_{z_k} \geqslant 0 \quad \textit{in } (1.1) \textit{ for } x > 0 \textit{ and } i \neq k.$$

Under the above assumptions

$$(1.11) \qquad u^i(x, y) < v^i(x, y) \quad \textit{in } (1.1), \ i = 1, \ldots, m,$$

$$(1.12) \qquad v^i_x < g^i(x, y, v, v^i_y) \quad \textit{in } (1.1) \textit{ for } x > 0, \ i = 1, \ldots, m .$$

Proof. (1.11) is an immediate consequence of the assumptions and of Theorem 1.1. It follows now from (B) that

$$(1.13)\quad g^i(x,y,v,v_y^i)-g^i(x,y,u,u_y^i)-\\ -\sum_{k|1}^{m} g_{z_k}^i(x,y,u,u_y^i)(v^k-u^k)-\sum_{j|1}^{n} g_{q_j}^i(x,y,u,u_y^i)(v_{y_j}^i-u_{y_j}^i)\\ = g^i(x,y,v,v_y^i)-v_x^i=\tfrac{1}{2}\sum \tilde{g}_{p_kp_j}^i(x,y,\bar{z},\bar{q})\lambda_k\lambda_j>0\,,$$

where $\lambda_1=v^1-u^1,\ldots,\lambda_m=v^m-u^m$, $\lambda_{m+1}=v_{y_1}^i-u_{y_1}^i,\ldots,\lambda_{m+n}=v_{y_n}^i-u_{y_n}^i$ and $(\bar{z},\bar{q})$ is the suitable intermediate point. Inequalities (1.12) follow from (1.6) and (1.13).

Suppose we are given a sequence $\overset{\nu}{u}{}^i$ $(i=1,\ldots,m)$ of systems of regular functions such that

$$(1.14)\qquad \overset{\nu}{u}{}_x^i=G^i(x,y\colon \overset{\nu-1}{u},\overset{\nu-1}{u}{}_y^i\colon \overset{\nu}{u},\overset{\nu}{u}{}_y^i)\qquad (i=1,\ldots,m)\,,$$

where by definition

$$G^i(x,y\colon u,q\colon v,p)\\ = g^i(x,y,u,q)+\sum_{k|1}^{m} g_{z_k}^i(x,y,u,q)(v^k-u^k)+\\ +\sum_{j|1}^{n} g_{y_j}^i(x,y,u,q)(p_j-q_j)\,.$$

An easy induction shows that the following theorem holds true:

THEOREM 1.3. *Suppose that*

$$(1.15)\qquad \overset{\nu}{u}{}^i(0,y)<\overset{\nu+1}{u}{}^i(0,y)\,,\quad i=1,\ldots,m\,,\ \nu=0,1,\ldots\ \textit{for}\ |y_j|\leqslant b_j\ (j=1,\ldots,n)\,,$$

$$(1.16)\qquad \overset{0}{u}{}_x^i<g^i(x,y,\overset{0}{u},\overset{0}{u}{}_y^i)\quad \textit{in}\ (1.1)\ \textit{for}\ x>0$$

and let g^i satisfy (1.9) *and* (1.10) *of Theorem* 1.2. *We assume that g^i satisfy Condition* (B). *Then*

$$(1.17)\qquad \overset{\nu}{u}{}^i(x,y)<\overset{\nu+1}{u}{}^i(x,y)\quad \textit{for}\ i=1,\ldots,m\,,\ \nu=0,1,\ldots\ \textit{in}\ (1.1)\,,$$

$$(1.18)\qquad \overset{\nu}{u}{}_x^i(x,y)<g^i(x,y,\overset{\nu}{u},\overset{\nu}{u}{}_y^i)\quad \textit{for}\ i=1,\ldots,m\,,\ \nu=0,1,\ldots\ \textit{in}\ (1.1)\ \textit{for}\ x>0\,.$$

The sequence $\overset{\nu}{u}$ which satisfies (1.14) is called the *Chaplighin sequence.* Theorem 1.3 shows that under suitable conditions, assumptions (1.15) and (1.16) imply that the Chaplighin sequence satisfies (1.17) and (1.18).

Suppose that there are regular functions v^i such that

$$\overset{\nu}{u}{}^i(0, y) < v^i(0, y) \quad \text{for } i = 1, \dots, m,\ \nu = 0, 1, \dots,\ |y_j| \leqslant b_j \quad (j = 1, \dots, n), \tag{1.19}$$

$$v^i_x > g^i(x, y, v, v^i_y) \quad \text{in (1.1) for } x > 0. \tag{1.20}$$

Note that (1.9) and (1.10) imply Condition (A) for g^i. Hence, by Lemma 1.1, (1.18) (1.19) and (1.20) we get

$$\overset{\nu}{u}{}^i(x, y) < v^i(x, y) \quad \text{in (1.1) for } i = 1, \dots, m,\ \nu = 0, 1, \dots \tag{2.21}$$

It is thus seen that the following boundedness theorem holds true:

THEOREM 1.4. *Let the assumptions of Theorem* 1.3 *be satisfied and suppose that* (1.19) *and* (1.20) *hold true. Then the Chaplighin sequence* $\overset{\nu}{u}$ *satisfies* (1.21).

2. The investigations of section 1 were based essentially on Lemma 1.1. We dealed with strict inequalities. The present section concerns "weak" inequalities.

To begin with we formulate the following condition:

Condition (C). The non-negative functions $\sigma_i(x, u_1, \dots, u_m)$, $i = 1, \dots, m$, are continuous for $0 \leqslant x \leqslant a$ $(a > a)$ and $u_i \geqslant 0$. Moreover, $\sigma_i(x, u_1, \dots, u_m)$ increases in u_j for $i \neq j$ for each i. We suppose also that the right-hand maximal solution $\omega_i(x)$ $(i = 1, \dots, m)$ of the system ([1])

$$u_i' = \sigma_i(x, u_1, \dots, u_m), \quad i = 1, \dots, m,$$

such that $\omega_i(0) = 0$ for $i = 1, \dots, m$ exists in $\langle 0, a\rangle$ and $\omega_i(x) \equiv 0$ in $\langle 0, a\rangle$ for $i = 1, \dots, m$.

The key lemma is the following one ([2]):

LEMMA 2.1. *Suppose that* f^i *satisfy Condition* (W) *and that Condition* (C) *holds true. We suppose that*

$$\begin{aligned} &\big|f^i\big(x, y, u(x, y), u^i_y(x, y)\big) - f^i\big((x, y)v(x, y), v^i_y(x, y)\big)\big| \\ &\quad \leqslant \sigma_i\big(x, |u^1(x, y) - v^1(x, y)|, \dots, |u^m(x, y) - v^m(x, y)|\big) + M \sum_{k|1}^{n} |u^i_{y_k} - v^i_{y_k}|, \end{aligned} \tag{2.1}$$

where u^i, v^i *are regular functions which satisfy*

$$u^i_y \leqslant f^i(x, y, u, u^i_y) \quad \textit{in (1.1) for } x > 0, \tag{2.2}$$

$$v^i_x \geqslant f^i(x, y, v, v^i_y) \quad \textit{in (1.1) for } x > 0, \tag{2.3}$$

$$u^i(0, y) \leqslant v^i(0, y) \quad \textit{for } |y_j| \leqslant b_j \ (j = 1, \dots, n), \tag{2.4}$$

([1]) For the theory of maximal solutions of ordinary differential equations, see [10].

([2]) This lemma belongs to A. Pliś, see also [9].

for $i = 1, \ldots, m$. *Then*

$$u^i(x, y) \leqslant v^i(x, y)$$

in (1.1) *for* $i = 1, \ldots, m$.

Lemma 2.1 is used in the proof of the following theorem:

THEOREM 2.1. *Let the functions* $g^i(x, y, z, q)$ *satisfy in* (1.1) *the following conditions*:

$$|g^i_{q_j}| \leqslant M, \quad i = 1, \ldots, m, \ j = 1, \ldots, n, \tag{2.5}$$

$$g^i_{z_k} \geqslant 0 \quad \text{for } i \neq k. \tag{2.6}$$

The functions $\hat{g}^i(x, y, p) = g^i(x, y, z, q)$ $(p_i = z_i$ *for* $i = 1, \ldots, m$, $p_i = q_i$ *for* $i = m+1, \ldots, m+n)$ *have continuous derivatives* $\hat{g}^i_{p_k p_j}$ *and the forms*

$$\sum \tilde{g}^i_{p_k p_j} \lambda_k \lambda_j \geqslant 0 \quad (i = 1, \ldots, m). \tag{2.7}$$

Suppose that the Chaplighin sequence $\overset{\nu}{u} = (\overset{\nu}{u}{}^1, \ldots, \overset{\nu}{u}{}^m)$ *satisfies*

$$\overset{\nu}{u}{}^i(0, y) \leqslant \overset{\nu+1}{u}{}^i(0, y) \quad \text{for } i = 1, \ldots, m, \ |y_j| \leqslant b_j \ (j = 1, \ldots, n). \tag{2.8}$$

Assume that there are constants N_ν *such that*

$$|g^i_{zk}(x, y, \overset{\nu}{u}, \overset{\nu}{u}{}^i_y)| \leqslant N_\nu \quad \text{in (1.1) for } x > 0, \ i = 1, \ldots, m, \ k = 1, \ldots, n, \ \nu = 0, 1, \ldots \tag{2.9}$$

We assume that $\overset{0}{u}$ *satisfies*

$$\overset{0}{u}{}^i_x \leqslant g^i(x, y, \overset{0}{u}, \overset{0}{u}{}^i_y) \quad \text{for } i = 1, \ldots, m \text{ in (1.1) for } x > 0. \tag{2.10}$$

Under the above assumptions

$$\overset{\nu}{u}{}^i(x, y) \leqslant \overset{\nu+1}{u}{}^i(x, y) \quad \text{in (1.1) } (i = 1, \ldots, m; \ \nu = 0, 1, \ldots), \tag{2.11}$$

$$\overset{\nu}{u}{}^i_x \leqslant g^i(x, y, \overset{\nu}{u}, \overset{\nu}{u}{}^i_y) \quad \text{in (1.1) for } x > 0 \ (i = 1, \ldots, m, \ \nu = 0, 1, \ldots,). \tag{2.12}$$

Proof. We have for arbitrary ν

$$g^i(x, y, \overset{\nu+1}{u}, \overset{\nu+1}{u}{}^i_y) - G^i(x, y, \overset{\nu}{u}, \overset{\nu}{u}{}^i_y: \overset{\nu+1}{u}, \overset{\nu+1}{u}{}^i_y) = \tfrac{1}{2} \sum \tilde{g}^i_{p_k p_j}(x, y, \tilde{p}) \lambda_k \lambda_j \geqslant 0,$$

where $\tilde{p}$ is the suitable intermediate point. It follows that (2.12) holds true. Now we write in Lemma 2.1

$$f^i(x, y, z, q) = G^i(x, y: \overset{\nu}{u}, \overset{\nu}{u}{}^i_y: z, q), \quad u^i = \overset{\nu}{u}{}^i, \ v^i = \overset{\nu+1}{u}{}^i$$

and $\sigma_i = N_\nu \sum_{k|1}^{m} u_k$, and just conclude that the assumptions of the lemma are satisfied for every ν. Indeed, (2.1) follows from (2.9), (A) follows from (2.5) and Condition (W) follows from (2.6). On the other hand, (2.8) implies (2.4) and (2.2) is a consequence of the above part of the proof in case $\nu > 0$ and of (2.10) in case $\nu = 0$. Relation (2.3) follows from the definition of G^i and from the definition of the Chaplighin sequence. Applying Lemma 2.1 we conclude that (2.11) holds true.

The next theorem deals with the equiboundedness of the sequence $\overset{\nu}{u}{}^i$ $(i = 1, \ldots, m, \nu = 0, 1, \ldots)$ which satisfies (2.12). We will prove the following theorem:

THEOREM 2.2. *Suppose that g^i satisfy* (2.5) *and* (2.6) *of Theorem* 2.1. *Let the derivatives $g^i_{z_k}$ be continuous with regard to all variables and suppose that $|\dot{g}^i_{q_j}| \leqslant M$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$. Let u^i, v^i $(i = 1, \ldots, m)$ be of class C^1 and suppose that*

$$u^i_x \leqslant g^i(x, y, u, u^i_y), \qquad v^i_x \geqslant g^i(x, y, v, v^i_y)$$

for $i = 1, \ldots, m$ in (1.1) *and $u^i(0, y) \leqslant v^i(0, y)$ for $i = 1, \ldots, m$, $|y_j| \leqslant b_j$ $(j = 1, \ldots, n)$.*

Then $u^i(x, y) \leqslant v^i(x, y)$ in (1.1) *for $i = 1, \ldots, m$.*

Proof. It is sufficient to verify that (2.1) of Lemma 2.1 holds true. To do this we notice that

$$\left|g^i\big(x, y, u(x, y), u^i_y(x, y)\big) - g^i\big(x, y, v(x, y), v^i_y(x, y)\big)\right| \leqslant K \sum_{k|1}^{m} |u^k(x, y) - v^k(x, y)| + M \sum_{k|1}^{n} |u^i_{y_k} - v^i_{y_k}| ,$$

where $K = \sup_{i,k,\Omega} |g^i_{z_k}|$ and Ω is the compact convex hull of points $\big(x, y, u(x, y), u^i_y(x, y)\big)$, $\big(x, y, v(x, y), v^i_y(x, y)\big)$ with (x, y) varying over (1.1) and $1 \leqslant i \leqslant m$. This completes the proof of the theorem.

We can combine Theorem 2.1 and Theorem 2.2 and thus get the sufficient conditions for equiboundedness of the Chaplighin sequence, provided the suitable functions in question are in C^1. No uniform bounds for $g^i_{z_k}$ are needed in this case.

3. In what follows we shall consider instead of (1.1) some more general domains related to the hyperbolic equations and inequalities. We will appropriately specialize the basic lemmas to our situation.

First of all here is (see [6], Theorem 3):

Condition ($\mathcal{A}$). Given the set

(3.1) $\quad 0 \leqslant x < a \ (a > 0); \qquad B_\mu(x, y_1, y_2, \ldots, y_n) \geqslant 0 \ (\mu = 1, \ldots, 2n)$,

the functions B being of class C^1 for $0 \leqslant x < a$ and arbitrary y_i.

Suppose that for every set of indices $\mu_1, \mu_2, \ldots, \mu_r$, $r \leqslant n$, and for every point simultaneously situated on the surfaces

$$B_{\mu_a}(x, y) = 0\,, \quad a = 1, \ldots, r\,, \tag{3.2}$$

the rank of the matrix

$$\left\| \frac{\partial B}{\partial J_i} \right\|$$

is r.

The functions $\lambda_{ij}(x, y)$ $(i = 1, \ldots, m;\ j = 1, \ldots, n)$ defined in (3.1) are such that for each point belonging to (3.2) the inequality

$$-\sum_{j=1}^{n} \lambda_{ij}(x, y) \sum_{a=1}^{r} l_a \frac{\partial B_{\mu_a}}{\partial J_j} \geqslant \sum_{a=1}^{r} l_a \frac{\partial B_{\mu_a}}{\partial x}\,, \quad i = 1, \ldots, m\,, \tag{3.3}$$

holds for every sequence of numbers $l_a \geqslant 0$.

Now we state the first of the above-mentioned lemmas.

LEMMA 3.1. *Let Conditions* ($\mathcal{A}$) *and* (W) *hold and suppose that the functions* $f^i(x, y, z)$, $i = 1, \ldots, m$, *are defined in the domain* Δ *of the points* (x, y, z) *such that* (x, y) *belongs to* (3.1) *and* z *is arbitrary.*

Assume that $u^i(x, y), v^i(x, y)$ (see Lemma 1.1) *are continuous in* (3.1) *and*

$$u^i(0, y) < v^i(0, y)\,, \quad i = 1, \ldots, m\,.$$

We also suppose that u *and* v *have total differentials in* (3.1) *for* $x > 0$ *and satisfy there the inequalities*:

$$u_x^i + \sum_{j=1}^{n} \lambda_{ij}(x, y)\, u_{y_j}^i < f^i(x, y, u)\,, \quad i = 1, \ldots, m\,,$$

$$v_x^i + \sum_{j=1}^{n} \lambda_{ij}(x, y)\, v_{y_j}^i \geqslant f^i(x, y, v)\,, \quad i = 1, \ldots, m\,.$$

Then

$$u^i(x, y) < v^i(x, y) \quad \textit{in } (3.1) \textit{ for } i = 1, \ldots, m\,.$$

In the sequel we shall make use of the concept of proper set defined for the plane case in [3].

DEFINITION 3.1. The functions $\bar{\varphi}_i(x)$ and $\bar{\bar{\varphi}}_i(x)$, $i = 1, \ldots, n$, are continuous in $[0, a)$ and in the same interval

$$\bar{\varphi}_i(x) < \bar{\bar{\varphi}}_1(x)\,, \quad i = 1, \ldots, n\,.$$

The set

$$R(\bar{\varphi}, \bar{\bar{\varphi}}) = \{(x, y);\ 0 \leqslant x < a, \bar{\varphi}_1(x) \leqslant y_i \leqslant \bar{\bar{\varphi}}_i(x)\}\,.$$

is said to be a *proper* one if:

1° There exist n couples of sequences $\bar{\varphi}_i^\nu(x), \bar{\bar{\varphi}}_i^\nu(x)$ continuously differentiable in $[0, a)$ and

$$\bar{\varphi}_1(x) < \bar{\varphi}_1^\nu(x) < \bar{\bar{\varphi}}_i^\nu(x) < \bar{\bar{\varphi}}_i(x) ,$$

for $x \in [0, a)$ $(i = 1, \dots, n;\ \nu = 1, 2, \dots)$.

2° $\lim\limits_{\nu\to\infty} \bar{\varphi}_i^\nu(x) = \bar{\varphi}_i(x)$, $\lim\limits_{\nu\to\infty} \bar{\bar{\varphi}}_i^\nu(x) = \bar{\bar{\varphi}}_i(x)$, almost uniformly in $[0, a)$.

3° Suppose that $\lambda_{ij}(x, y)$ $(i = 1, \dots, m;\ j = 1, \dots, n)$ are continuous functions in $R(\bar{\varphi}, \bar{\bar{\varphi}})$ and let $\bar{\lambda}_j(x, y) = \max\limits_i \lambda_{ij}(x, y)$; $\underline{\lambda}_j(x, y) = \min\limits_i \lambda_{ij}(x, y)$. Then

$$\frac{d\bar{\varphi}_j^\nu(x)}{dx} \geqslant \bar{\lambda}_j\big(x, \bar{\varphi}^\nu(x)\big); \qquad \frac{d\bar{\bar{\varphi}}_j^\nu(x)}{dx} \leqslant \underline{\lambda}_j\big(x, \bar{\bar{\varphi}}^\nu(x)\big)$$

for $x \in [0, a)$ $(\nu = 1, 2, \dots,\ j = 1, \dots, n)$.

Example. Suppose $\lambda_{ij}(x, y)$ to be bounded in a domain D which contains the hyperparallelepiped

$$O \times [a_1, b_1] \times \dots \times [a_n, b_n] .$$

Define

$$A = \sup_{i,j;D} \lambda_{ij}(x, y) , \qquad B = \inf_{i,j;D} \lambda_{ij}(x, y) ,$$

$$T = \{(x, y)\colon\ a_i + Ax \leqslant y_i \leqslant b_i + Bx ,\ 0 \leqslant x < a\} .$$

Assume that $T \subset D$. The functions $\bar{\varphi}_i^\nu(x) = a_i + 1/\nu + Ax$, $\bar{\bar{\varphi}}_i^\nu(x) = b_i - 1/\nu + Bx$, satisfy the above requirements if $\bar{\varphi}_i(x) = a_i + Ax$ and $\bar{\bar{\varphi}}_i(x) = b_i + Bx$.

Lemma 3.2. *Suppose $\bar{\varphi}_i(x)$ and $\bar{\bar{\varphi}}_i(x)$ to be of class C^1 in $[0, a)$ and*

$$\bar{\varphi}'(x) \geqslant \bar{\lambda}_i\big(x, \bar{\varphi}(x)\big) , \qquad \bar{\bar{\varphi}}'(x) \leqslant \underline{\lambda}_i\big(x, \bar{\bar{\varphi}}(x)\big) . \tag{3.4}$$

Assume that for $\bar{y}_k \geqslant \bar{\bar{y}}_k$, $k \neq i$,

$$\bar{\lambda}_i(x, \bar{y}) \geqslant \bar{\lambda}_i(x, \bar{\bar{y}}_1, \dots, \bar{y}_i, \dots, \bar{\bar{y}}_n) , \qquad \underline{\lambda}_i(x, \bar{y}) \leqslant \underline{\lambda}_i(x, \bar{\bar{y}}_1, \dots, \bar{y}_i, \dots, \bar{\bar{y}}_n) .$$

Then a proper set $R(\bar{\varphi}, \bar{\bar{\varphi}})$ satisfies Condition $(\mathcal{A})$ with

$$B_\mu = y_\mu - \bar{\varphi}_\mu(x) , \qquad B_{\mu+n} = \bar{\bar{\varphi}}_\mu(x) - y_\mu , \qquad \mu = 1, \dots, n ,$$

and

$$\varphi_i(x) = \begin{cases} \bar{\varphi}_i(x) & \text{if} \quad i \leqslant n , \\ \bar{\bar{\varphi}}_{i-n}(x) & \text{if} \quad i > n . \end{cases}$$

Proof. We have

$$\frac{\partial B_\mu}{\partial y_i} = \varepsilon_\mu \delta_{\mu i}\,, \qquad i = 1, \dots, n\,,$$

$$\frac{\partial B_\mu}{\partial x} = -\varepsilon_\mu \varphi'_\mu(x)\,, \qquad \mu = 1, \dots, 2n\,,$$

where

$$\varepsilon_\mu = \begin{cases} 1 & \text{if } \mu \leqslant n\,, \\ -1 & \text{if } \mu > n\,, \end{cases}$$

and δ_{jk} is the Kronecker symbol.

The inequality

$$\sum_{a=1}^{r} l_a \varepsilon_{\mu_a} \lambda_{i\mu_a}\big(x, \dots, \varphi_{\mu_1}(x), \dots, \varphi_{\mu_r}(x), \dots\big) = \sum_{a=1}^{r} l_a \varepsilon_{\mu_a} \varphi'_{\mu_a}(x)$$

holds true. Indeed, if $\mu_a \leqslant n$, then $\varepsilon_{\mu_a} = 1$ and

$$\varphi'_{\mu_a}(x) = \bar{\varphi}'_{\mu_a}(x) \geqslant \bar{\lambda}_{\mu_a}\big(x, \bar{\varphi}(x)\big) = \bar{\lambda}_{\mu_a}\big(x, \dots, \varphi_{\mu_1}(x), \dots, \varphi_{\mu_r}(x), \dots\big)$$
$$\geqslant \lambda_{i\mu_a}\big(x, \dots, \varphi_{\mu_1}(x), \dots, \varphi_{\mu_r}(x), \dots\big)\,.$$

If $\mu_a > n$, $\varepsilon_{\mu_a} = -1$ and

$$\varphi'_{\mu_a}(x) = \bar{\bar{\varphi}}_{\mu_a}(x) \leqslant \underline{\lambda}_{\mu_a}\big(x, \bar{\bar{\varphi}}(x)\big) \leqslant \lambda_{\mu_a}\big(x, \dots, \varphi_{\mu_1}(x), \dots, \varphi_{\mu_r}(x), \dots\big)$$
$$\leqslant \lambda_{i\mu_a}\big(x, \dots, \varphi_{\mu_1}(x), \dots, \varphi_{\mu_r}(x), \dots\big)\,.$$

Remark 3.1. The functions $\bar{\varphi}_i(x), \bar{\bar{\varphi}}_i(x)$ from the example, satisfy Condition $(\mathcal{A})$.

Take $B_\mu = y_\mu - a_i - Ax$, $B_{\mu+n} = b_i + Bx - y_\mu$ $(\mu = 1, 2, \dots, n)$. Then (3.1) coincides with T.

DEFINITION 3.2. A proper set is called a *regular domain* if it satisfies Condition $(\mathcal{A})$.

LEMMA 3.3. *Suppose that Conditions* (C) *and* (W) *hold for a regular domain* R.

Assume that $u^i(x, y)$, $v^i(x, y)$, $i = 1, \dots, m$, *are regular functions in* R (see Definition 4.1) *such that*:

$$|f^i(x, y, \bar{u}^1, \dots, \bar{u}^m) - f^i(x, y, \bar{\bar{u}}^1, \dots, \bar{\bar{u}}^m)| \leqslant \sigma_i(x, |\bar{u}^1 - \bar{\bar{u}}^1|, \dots, |\bar{u}^m - \bar{\bar{u}}^m|)\,,$$

$$u^i_x + \sum_{j=1}^{n} \lambda_{ij}(x, y) u^i_{y_j} = f^i(x, y, u) \quad \textit{in } R \textit{ for } x > 0,\ i = 1, \dots, m\,,$$

$$v^i_y + \sum_{j=1}^{n} \lambda_{ij}(x, y) v^i_{y_j} = f^i(x, y, v) \quad \textit{in } R \textit{ for } x > 0\,,\ i = 1, \dots, m\,,$$

$$u^i(0, y) \leqslant v^i(0, y)\,, \qquad (0, y) \in R\,,\ i = 1, \dots, m\,.$$

Our assumptions imply that

$$u^i(x, y) \leqslant v^i(x, y) \quad \text{in } R .$$

Here $\lambda_{ij}(x, y)$ *are, as it is supposed in Definition* 3.1, *continuous functions in R.*

Proof. Let ν be an arbitrary (fixed) subscript and put:

$$\psi_i(x, y) = \frac{|u^i - v^i| + (u^i - v^i)}{2};$$

$$\varrho_i(x) = \max_{\bar{\varphi}^\nu \leqslant y \leqslant \bar{\bar{\varphi}}^\nu} \psi_i(x, y)$$

($\bar{\varphi}^\nu$ and $\bar{\bar{\varphi}}^\nu$ appear in Definition 3.1).

Notice that $\varrho(0) = 0$. Let us show that if for an index i_0 and a point x_0 $\varrho_{i_0}(x_0) > 0$, then

(3.5) $$\bar{D}_- \varrho_{i_0}(x_0) \leqslant \sigma_{i_0}\big(x_0, \varrho_1(x_0), \ldots, \varrho_m(x_0)\big) .$$

Let $\bar{y}$ be such that

$$\varrho_{i_0}(x_0) = \psi_{i_0}(x_0, \bar{y}) .$$

In order to prove (3.5) we shall consider the following two cases:

a) $\bar{\varphi}^\nu(x_0) < \bar{y} < \bar{\bar{\varphi}}^\nu(x_0)$.

Since, in $(x_0, \bar{y})$, $\psi_{i_0} = u^{i_0} - v^{i_0}$, it follows that on this maximum point $u^{i_0}_{y_j} = v^{i_0}_{y_j}$, $j = 1, 2, \ldots, n$. Hence (see Lemma 1.1 from [7])

$$\bar{D}_- \varrho_{i_0}(x_0) \leqslant \frac{\partial(u^{i_0} - v^{i_0})}{\partial x}\bigg|_{(x_0, \bar{y})},$$

so that

$$\bar{D}_- \varrho_{i_0}(x_0) \leqslant f^{i_0}\big(x_0, \bar{y}, u(x_0, \bar{y})\big) - f^{i_0}\big(x_0, \bar{y}, v(x_0, \bar{y})\big) .$$

Now if write

$$U_j = \max\{u^j(x_0, \bar{y}), v^j(x_0, \bar{y})\} ,$$

we are lead to (3.5) by the inequalities

$$\begin{aligned}\bar{D}_- \varrho_{i_0}(x_0) &\leqslant f^{i_0}(x_0, \bar{y}, U) - f^{i_0}\big(x, \bar{y}, v(x_0, \bar{y})\big)\\ &\leqslant \sigma_{i_0}\big(x_0, \psi_1(x_0, \bar{y}), \ldots, \psi_m(x_0, \bar{y})\big) \leqslant \sigma_{i_0}\big(x_0, \varrho_1(x_0), \ldots, \varrho_m(x_0)\big) .\end{aligned}$$

b) $\bar{y}$ is a boundary point. Take for instance

$$\bar{y}_1 = \bar{\varphi}^\nu_1(x_0) , \quad \bar{y}_2 = \bar{\bar{\varphi}}^\nu_2(x_0) , \quad \bar{\varphi}^\nu_j(x_0) < \bar{y}_j < \bar{\bar{\varphi}}_j(x_0) , \quad j = 3, 4, \ldots, n .$$

Thus

$$\varrho_{i_0}(x_0) = \psi_{i_0}\big(x_0, \bar{\varphi}^\nu_1(x_0), \bar{\bar{\varphi}}^\nu_2(x_0), \bar{y}_3, \ldots, \bar{y}_n\big) .$$

If h is sufficiently small:

$$\bar{\varphi}_j^\nu(x_0+h) \leqslant \bar{y}_j \leqslant \bar{\bar{\varphi}}_j^\nu(x_0+h)\,, \qquad j = 2, 3, \ldots, n\,,$$

and

$$\psi_{i_0}\big(x_0+h, \bar{\varphi}_1^\nu(x_0+h), \varphi_2^\nu(x_0+h), \bar{y}_3, \ldots, \bar{y}_n\big) \leqslant \varrho_{i_0}(x_0+h)\,.$$

From this inequality we get for $h<0$ (ψ_{i_0} is derivable in $(x_0, \bar{y})$):

$$\bar{D}_- \varrho_{i_0}(x_0) \leqslant \frac{d}{d\xi}\big[\psi_{i_0}\big(\xi, \bar{\varphi}_1^\nu(\xi), \bar{\bar{\varphi}}_2^\nu(\xi), \bar{y}_3, \ldots, \bar{y}_n\big)\big]\Big|_{\xi=x_0}$$

$$= \frac{\partial\psi_{i_0}}{\partial x}\Big|_{(x_0,\bar{y})} + \bar{\varphi}_1'^\nu(x_0)\frac{\partial\psi_{i_0}}{\partial y_1}\Big|_{(x_0,\bar{y})} + \bar{\bar{\varphi}}_2'^\nu(x_0)\frac{\partial\psi_{i_0}}{\partial y_2}\Big|_{(x_0,y)}\,.$$

But

$$\frac{\partial\psi_{i_0}}{\partial y_1}\Big|_{(x_0,\bar{y})} \leqslant 0\,, \qquad \frac{\partial\psi_{i_0}}{\partial y_2}\Big|_{(x_0,\bar{y})} \geqslant 0$$

and

$$(\bar{\varphi}_1^\nu)' \geqslant \bar{\lambda}_1\big(x, \bar{\varphi}_1^\nu(x)\big) \geqslant \lambda_{i_01}\big(x, \bar{\varphi}_1^\nu(x)\big)\,,$$

$$(\bar{\bar{\varphi}}_2^\nu)' \leqslant \underline{\lambda}_2\big(x, \bar{\varphi}_2^\nu(x)\big) \leqslant \lambda_{i_02}\big(x, \bar{\bar{\varphi}}_2^\nu(x)\big)\,.$$

Obviously, then

$$\bar{D}_- \varrho_{i_0}(x) = \frac{\partial(u_{i_0}-v_{i_0})}{\partial x}\Big|_{(x_0,\bar{y})} + \lambda_{i_01}\big(x, \bar{\varphi}_1^\nu(x)\big)\frac{\partial(u_{i_0}-v_{i_0})}{\partial y_1}\Big|_{(x_0,\bar{y})} +$$

$$+ \lambda_{i_02}\big(x, \bar{\bar{\varphi}}_2^\nu(x)\big)\frac{\partial(u_{i_0}-v_{i_0})}{\partial y_2}\Big|_{(x_0,\bar{y})}\,.$$

Because

$$\frac{\partial(u_{i_0}-v_{i_0})}{\partial y_j} = 0\,, \qquad j = 3, 4, \ldots, n\,,$$

$$\bar{D}_- \varrho_{i_0}(x_0) \leqslant f^{i_0}\big(x_0, \bar{y}, u_1(x_0,\bar{y}), \ldots, u_m(x_0,\bar{y})\big) -$$

$$-f^{i_0}\big(x_0, \bar{y}, v_1(x_0,\bar{y}), \ldots, v_m(x_0,\bar{y})\big)$$

and we are lead to the same inequality as in a).

Since in both cases (3.5) holds, we conclude by Lemma 1.2 from [7] that

$$\varrho_i(x) \leqslant 0\,, \qquad i = 1, 2, \ldots, m\,.$$

This implies

$$\varrho_i(x) = 0\,, \qquad i = 1, 2, \ldots, m\,,$$

i.e.

$$u^i(x,y) \leqslant v^i(x,y) \quad \text{in } R \text{ for } i = 1, 2, \ldots, m\,.$$

4. The corresponding theorem to 1.1 is the following:

THEOREM 4.1. *Let the functions* $u^i(x,y)$, $i = 1, \ldots, m$, *be regular in* (3.1).

Assume that, for (x, y) *belonging to the same domain and* $x > 0$,

$$u^i_x + \sum_{j=1}^{n} \lambda_{ij}(x, y) u^i_{y_j} < g^i(x, y, u), \quad i = 1, \ldots, m.$$

Suppose that

$$g^i_{z_k}(x, y, u(x, y)) \geqslant 0 \quad \text{if } i \neq k \text{ and } x > 0.$$

Let $v^i(x, y)$, $i = 1, \ldots, m$, *be regular functions in* (3.1) *satisfying for* $x > 0$ *the equations*

$$(4.1) \qquad v^i_x + \sum_{j=1}^{n} \lambda_{ij}(x, y) v^i_{y_j} = g^i(x, y, u) + \sum_{k=1}^{m} g^i_{z_k}(x, y, u)(v^k - u^k).$$

If $u^i(0, y) < v^i(0, y)$, $i = 1, \ldots, m$, *and* $(0, y)$ *belongs to* (3.1), *the above assumptions imply that* $u^i(x, y) < v^i(x, y)$, $i = 1, \ldots, m$, *in* (3.1).

Further we shall employ for the right-hand side of (4.1) the symbol $G^i(x, y, u; v)$. We shall also need the following

Condition ($\mathfrak{B}$). Let $g^i(x, y, z)$, $i = 1, \ldots, m$, be defined in Δ (see Lemma 3.1). Suppose that $g^i_{z_k}$ are continuous in Δ with respect to z, non-decreasing in z_j if $j \neq k$ and increasing if $j = k$. We shall now prove

THEOREM 4.2. *Suppose we are given two regular systems of functions* $u^i(x, y)$ *and* $v^i(x, y)$, $i = 1, \ldots, m$.

Suppose that $u^i(0, y) < v^i(0, y)$ *for* $(0, y)$ *belonging to* (3.1), $i = 1, \ldots, m$,

$$u^i_x + \sum_{j=1}^{n} \lambda_{ij}(x, y) u^i_{y_j} < g^i(x, y, u) \quad \text{in (3.1) for } x > 0,$$

$i = 1, \ldots, m$, *and that* $v^i(x, y)$ *satisfy* (4.1).

Assume that Condition ($\mathfrak{B}$) *holds and that* $g^i_{z_k} \geqslant 0$ *in* Δ.

Then $u^i(x, y) < v^i(x, y)$ *in* (3.1), $i = 1, \ldots, m$, *and*

$$v^i_x + \sum_{j=1}^{n} \lambda_{ij}(x, y) v^i_{y_j} < g^i(x, y, v), \quad x > 0 \ (i = 1, \ldots, m).$$

Proof. The first inequality is a consequence of Theorem 4.1. On the other hand,

$$\begin{aligned} g^i(x, y, v) - v^i_x - \sum_{j=1}^{n} \lambda_{ij}(x, y) v^i_{y_j} \\ = g^i(x, y, v) - g^i(x, y, u) - \sum_{k=1}^{m} g^i_{z_k}(x, y, u)(v^k - u^k) \end{aligned}$$

$$= g^i(x,y,v) - g^i(x,y,u^1,v^2,\ldots,v^m) -$$
$$- g^i_{z_1}(x,y,u)(v^1-u^1) + g^i(x,y,u^1,v^2,\ldots,v^m) -$$
$$- g^i(x,y,u^1,u^2,v^3,\ldots,v^m) - g^i_{z_2}(x,y,u)(v^2-u^2) + \ldots +$$
$$+ g^i(x,y,v^1,\ldots,v^{m-1},u^m) - g^i(x,y,u) - g^i_{z_m}(x,y,u)(v^m-u^m)$$
$$\geqslant g^i(x,y,v) - g^i(x,y,u^1,v^2,\ldots,v^m) -$$
$$- g^i_{z_1}(x,y,u^1,v^2,\ldots,v^m) + \ldots + g^i(x,y,u^1,\ldots,u^{m-1},v^m) -$$
$$- g^i(x,y,u) - g^i_{z_m}(x,y,u)(v^m-u^m) > 0 \, .$$

The proofs of the following theorems are the same as in § 1.

THEOREM 4.3. *Suppose that in* (3.1)

$$\overset{\nu}{u}{}^i(0,y) < \overset{\nu+1}{u}{}^i(0,y)\,, \qquad i = 1,\ldots,m;\ \nu = 0,1,\ldots,$$

$$\overset{0}{u}{}^i_x + \sum_{j=1}^{n} \lambda_{ij}(x,y)\overset{0}{u}{}^i_{y_j} < g^i(x,y,\overset{0}{u}) \qquad \textit{for } x > 0\,.$$

Assume that $g^i_{z_k} \geqslant 0$ *and that* ($\mathfrak{B}$) *holds.*

Then the Chaplighin sequence $\overset{\nu}{u}(x,y)$ *satisfies the following inequalities:*

(4.2) $$\overset{\nu}{u}{}^i(x,y) < \overset{\nu+1}{u}{}^i(x,y) \ \textit{in}\ (3.1)\,,\ i = 1,\ldots,m\,,\ \nu = 0,\ldots$$

(4.3) $$u^i_x(x,y) + \sum_{j=1}^{n} \lambda_{ij}(x,y) u^i_{y_j} < g^i(x,y,u)$$

in (3.1) *for* $i = 1,\ldots,m$, $\nu = 0,1,\ldots$ *and* $x > 0$.

THEOREM 4.4. *Suppose that the assumptions of Theorem* 4.3 *hold true.*

Assume that in (3.1)

$$\overset{\nu}{u}{}^i(0,y) < v^i(0,y)\,,$$

$$v^i_x + \sum_{j=1}^{n} \lambda_{ij}(x,y) v^i_{y_j} > g^i(x,y,v)\,, \qquad \textit{for } x > 0 \textit{ and } i = 1,\ldots,m\,.$$

Then $\overset{\nu}{u}{}^i(x,y) < v^i(x,y)$ *in* (3.1) *for* $i = 1,\ldots,m$ *and* $\nu = 0,1,\ldots$

We now pass to weak inequalities.

THEOREM 4.5. *The functions* $g^i(x,y,z)$ *are such that, in the regular domain* R *and arbitrary* z, $g^i_{z_k} \geqslant 0$, *if* $i \neq k$ *and* ($\mathfrak{B}$) *holds.*

Suppose that the Chaplighin sequence $\overset{\nu}{u}(x,y)$ *is non-decreasing for* $x = 0$, *i.e.* $\overset{\nu}{u}(0,y) \leqslant \overset{\nu+1}{u}(0,y)$, *for* $(0,y) \in R$, *and that there exist constants* N_ν *so that* $|g^i_{z_k}(x,y,\overset{\nu}{u})| \leqslant N_\nu$ *in* R *for* $x > 0$, $i = 1,\ldots,m$, $k = 1,\ldots,n$, $\nu = 0,1,\ldots$

Assume finally that for $\overset{0}{u}$ *we have*

$$\overset{0}{u}{}^i_x + \sum_{j=1}^{n} \lambda_{ij}(x, y)\overset{0}{u}{}^i_{y_j} \leqslant g^i(x, y, \overset{0}{u}) \quad \textit{in } (3.1) \textit{ for } x > 0 \ (i = 1, \ldots, m) .$$

Under these assumptions the following inequalities hold:

$$\overset{\nu}{u}{}^i(x, y) \leqslant \overset{\nu+1}{u}{}^i(x, y) \quad \textit{in } R \ (i = 1, \ldots, m;\ \nu = 0, 1, \ldots) ,$$

$$\overset{\nu}{u}{}^i_x + \sum_{j=1}^{n} \lambda_{ij}(x, y)\overset{\nu}{u}{}^i_{y_j} \leqslant g^i(x, y, \overset{\nu}{u}) \quad \textit{in } R \textit{ for } x > 0$$

$$(i = 1, \ldots, m ,\ \nu = 0, 1, \ldots) .$$

The equiboundedness theorem may be stated as follows:

THEOREM 4.6. *Suppose* $g^i_{z_k}$ *to be continuous with regard to all variables and* $g^i_{z_k} \geqslant 0$.

Let $u^i(x, y)$, $v^i(x, y)$ *be of class* C^1 *in the regular domain* R *and such that in* R *for* $i = 1, \ldots, m$,

$$\left.\begin{aligned} u^i_x + \sum_{j=1}^{n} \lambda_{ij}(x, y) u^i_{y_j} &\leqslant g^i(x, y, u) \\ v^i_x + \sum_{j=1}^{n} \lambda_{ij}(x, y) v^i_{y_j} &\geqslant g^i(x, y, v) \end{aligned}\right\} \quad \textit{for } x > 0 ,$$

$$u^i(0, y) \leqslant v^i(0, y) ,$$

$$u^i(x, y) \leqslant v^i(x, y) , \quad \textit{in } R ,\ i = 1, \ldots, m .$$

5. In this paragraph we shall investigate the problem of convergence of the Chaplighin sequences to the exact solution. We shall also give estimates of the difference between the exact and the approximate solutions. In what follows R_α $(\alpha < a)$ will denote the set

$$\{0 \leqslant x \leqslant \alpha;\ \bar{\varphi}_i(x) \leqslant y_i \leqslant \bar{\bar{\varphi}}_i(x)\} .$$

THEOREM 5.1. *Let the assumptions of Theorems* 4.5 *and* 4.6 *be fulfilled with* $u = \overset{\nu}{u}$, $\nu = 1, 2, \ldots,$ *and* v *continuous in* R_α.

Write $u^i(0, y) = \varphi_i(y)$, $y \in I$; $I = [\bar{\varphi}(0), \bar{\bar{\varphi}}(0)]$ *and suppose that* $\overset{\nu}{\omega}_i(y) \Rightarrow \omega^i(y)$ *for* $y \in I$.

Suppose that the system

$$u^i_x + \sum_{j=1}^{n} \lambda_{ij}(x, y) u^i_{y_j} = f^i(x, y, u) , \quad i = 1, \ldots, m ,$$

has a unique solution in R_α *which satisfies the initial conditions*

$$u^i(0, y) = \omega^i(y) , \quad y \in I ,\ i = 1, \ldots, m .$$

Assume finally that $\lambda_{ij}(x, y)$ are such that through each point of R_a it passes a unique characteristic intersecting $x = 0$ in I.

Under the above assumptions

$$\overset{\nu}{u}{}^i(x, y) \Rightarrow u^i(x, y) \quad \textit{in } R_a .$$

Proof. Denote the characteristic which passes through (x, y), i.e. the solution of the system

$$\frac{dy_j^{(i)}}{dx} = \lambda_{ij}(\xi, y_1^{(i)}, \ldots, y_n^{(i)}) \quad (j = 1, \ldots, n;\ i = 1, \ldots, n),$$

by $y^{(i)}(\xi; x, y)$.

By Theorem 4.6 the sequences $\overset{\nu}{u}{}^i(x, y)$ are equibounded in R_a. It is clear from

$$\overset{\nu+1}{u}(x, y) = \omega_{\nu+1}[y(0; x, y)] + \int_0^x f^i\big(\xi, y(\xi; x, y), \overset{\nu}{u}\big)\, d\xi + \\ + \sum_{k=1}^{m} \int_0^x f^i_{z_k}\big(\xi, y(\xi; x, y), \overset{\nu}{u}\big)\big(\overset{\nu+1}{u}{}^k - \overset{\nu}{u}{}^k\big)\, d\xi$$

that they are equicontinuous. Since the sequences $\overset{\nu}{u}{}^i$ are monotonic, they are uniformly convergent.

Remark 5.1. Under certain conditions the Chaplighin sequences are all defined in a domain R_a, $a > 0$, independent of ν. Take for instance the case $n = 1$. Suppose f^i and $f^i_{z_k}$ to be of class C^1 for $(x, y) \in R$ and z arbitrary and $\omega_i(y)$ of the same class in I. Since the existence conditions in [5] are fulfilled, the solutions $u^i(x, y)$ are all defined in the same domain R_a.

In order to give estimates of the errors we first formulate:

LEMMA 5.1. *Suppose $\sigma(x, z)$ to be a continuous function of two variables for $x \in [0, a)$ and $z \geqslant 0$.*

Denote by $y = \omega(x, \xi)$ the right-hand maximum solution of $dy/dx = \sigma(x, y)$ passing through $(0, \xi)$. Let it exist in $[0, a)$.

Assume $u_s^i(x, y)$, where $s = 1, 2$ and $i = 1, \ldots, m$, to be regular solutions of the systems

$$\frac{\partial u_s^i}{\partial x} + \sum_{j=1}^{n} \lambda_{ij}(x, y) \frac{\partial u_s^i}{\partial y_j} = f_s^i(x, y, u_s), \quad s = 1, 2;\ i = 1, \ldots, n,$$

in the regular domain R, for $x > 0$.

Suppose also that in the same domain for $i = 1, 2, \ldots, m$:

$$|f_1^i(x, u_1^1, \ldots, u_1^m) - f_2^i(x, u_2^1, \ldots, u_2^m)| \leqslant \sigma\big(x, \max_i |u_1^i(x, y) - u_2^i(x, y)|\big).$$

Then we have $|u_1^i(x,y)-u_2^i(x,y)| \leqslant \omega(x,\xi)$ *for* $\xi = \max\limits_i \xi_i$, *where* $\max |u_1^i(0,y)-u_2^i(0,y)| \leqslant \xi_i$.

The proof of the lemma is similar to that of Lemma 3.3, for

$$\xi = \max_i \xi_i \quad \text{and} \quad \xi_i \geqslant \max_I |u_1^i(0,y)-u_2^i(0,y)| .$$

THEOREM 5.2. *Let the conditions of Theorem* 5.1 *be fulfilled. Suppose that* $|f_{z_k}^i(x,y,\bar{z}) - f_{z_k}^i(x,y,\bar{\bar{z}})| \leqslant \sigma(x, \max\limits_j |\bar{z}_j - \bar{\bar{z}}_j|)$, *where* $i, k = 1, \ldots, m$ *for* $u \leqslant \bar{z}$, $\bar{\bar{z}} \leqslant v$; $(x,y) \in R$ *and* $i = 1, \ldots, m$.

Write $\tau_0(x) \geqslant \max\limits_{R_a} (v^i - \overset{0}{u}{}^i)$ *for* $x \in [0, a)$

$$D = \{(x,y) \in R;\ \overset{0}{u}(x,y) \leqslant z \leqslant v(x,y)\}, \qquad K = \max_{i,k;D} |f_{z_k}^i(x,y,z)| .$$

Assume that $\max\limits_{i;R_a} \big(u^i(0,y) - \overset{\nu}{u}{}^i(0,y)\big) \leqslant \varepsilon$, *and put*

$$\tau_{\nu+1}(x) = \varepsilon_{\nu+1} e^{mKx} + m \int_0^x e^{mK(x-\xi)} \omega\big(\xi, \tau_\nu(\xi)\big) \tau_\nu(\xi)\, d\xi .$$

Then $u(x,y) - \overset{\nu}{u}(x,y) \leqslant \tau_\nu(x)$ *for* $\nu = 0, 1, \ldots$ *and* $(x,y) \in R_a$.

Proof. For $\nu = 0$ the assertion is evident. Let us show that if $u^i(x,y) - \overset{\nu}{u}{}^i(x,y) \leqslant \tau_\nu(x)$, then $\tau_{\nu+1}(x)$ yields an estimate for $u^i(x,y) - \overset{\nu+1}{u}{}^i(x,y)$. Following the technique used in [4] we are lead to the relation

$$\Big| f^i(x,y,u) - f^i(x,y,\overset{\nu}{u}) - \sum_{k=1}^{m} f_{z_k}^i(x,y,\overset{\nu}{u})(\overset{\nu+1}{u}{}^k - \overset{\nu}{u}{}^k) \Big| \leqslant m\sigma\big(x, \tau_\nu(x)\big)\tau_\nu(x) + mK\tau_{\nu+1}(x) .$$

Now if $y = \tau(x, \varepsilon_{\nu+1})$ is the solution of the problem

$$y' = mKy + m\sigma\big(x, \tau_\nu(x)\big)\tau_\nu(x) , \qquad \tau(0, \varepsilon_{\nu+1}) = \varepsilon_{\nu+1}$$

our inequality follows from Lemma 5.1.

Remark 5.2. Put $H = \max\limits_D |f_{z_i z_k}(x,y,u)|$, $\lambda = mHe^{Ka}$ and take $\sigma(x,y) = Hy$. Then for $\varepsilon_\nu = 0$, $\nu = 0, 1, \ldots$,

$$\tau_{\nu+1}(x) < \lambda \int_0^x \tau_\nu^2(\xi)\, d\xi$$

if $\tau_0(x) = 1/2\lambda a$.

The considered Chaplighin sequences have the same rate of convergence as in the case of ordinary differential equations (see [2]):

$$\tau_n(x) = \frac{C}{2^{2^n}} ,$$

C being a constant independent of n.

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