IRREGULAR CONVEX SETS WITH FIXED-POINT PROPERTY
FOR NON-EXPANSIVE MAPPINGS

BY

K. GOEBEL AND T. KUCZUMOW (LUBLIN)

Let $X$ be a Banach space with norm $\| \cdot \|$ and let $C$ be a non-empty bounded closed subset of $X$. The mapping $T: C \rightarrow C$ is said to be non-expansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for} \ x, y \in C.$$

We say that $C$ has the fixed point property for non-expansive mappings (shortly, f.p.p.) if each non-expansive mapping $T: C \rightarrow C$ has at least one fixed point.

It is known that f.p.p. for the set $C$ depends strongly on the "nice" geometrical properties of the space $X$ or on the set $C$ itself. Among bounded closed convex sets having f.p.p. there are, for example, all compact ones, all sets in uniformly convex space $X$ ([1], [3]), all weakly compact sets having normal structure [6], all weakly compact sets in the space $X$ satisfying so-called Opial's condition [7], all weak-star compact subsets of $l^1$ [4], and all sets in a very special "geometrically bad" but still reflexive space $X_F$ [5].

Our aim here is not to prove any new sufficient conditions for f.p.p. but to construct several examples of very irregular sets still having f.p.p. but not satisfying the above-mentioned conditions. Our sets are not even weak-star compact. We would like to turn attention to some singularities occurring in this field.

Our method is based on the notion of asymptotic center [2]. Let $C$ and $X$ be as above and let $\{x_n\}$ be a bounded sequence of elements of $X$. Consider a function $r: X \rightarrow [0, \infty)$ such that

$$r(y) = \limsup_{n} \|x_n - y\|.$$

It is a convex function of $y$ depending obviously also on the sequence $\{x_n\}$. However, we skipped this dependence in the notation, which should not lead to any misunderstanding in this paper. The value $r(y)$ is called the asymptotic radius of $\{x_n\}$ at $y$, and the number

$$r(C) = \inf \{r(y) : y \in C\}$$
is the asymptotic radius of \( \{x_n\} \) with respect to \( C \). The set
\[
A(C) = \{ y \in C : r(y) = r(C) \}
\]
is called the asymptotic center of \( \{x_n\} \) in \( C \). Obviously, \( A(C) \) is closed and convex but it may be empty.

The connection between f.p.p. and the asymptotic center is given by the following easy lemmas:

**Lemma 1.** If \( T : C \to C \) is a non-expansive mapping, then
\[
\inf \| x - Tw \| : x \in C = 0.
\]

**Lemma 2.** If \( T : C \to C \) is non-expansive and \( \{x_n\} \) is a sequence of elements of \( C \) such that \( x_n - Tw_n \to 0 \), then \( A(C) \) is invariant under \( T \). Especially, if \( A(C) \) contains exactly one point, then it is fixed under \( T \).

The proofs are standard. The next observation is the following

**Trivial Theorem.** If each sequence \( \{x_n\} \) of elements of \( C \) contains a subsequence whose asymptotic center in \( C \) is non-empty and has f.p.p., then \( C \) has f.p.p.

The theorem follows immediately from our lemmas.

We shall construct our examples for \( X = l^1 \), so denote by \( \{e^i\} \) the standard basis \( e^i = \{\delta_j\} \), by \( P_i \) the natural projection of \( l^1 \) on the space spanned by \( e^1, e^2, \ldots, e^i \), and let \( I \) be the identity. For any set \( K \), Conv \( K \) will denote the closed convex envelope of \( K \).

**Lemma 3.** If \( \{x_n\} \) is a sequence in \( l^1 \) converging to \( x \) in weak-star topology, then for any \( y \in l^1 \)
\[
r(y) = r(x) + \|y - x\|.
\]

**Proof.** Notice first that for any \( i = 1, 2, \ldots \)
\[
r(x) = \limsup_{n \to \infty} \|(I - P_i)(x_n - x)\|,
\]
\[
\|x - y\| = \lim_{i \to \infty} \lim_{n \to \infty} \|P_i(x_n - y)\|,
\]
\[
\lim_{i \to \infty} \|(I - P_i)(x - y)\| = 0.
\]

Then the thesis follows from the inequality
\[
\|P_i(x_n - y)\| + \|(I - P_i)(x_n - x)\| - \|(I - P_i)(x - y)\| \leq \|x_n - y\|
\]
\[
\leq \|P_i(x_n - y)\| + \|(I - P_i)(x_n - x)\| + \|(I - P_i)(x - y)\|
\]
by passing to infinity first with \( n \) and then with \( i \).

**Lemma 4.** Let \( \{x_n\} \) be a sequence of elements of \( l^1 \) converging to \( x \) in weak-star topology. Then
\[
A(C) = \text{Proj}_x, \quad \text{where} \quad \text{Proj}_x = \{ y \in C : \|x - y\| = \text{dist}(x, C) \}.
\]
Lemma 4 is an immediate consequence of Lemma 3. This lemma is also true without assumption of convexity of $\mathcal{C}$. Also it is worth to notice that the set $\text{Proj} w$ may be empty.

Let us construct now a special set $\mathcal{C}$. Take any bounded sequence of non-negative reals $\{a_i\}$ and put $f^i = (1 + a_i)e^i$.

Let

$$\mathcal{C} = \text{Conv} \{f^i\} = \left\{ x = \sum_{i=1}^{\infty} \lambda_i f^i : \lambda_i \geq 0, \sum_{i=1}^{\infty} \lambda_i = 1 \right\}.$$ 

The set $\mathcal{C}$ is closed convex but it is not weak-star compact, since the weak-star limit of $\{f^i\}$ is the origin which does not belong to $\mathcal{C}$. The weak-star closure of $\mathcal{C}$ is

$$\overline{\mathcal{C}} = \left\{ x = \sum_{i=1}^{\infty} \mu_i f^i : \mu_i \geq 0, \sum_{i=1}^{\infty} \mu_i \leq 1 \right\}.$$ 

For any $x \in \overline{\mathcal{C}}$ put

$$\delta_x = 1 - \sum_{i=1}^{\infty} \mu_i.$$ 

Obviously, $\delta_x$ is well defined, since the representation of $x$ as a combination of $\{f^i\}$ is unique. Finally, let $a = \inf a_i$ and $N_0 = \{i : a_i = a\}$.

**Lemma 5.** For any $x \in \overline{\mathcal{C}}$

$$\text{dist}(x, \mathcal{C}) = \delta_x (1 + a)$$

and

$$\text{Proj} w = \text{Conv} [x + \delta_x f^i : i \in N_0].$$

**Proof.** We have

$$\text{dist}(x, \mathcal{C}) \leq \inf [(\|x - (x + \delta_x f^i)\| : i = 1, 2, \ldots)]$$

$$= \inf [(1 + a_i) \delta_x : i = 1, 2, \ldots] = \delta_x (1 + a).$$

On the other hand, if

$$y = \sum_{i=1}^{\infty} \lambda_i f^i \in \mathcal{C} \quad \text{and} \quad x = \sum_{i=1}^{\infty} \mu_i f^i \in \overline{\mathcal{C}},$$

then

$$\|x - y\| = \sum_{i=1}^{\infty} |\lambda_i - \mu_i| (1 + a_i) \geq \left| \sum_{i=1}^{\infty} \lambda_i - \sum_{i=1}^{\infty} \mu_i \right| (1 + a) = \delta_x (1 + a).$$

It shows also that for $y \in \mathcal{C} \setminus \text{Conv} [x + \delta_x f^i : i \in N_0]$

$$\|x - y\| > \delta_x (1 + a).$$

**Example 1.** The set $\mathcal{C}$ described above has f.p.p. if and only if $N_0$ is non-empty but finite.
Proof. Take \( \{x_n\} \) to be a weak-star convergent sequence of elements of \( C \) and let \( x \) be its limit. If \( N_0 \) is non-empty but finite, then — in view of Lemmas 4 and 5 — \( A(C) \) is compact and, since each sequence contains a weak-star convergent subsequence, the assumptions of our Trivial Theorem are fulfilled.

Suppose now that \( N_0 \) is empty. Take \( \{a_{i_k}\} \) to be a strictly decreasing subsequence of \( \{a_i\} \) such that

\[
\lim_{k \to \infty} a_{i_k} = a.
\]

Let

\[
N_k = [i: a_{i_k} \leq a_i < a_{i_{k-1}}] \quad \text{for} \quad k = 1, 2, \ldots \quad (a_{i_0} = +\infty).
\]

Define \( T: C \to C \) by

\[
(*) \quad T: x = \sum_{i=1}^{\infty} \lambda_i x^i \mapsto \sum_{k=1}^{\infty} \mu_k x^{i_{k+1}} \text{, where } \mu_k = \sum_{i \in N_k} \lambda_i.
\]

It is non-expansive and fixed point free.

Now, let \( N_0 \) be infinite and let \( N_0 = [i_1, i_2, \ldots] \). Put

\[
N_k = [i: i_{k-1} < i \leq i_k] \quad \text{for} \quad k = 1, 2, \ldots \quad (i_0 = 0).
\]

We see that \( T \) defined by \( (*) \) is also non-expansive and fixed point free.


Proof. Take any \( b > 0 \) and construct the set \( C_1 \) in the way described above putting \( a_1 = 0 \) and \( a_i = b \) for \( i = 2, 3, \ldots \). Then construct the set \( C_2 \) in the same way putting \( a_1 = \frac{1}{2} b \) and \( a_i = b \) for \( i = 2, 3, \ldots \). Then \( C_1 \) and \( C_2 \) have f.p.p. but

\[
C_3 = C_1 \cap C_2 = \text{Conv}[(1+b) e^i: i = 2, 3, \ldots]
\]

fails to have it.

Example 3. There exists a sequence of sets \( \{C_n\} \) such that \( C_1 \supset C_2 \supset C_3 \supset \ldots \), and \( C_n \) has f.p.p. for \( n = 1, 3, 5, \ldots \) and does not have it for \( n = 2, 4, \ldots \) Moreover, this sequence may be such that

\[
C_\infty = \bigcap_{n=1}^{\infty} C_n
\]

is non-empty and does have or does not have f.p.p. up to our choice.

Proof. Take any bounded increasing sequence \( \{b_n\} \) of positive reals and then take the double indexed sequence \( \{a_{in}\} \) of positive reals such that

\[
\ldots < a_{21} < a_{21} < a_{11} < b_1 < \ldots < a_{22} < a_{12} < b_2 < \ldots
\]
Use the sequence \( \{a_{in}\} \) to construct the set \( C \) in the following way.
Let \( N \) denote the set of integers and let \( \varphi: N \times N \to N \) be a 1-1 correspondence. Put
\[
f^{t,i,n} = (1 + a_{in})\varphi^{t,i,n}
\]
and select any sequence \( \{a_{tn}\} \). Now put
\[
C_{2n-1} = \text{Conv} [f^{t,i,n}: a_{ik} \geq a_{tn}] ,
C_{2n} = \text{Conv} [f^{t,i,n}: a_{ik} > b_n].
\]

Then the first part of our statement is proved.

To get the second part it is enough to take first a non-empty subset \( N_{\infty} \subset N \) such that \( N \setminus N_{\infty} \) is infinite, then repeat our construction on the basis vectors with indices \( i \notin N_{\infty} \), and put \( f^i = (1 + b)\varphi^i \) with \( b > b_n \) for all \( n \) if \( i \in N_{\infty} \). Then
\[
\bigcap_{n=1}^{\infty} C_n = \text{Conv} [f^i: i \in N_{\infty}]
\]
has f.p.p. if \( N_{\infty} \) is finite and does not have it if \( N_{\infty} \) is infinite.

The last example is of a little different nature. Let \( X_1 \) be an arbitrary uniformly convex Banach space and put \( X = X_1 \times l^1 \) with the norm
\[
\|(x, y)\|_X = \max [\|x\|_{X_1}, \|y\|_{l^1}].
\]

Let \( C_1 \) be an arbitrary non-empty bounded closed and convex subset of \( X_1 \) and let \( B \) be a unit ball in \( l^1 \). Put \( C = C_1 \times B \).

Example 4. \( C \) has f.p.p.

Proof. First notice that each sequence \( \{(x_n, y_n)\} \) of elements of \( C \) contains a subsequence with \( \{y_n\} \) weak-star convergent. Moreover, the asymptotic center of any sequence in a uniformly convex space contains exactly one point [2]. Let then \( \{(x_n, y_n)\} \) be such that \( \text{w}^*\text{-}\text{lim} y_n = y \). The asymptotic radius of this sequence with respect to \( C \) is equal to
\[
r = \max [r_1, r_2],
\]
where \( r_1 \) is the asymptotic radius of \( \{x_n\} \) in \( C_1 \) and \( r_2 \) is the asymptotic radius of \( \{y_n\} \) in \( B \). Let \( \{x\} \) be the asymptotic center of \( \{x_n\} \) in \( C_1 \). The asymptotic center \( \text{A}(C) \) of \( \{(x_n, y_n)\} \) in \( C \) is equal to \( \{(x, y)\} \) if \( r_1 = r_2 \). However, if \( r_1 < r_2 \), then
\[
\text{A}(C) = \{z \in C_1: \limsup_{n} \|x_n - z\|_{X_1} \leq r_2\} \times \{y\},
\]
which has f.p.p. as it is isometric to the closed convex and bounded subset of \( X_1 \). On the other hand, if \( r_1 > r_2 \), then
\[
\text{A}(C) = \{x\} \times \{z \in B: \limsup_{n} \|y_n - z\|_{l^1} \leq r_1\}
\]
\[
= \{x\} \times \{z \in B: \|z - y\|_{l^1} \leq r_1 - r_2\}.
\]
according to Lemma 3. This set is isometric to the intersection of two balls in $l^1$ and such an intersection is weak-star compact. So, in view of [4], it has f.p.p. As we see, the assumptions of Trivial Theorem are fulfilled, so $C$ has f.p.p.

Let us finish with the metamathematical statement, not quite clear but in our opinion in some sense true:

*For any sufficient condition for f.p.p. there exists a set having f.p.p., which does not satisfy it.*

REFERENCES


M. CURIE-SKŁODOWSKA UNIVERSITY
INSTITUTE OF MATHEMATICS, LUBLIN

*Reçu par la Rédaction le 20. 4. 1977*