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# ON $\mathscr{L}$-SPLINE INTERPOLATION AND APPROXIMATION ON THE WHOLE REAL LINE

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Let $\mathscr{D} = d/dx$ and let

$$\mathscr{L}_n(\mathscr{D}) \equiv \mathscr{D}^n + a_{n-1}\mathscr{D}^{n-1} + \ldots + a_1\mathscr{D} + a_0$$

be an arbitrary $n$th order linear differential operator with constant real coefficients. Let $p_n$ denote the characteristic polynomial of $\mathscr{L}_n(\mathscr{D})$ and $T_n = \{t_1, \ldots, t_n\}$ the set of its zeros, with each zero repeated according to its multiplicity. Let $m \in N$, $h = \pi/m$, and denote by $\Delta_\infty = \{\nu h\colon \nu \in \mathbf{Z}\}$ the uniform mesh on the whole real line with step $h$.

DEFINITION. A function $s_n$ satisfying the conditions:

1) $s_n \in C^{(n-2)}(\mathbf{R})$,

2) $\mathscr{L}_n(\mathscr{D})s_n(x) = 0$ for $x \in (jh, (j+1)h)$ $(j \in \mathbf{Z})$,

is called an *$\mathscr{L}$-spline corresponding to the operator* $\mathscr{L}_n(\mathscr{D})$ with knots at the points of $\Delta_\infty$.

The set of all $\mathscr{L}$-splines corresponding to $\mathscr{L}_n(\mathscr{D})$ with knots at the points of $\Delta_\infty$ will be denoted by $S(\mathscr{S}_n, \Delta_\infty)$.

Note that for $\mathscr{L}_n(\mathscr{D}) = \mathscr{D}^n$, $\mathscr{L}$-splines are polynomial splines. Surveys of results concerning polynomial splines can be found e.g. in [6], [12], [17].

It is remarkable that an $\mathscr{L}$-spline corresponding to an arbitrary linear differential operator $\mathscr{L}_n(\mathscr{D})$ with constant coefficients is not, in general, a Chebyshev spline (for the definition see e.g. [4]), since the basis of the space of solutions of the equation $\mathscr{L}_n(\mathscr{D})f = 0$ is not a Chebyshev system on $\mathbf{R}$. However, for the operators whose characteristic polynomials have only real roots, the corresponding $\mathscr{L}$-splines are in fact Chebyshev. Such splines were studied in [8], [13] and in other papers.

We define

$$\mathscr{K}_\infty(\mathscr{L}_n) = \{f\colon f^{(n-1)} \text{ locally absolutely continuous}, \\ \|f\|_{C(\mathbf{R})} < +\infty,\ \|\mathscr{L}_n(\mathscr{D})f\|_{L_\infty(\mathbf{R})} \leqslant 1\}.$$

The present paper concerns the approximation of the class $\mathcal{K}_\infty(\mathcal{L}_n)$ by interpolatory $\mathcal{L}$-splines and by $\mathcal{L}$-splines of best approximation in the uniform norm. We show that the interpolation knots may be chosen in such a way that the interpolatory $\mathcal{L}$-splines approximate the class $\mathcal{K}_\infty(\mathcal{L}_n)$ with the order of best approximation, and we calculate exactly the least upper bounds for the corresponding deviations.

Let $h_0 = \pi(\max_{1\leqslant j\leqslant n} \operatorname{Im} t_j)^{-1}$ and

$$\mathcal{L}_r(\mathcal{D}) = \begin{cases} \mathcal{L}_n(\mathcal{D}), & r = n, \\ \mathcal{D}\mathcal{L}_n(\mathcal{D}), & r = n+1. \end{cases}$$

Denote by $p_r = p_r(z)$ the characteristic polynomial of $\mathcal{L}_r(\mathcal{D})$ $(r = n, n+1)$ and write $T_r = T_n$ for $r = n$, and $T_r = T_n \cup \{0\}$ for $r = n+1$. Following Micchelli [8], for $0 < h < h_0$ we define

$$\mathcal{A}_r(x) = \frac{1}{2\pi i}\int_C \frac{e^{xz}}{(e^{zh}+1)\,p_r(z)}\,dz, \qquad x\in[0, h) \ (r = n, n+1), \tag{1}$$

where $C$ is any rectifiable contour in $\boldsymbol{C}$ without selfinter-sections, bounding a domain which contains the set $T_r$ of zeros of $p_r$, but which contains no poles of the meromorphic function $e^{xz}(e^{zh}+1)^{-1}$. The condition $0 < h < h_0$ guarantees the existence of such a contour.

LEMMA 1 ([8], [15], [10]). *The function $\mathcal{A}_r$ has the following properties:*
1) $\mathcal{A}_r^{(j)}(h^-) = -\mathcal{A}_r^{(j)}(0^+) + \delta_{j,r-1}$ $(j = 0, 1, \ldots, r-1$; $\delta_{jl}$ is the Kronecker symbol).
2) $\mathcal{L}_r(\mathcal{D})\,\mathcal{A}_r(x) = 0$ *for* $x\in(0, h)$.
3) $\mathcal{A}_r$ *has a unique zero* $\xi_r$ *in* $[0, h)$; *this zero is simple.*

Let $\gamma \geqslant 0$. We define a class of sequences by

$$Y_\gamma = \{(y_\nu)_{-\infty}^{\infty}\colon\ y_\nu = O(|\nu|^\gamma) \text{ as } \nu \to \pm\infty\},$$

and a class of $\mathcal{L}$-splines by

$$\mathcal{S}_{\mathcal{L}_n,\gamma}(\Delta_\infty) = \{s\in S(\mathcal{L}_n, \Delta_\infty)\colon\ s(x) = O(|x|^\gamma) \text{ as } x \to \pm\infty\}.$$

Let $\alpha\in[0, h)$ be arbitrary.

THEOREM 1. *If* $0 < h < h_0$, *then for every sequence of real numbers* $(y_\nu)_{-\infty}^{\infty}\in Y_\gamma$ *there is a unique* $\mathcal{L}$*-spline* $s_n\in\mathcal{S}_{\mathcal{L}_n,\gamma}(\Delta_\infty)$ *such that*

$$s_n(\alpha+\nu h) = y_\nu, \qquad \nu\in\boldsymbol{Z},$$

*if and only if* $\alpha \neq \xi_n$.

*Remark.* For polynomial splines Theorem 1 was proved by Yu. N. Subbotin [18] and I. Schoenberg [12], and for $\mathcal{L}$-splines under the assumption that $T_n \subset \boldsymbol{R}^n$ by I. Schoenberg [13]. V. T. Shevaldin [15] proved it assuming that $0 \in T_n$, $T_n \subset C^n$, but the proof is essentially the same if instead of $0 \in T_n$ one assumes that $T_n$ contains at least one real number. Note that the proof for the class of periodic sequences and for any linear differential operator with constant real coefficients was given by V. T. Shevaldin [16].

The proof of Theorem 1 is based on Lemmas 1 and 2 of [15] and in fact repeats word by word the considerations from Schoenberg [13]; therefore we omit it.

Write $\mathscr{S}_{\mathscr{L}_n,0}(\Delta_\infty) = \mathscr{S}_{\mathscr{L}_n}(\Delta_\infty)$. This is the set of all $\mathscr{L}$-splines bounded on $\boldsymbol{R}$, corresponding to the differential operator $\mathscr{L}_n(\mathscr{D})$, with knots at the points of $\Delta_\infty$.

Consider the function $\mathscr{A}_{n+1}$ defined by (1) for $x \in [0, h)$, $0 < h < h_0$. By Lemma 1 (assertion 1)), it can be extended to $\boldsymbol{R}$ by setting $\mathscr{A}_{n+1}(x+h) = -\mathscr{A}_{n+1}(x)$, $x \in \boldsymbol{R}$; then $\mathscr{A}_{n+1} \in C^{(n-1)}(\boldsymbol{R})$. Moreover, from (1) by the residue theorem we obtain

$$\mathscr{L}_n(\mathscr{D})(2\mathscr{A}_{n+1}(x)) = \frac{1}{\pi i}\int_C \frac{e^{xz}}{z(e^{zh}+1)}dz = 2\operatorname*{res}_{z=0}\frac{e^{xz}}{z(e^{zh}+1)} = 1, \qquad x \in (0, h).$$

Consequently,

$$\mathscr{L}_n(\mathscr{D})(2\mathscr{A}_{n+1}(x)) = \operatorname{sign}\sin mx, \qquad x \in \boldsymbol{R}.$$

Thus the function $2\mathscr{A}_{n+1}$ is an Euler $\mathscr{L}$-spline corresponding to the operator $\mathscr{L}_n(\mathscr{D})$.

Let $\xi_{n+1}$ be the unique zero of $\mathscr{A}_{n+1}$ in $[0, h)$, $0 < h < h_0$; it exists by Lemma 1 (assertion 3)). Then

$$\Delta'_\infty = \{\xi_{n+1} + jh \colon j \in \boldsymbol{Z}\}$$

is the set of all zeros of $\mathscr{A}_{n+1}$ in $\boldsymbol{R}$. Since $\xi_{n+1}$ is a simple zero and $\mathscr{A}'_{n+1}(x) = \mathscr{A}_n(x)$, $x \in \boldsymbol{R}$, as can be easily verified, it follows from Theorem 1 that for $0 < h < h_0$ and for every bounded sequence $(y_j)_{-\infty}^{\infty}$ there is a unique $\mathscr{L}$-spline $s_n \in \mathscr{S}_{\mathscr{L}_n}(\Delta_\infty)$ such that $s_n(\xi_{n+1}+jh) = y_j$, $j \in \boldsymbol{Z}$.

We define the quantities

$$U(\mathscr{K}_\infty(\mathscr{L}_n); \Delta_\infty, \Delta'_\infty)_\infty = \sup_{f \in \mathscr{K}_\infty(\mathscr{L}_n)} \|f - (s_n f)\|_{C(\boldsymbol{R})}, \tag{2}$$

where $(s_n f)$ is the interpolatory $\mathscr{L}$-spline of class $\mathscr{S}_{\mathscr{L}_n}(\Delta_\infty)$ interpolating $f$ on $\Delta'_\infty$, and

$$E(\mathscr{K}_\infty(\mathscr{L}_n); \mathscr{S}_{\mathscr{L}_n}(\Delta_\infty))_\infty = \sup_{f \in \mathscr{K}_\infty(\mathscr{L}_n)} \inf_{s_n \in \mathscr{S}_{\mathscr{L}_n}(\Delta_\infty)} \|f - s_n\|_{C(\boldsymbol{R})}. \tag{3}$$

(2) is the accuracy of approximation of the class $\mathscr{K}_\infty(\mathscr{L}_n)$ by interpolating $\mathscr{L}$-splines with our choice of the interpolation knots, and (3) is the error of the $\mathscr{L}$-spline best approximation on $\mathscr{K}_\infty(\mathscr{L}_n)$.

The main result of this paper is the following:

THEOREM 2. *Let $k$ be the number of pairs of complex roots of the polynomial $p_n$ with nonzero imaginary part. Then for* $0 < h < 3^{-(k-1)} h_0$

$$E(\mathscr{K}_\infty(\mathscr{L}_n);\ \mathscr{S}_{\mathscr{L}_n}(\Delta_\infty))_\infty = U(\mathscr{K}_\infty(\mathscr{L}_n);\ \Delta_\infty, \Delta'_\infty)_\infty = 2\|\mathscr{A}_{n+1}\|_{C(\boldsymbol{R})}.$$

*Remark.* For the operator $\mathscr{L}_n(\mathscr{D}) = \mathscr{D}^n$, Theorem 2 in the periodic case was proved by V. M. Tikhomirov [19]. In the nonperiodic case the value of $U(\mathscr{K}_\infty(\mathscr{L}_n);\ \Delta_\infty, \Delta'_\infty)$ was found by Schoenberg [14] for $n$ even and by de Boor and Schoenberg [2] for $n$ odd (see also [11]). For a formally selfadjoint operator with $T_n \subset \boldsymbol{R}^n$ and $n$ even, $U(\mathscr{K}_\infty(\mathscr{L}_n);\ \Delta_\infty, \Delta'_\infty)$ was computed by Micchelli [8], and under the same assumptions but for $n$ odd, Theorem 2 was proved by the present author [9]. Finally, in the periodic case with $T_n \subset \boldsymbol{C}^n$, Theorem 2 was established by the author ([10], Theorem 3.1).

Before proving Theorem 2, we formulate some auxiliary statements.

Let $N$ be an arbitrary positive integer and $K$ any real number. We set

$$KW_\infty(\mathscr{L}_n;\ N) = \{f \in C_{2\pi N}:\ f^{(n-1)} \in AC_{2\pi N},\ \|\mathscr{L}_n(\mathscr{D}) f\|_{L_\infty(\boldsymbol{R})} \leqslant K\},$$

where the subscript $2\pi N$ indicates that we consider $2\pi N$-periodic functions.

Denote by $(s_n f)^{\sim}$ the $2\pi N$-periodic $\mathscr{L}$-spline corresponding to $\mathscr{L}_n(\mathscr{D})$ with knots at the points of $\Delta_\infty$ which interpolates the $2\pi N$-periodic function $f$ on $\Delta'_\infty$. The existence and uniqueness of such an $\mathscr{L}$-spline follows from Theorem 1 (see also [16]).

LEMMA 2 ([10]). *If* $0 < h < 3^{-(k-1)} h_0$, *then for* $f \in KW_\infty(\mathscr{L}_n;\ N)$ *and* $x \in \boldsymbol{R}$

$$|f(x) - (s_n f)^{\sim}(x)| \leqslant 2K |\mathscr{A}_{n+1}(x)|.$$

*Equality holds for* $f(x) = 2K\mathscr{A}_{n+1}(x),\ f \in KW_\infty(\mathscr{L}_n;\ N)$.

LEMMA 3 ([3]). *For any function $\varphi$ with a locally absolutely continuous $(n-1)$-th derivative and for $k < n$*

$$\|\varphi^{(k)}\|_{C(\boldsymbol{R})} \leqslant A_k \|\varphi\|_{C(\boldsymbol{R})} + B_k \|\mathscr{L}_n(\mathscr{D})\varphi\|_{L_\infty(\boldsymbol{R})},$$

*where $A_k$, $B_k$ are positive numbers depending on the operator $\mathscr{L}_n(\mathscr{D})$ and on $k$.*

LEMMA 4. *If* $0 < h < 3^{-(k-1)} h_0$, *then for* $f \in \mathscr{K}_\infty(\mathscr{L}_n)$ *and* $x \in \boldsymbol{R}$

$$|f(x) - (s_n f)(x)| \leqslant 2|\mathscr{A}_{n+1}(x)|,$$

*with equality holding for* $f(x) = 2\mathscr{A}_{n+1}(x) \in \mathscr{K}_\infty(\mathscr{L}_n)$.

The proof of Lemma 4 will be based on the corresponding result in the periodic case (Lemma 2); we will apply Bang's idea [1] (see also [7]) used by him to carry over the proof of Kolmogorov's inequality for norms of derivatives from the periodic case to the whole real line.

*Proof of Lemma* 4. It is not difficult to check (see e.g. [7]) that for every $\delta > 0$ there is a positive integer $N = N(\delta)$ and a function $v_\delta$ whose $(n-1)$th derivative is locally absolutely continuous on $\boldsymbol{R}$, such that:

1) $v_\delta(x) \equiv 1$ for $|x| \leqslant \pi$.
2) $v_\delta(x) \equiv 0$ for $|x| > \pi N$.
3) We have the estimate

$$|v_\delta^{(k)}(x)| \leqslant \delta, \quad k = 1, \ldots, n-1, \quad x \in \boldsymbol{R}. \tag{4}$$

Let $f \in \mathscr{K}_\infty(\mathscr{L}_n)$ and define $F_\delta(x) = f(x)\, v_\delta(x)$. It is clear that $F_\delta$ also has a locally absolutely continuous $(n-1)$th derivative and, by 1)–3):

a) $F_\delta(x) = f(x)$ for $|x| \leqslant \pi$.
b) $F_\delta(x) = 0$ for $|x| > \pi N$.

We now prove that

c) We have the estimate

$$\|\mathscr{L}_n(\mathscr{D})\, F_\delta\|_{L_\infty(\boldsymbol{R})} \leqslant 1 + \delta M, \tag{5}$$

where $M$ is a positive constant independent of $\delta$.

Indeed, (4) yields

$$\|\mathscr{L}_n(\mathscr{D})\, F_\delta\|_{L_\infty(\boldsymbol{R})} \leqslant \|v_\delta\|_{C(\boldsymbol{R})} \|\mathscr{L}_n(\mathscr{D})\, f\|_{L_\infty(\boldsymbol{R})} + \delta \sum_{j=0}^{n-1} \gamma_j \|f^{(j)}\|_{C(\boldsymbol{R})}, \tag{6}$$

where $f^{(0)} = f$, and $\gamma_j > 0$ is independent of $\delta$.

We assume for simplicity that $\|f\|_{C(\boldsymbol{R})} < B < +\infty$, where $B$ is a constant independent of $f$ (from what follows it is not difficult to deduce that this restriction is inessential). Applying Lemma 3 to the right-hand side of (6) and taking into account that $\|v_\delta\|_{C(\boldsymbol{R})} = 1$ and $\|\mathscr{L}_n(\mathscr{D})\, f\|_{L_\infty(\boldsymbol{R})} \leqslant 1$, we obtain (5).

Let $0 < h < 3^{-(k-1)} h_0$, $N_1 = N+1$. Consider the restriction of $F_\delta$ to $[-\pi N_1, \pi N_1]$ and extend it periodically (with period $2\pi N$) to $\boldsymbol{R}$. The resulting function will be denoted by $\tau_\delta$. By (5), $\tau_\delta \in K W(\mathscr{L}_n; N_1)$ with $K = 1 + \delta M$. Since $0 < h < 3^{-(k-1)} h_0$, Lemma 2 yields for $x \in \boldsymbol{R}$

$$|\tau_\delta(x) - (s_n \tau_\delta)^{\sim}(x)| \leqslant 2(1+\delta M)\, |\mathscr{A}_{n+1}(x)|.$$

Letting $\delta \to 0$ and using the equalities

$$\lim_{\delta \to 0} N(\delta) = +\infty, \quad \lim_{\delta \to 0} \tau_\delta(x) = f(x) \lim_{\delta \to 0} v_\delta(x) = f(x),$$

$$\lim_{\delta \to 0} (s_n \tau_\delta)^{\sim}(x) = (s_n f)(x),$$

we obtain the inequality of Lemma 4. Applying Theorem 1, it is not difficult to check that equality holds for $f(x) = 2\mathscr{A}_{n+1}(x) \in \mathscr{K}_\infty(\mathscr{L}_n)$. The proof of Lemma 4 is complete.

Write $1 \cdot W_\infty(\mathscr{L}_n; 1) = W_\infty(\mathscr{L}_n)$ and let $\tilde{S}(\mathscr{L}_n; \Delta_\infty)$ be the subset of $2\pi$-periodic functions from $S(\mathscr{L}_n; \Delta_\infty)$.

LEMMA 5 ([10], Theorem 3.1). *Let* $0 < h < 3^{-(k-1)}h_0$. *Then*

$$E(W_\infty(\mathcal{L}_n); \tilde{S}(\mathcal{L}_n; \Delta_\infty))_\infty = 2\|\mathcal{A}_{n+1}\|_{C(\mathbf{R})},$$

*and the extremal function is* $2\mathcal{A}_{n+1} \in W_\infty(\mathcal{L}_n)$.

*Proof of Theorem* 2. Lemma 4 shows that

$$(7)\quad E(\mathcal{K}_\infty(\mathcal{L}_n); \mathcal{S}_{\mathcal{L}_n}(\Delta_\infty))_\infty \leqslant U(\mathcal{K}_\infty(\mathcal{L}_n); \Delta_\infty, \Delta'_\infty)_\infty = 2\|\mathcal{A}_{n+1}\|_{C(\mathbf{R})}.$$

Let us estimate $E(\mathcal{K}_\infty(\mathcal{L}_n); \mathcal{S}_{\mathcal{L}_n}(\Delta_\infty))_\infty$ from below.

The set $\mathcal{S}_{\mathcal{L}_n}(\Delta_\infty)$ is closed and locally compact, and therefore it is a set of existence (see [5], p. 21). Let $\sigma_n$ be a function realizing the best approximation of $2\mathcal{A}_{n+1}$ by $\mathcal{S}_{\mathcal{L}_n}(\Delta_\infty)$. Since $2\mathcal{A}_{n+1}$ is $2\pi$-periodic, the function $\sigma_n(x+2\pi j)$, for any $j \in \mathbf{Z}$, also realizes its best approximation.

Consider the sequence of functions

$$\varphi_r(x) = \frac{1}{2r+1} \sum_{j=-r}^{r} \sigma_n(x+2\pi j), \qquad r = 0, 1, \ldots$$

It can easily be seen that $\varphi_r$ is an $\mathcal{L}$-spline of best approximation for $2\mathcal{A}_{n+1}$. Since the value of the best approximation is finite and $\|2\mathcal{A}_{n+1}\|_{C(\mathbf{R})} < +\infty$, the set of elements realizing the best approximation of $2\mathcal{A}_{n+1}$ is uniformly bounded. By the local compactness and closedness of $\mathcal{S}_{\mathcal{L}_n}(\Delta_\infty)$ there is $\varphi_* \in \mathcal{S}_{\mathcal{L}_n}(\Delta_\infty)$ such that

$$\lim_{r_\nu \to +\infty} \varphi_{r_\nu}(x) = \varphi_*(x), \qquad x \in \mathbf{R}.$$

The function $\varphi_*$ is $2\pi$-periodic, since for $x \in \mathbf{R}$

$$\begin{aligned}
&\varphi_*(x+2\pi) - \varphi(x) \\
&= \lim_{r_\nu \to +\infty} [\varphi_{r_\nu}(x+2\pi) - \varphi_{r_\nu}(x)] \\
&= \lim_{r_\nu \to +\infty} \left[\frac{1}{2r_\nu+1} \sum_{j=-r_\nu+1}^{r_\nu+1} \sigma_n(x+2j\pi) - \frac{1}{2r_\nu+1} \sum_{j=-r_\nu}^{r_\nu} \sigma_n(x+2j\pi)\right] \\
&= \lim_{r_\nu \to +\infty} \frac{1}{2r_\nu+1} \sigma_n(x+2\pi(r_\nu+1)) - \lim_{r_\nu \to +\infty} \frac{1}{2r_\nu+1} \sigma_n(x-2\pi r_\nu) \\
&= 0
\end{aligned}$$

by the uniform boundedness of the set of elements realizing the best approximation of $2\mathcal{A}_{n+1}$.

It is not difficult to see that $\varphi_*$ realizes the best approximation of $2\mathcal{A}_{n+1}$. Thus the $2\pi$-periodic function $2\mathcal{A}_{n+1}$ has in $\mathcal{S}_{\mathcal{L}_n}(\Delta_\infty)$ a $2\pi$-periodic element realizing the best approximation, and so Lemma 5 shows that

(8) $$E(\mathscr{K}_\infty(\mathscr{L}_n);\ \mathscr{S}_{\mathscr{L}_n}(\Delta_\infty))_\infty \geqslant \inf_{s_n \in \mathscr{S}_{\mathscr{L}_n}(\Delta_\infty)} \|2\mathscr{A}_{n+1} - s_n\|_{C(\mathbf{R})}$$
$$= \|2\mathscr{A}_{n+1} - \varphi_*\|_{C(\mathbf{R})} = 2\|\mathscr{A}_{n+1}\|_{C(\mathbf{R})}.$$

(7) and (8) yield the conclusion of Theorem 2.

Let $A = \{\alpha\colon \alpha \in [0, h),\ \alpha \neq \xi_n\}$, $f \in \mathscr{K}_\infty(\mathscr{L}_n)$, $0 < h < h_0$. By Theorem 1, for every $\alpha \in A$ there is a unique $\mathscr{S}$-spline $(s_{n,\alpha} f) \in \mathscr{S}_{\mathscr{L}_n}(\Delta_\infty)$ interpolating $f$ on $\Delta_\infty^\alpha = \{\alpha + jh\colon j \in \mathbf{Z}\}$. From Theorem 2 we obtain

COROLLARY 1. *For* $0 < h < 3^{-(k-1)} h_0$

$$\inf_{\alpha \in A} \sup_{f \in \mathscr{K}_\infty(\mathscr{L}_n)} \|f - (s_{n,\alpha} f)\|_{C(\mathbf{R})} = 2\|\mathscr{A}_{n+1}\|_{C(\mathbf{R})},$$

*and the interpolation mesh* $\Delta'_\infty = \{\xi_{n+1} + jh\colon j \in \mathbf{Z}\}$ *we have chosen earlier is optimal as regards the accuracy of approximation.*

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