

Convergence of generalized-approximate iterations for an abstract equation

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Many authors have been interested in the solution of an abstract equation

$$(1) \quad x = f(x).$$

In papers [1], [2], [4] it is shown that under certain quite general assumptions, there exists a unique solution \bar{x} of equation (1). In papers [2] and [3] conditions are given by which a sequence $\{x_n\}$ is convergent to a solution of equation (1), and in [1], [2], [3] and [4] it is shown in what way the sequences $\{x_n\}$ are created.

In this paper, we analyse the generalized-approximate process (see [1], p. 40) defined by the relation

$$(2) \quad y_n = f_{n,k}(y_n, y_{n-1}), \quad y_0 = x^*, \quad k \geq 1, \quad n = 0, 1, \dots,$$

where

$$(3) \quad \begin{cases} f_{n,1}(x, y) = f_n(x, y), & n = 0, 1, \dots, \\ f_{n,i}(x, y) = f_n(x, f_{n,i-1}(x, y)), & i = 2, 3, \dots, k, \quad n = 1, 2, \dots \end{cases}$$

Conditions are given under which the sequence $\{y_n\}$ is well-defined, convergent to a unique solution of equation (1), and also estimations of errors are given. Moreover, it is proved that for certain conditions the estimations of errors are not worse than the estimations for a certain sequence $\{x_n\}$ in paper [2].

For the case

$$(4) \quad f_{n,k}(x, y) = f_n(x, y),$$

the sequence $\{y_n\}$ is analysed in papers [1], [2] and [3], and for

$$(5) \quad f_{n,k}(x, y) = f_k(x, y), \quad k \geq 1,$$

where

$$\begin{aligned} f_1(x, y) &= f(x, y), \\ f_i(x, y) &= f(x, f_{i-1}(x, y)), \quad i = 2, 3, \dots, k, \end{aligned}$$

the sequence $\{y_n\}$ is considered in paper [1].

Assumptions and lemmas do not differ much from those given in the papers of Kwapisz, Wazewski and Kurpiel.

I. Assumptions, definitions and lemmas. We introduce the following

ASSUMPTION H_1 (see [4], [2]). 1° G is a partially ordered set (an ordering relation is denoted by $<$, we write $u \leq v$ iff $u < v$ or $u = v$), in G there exists an element 0 such that $0 \leq u$ for any $u \in G$;

2° for any $u, v \in G$ a relation $u + v$ is defined and has the following properties:

(a) if $u, v, w \in G$, then $u + v \in G$, $u + v = v + u$, $(u + v) + w = u + (v + w)$, $u + 0 = u$;

(b) if $u, v, w \in G$ and $u \leq v$, then $u + w \leq v + w$;

(c) if $u, v, w \in G$ and $u + v \leq w$, then $u \leq w$;

3° for any non-increasing sequence $\{u_n\}$, $u_n \in G$, $u_{n+1} \leq u_n$, $n = 0, 1, \dots$, there exists a unique element $u \in G$ called the *limit of the sequence* $\{u_n\}$ (we write $u = \lim_{n \rightarrow \infty} u_n$ or $u_n \searrow u$).

The limit has the following properties:

(a) $\lim_{n \rightarrow \infty} u_n$ is invariant with respect to the change of a finite number of elements of the sequence $\{u_n\}$,

(b) if $u_n = u$, $n = 0, 1, \dots$, then $\lim_{n \rightarrow \infty} u_n = u$,

(c) if $u_n \searrow u$, $v_n \searrow v$ and $u_n \leq v_n$, then $u \leq v$,

(d) if $u_n \searrow u$, $v_n \searrow v$, then $u_n + v_n \searrow u + v$,

ASSUMPTION H_2 (see [4]). The function $a(u)$ is defined for $u \in \Delta \subset G$ and has the following properties:

1° $a(\Delta) \subset G$, where $0 \in \Delta$ and if $k \in \Delta$, then $u \in \Delta$ for any $u \leq k$,

2° if $u, v \in \Delta$ and $u \leq v$, then $a(u) \leq a(v)$;

3° if $u_n \in \Delta$, $n = 0, 1, \dots$, and $u_n \searrow u$, then $a(u_n) \searrow a(u)$;

4° $u = 0$ is the only solution in Δ of the equation $u = a(u)$.

DEFINITION (see [4]). For any $u \in \Delta$ we define the sequence $\{a_n(u)\}$ of the iterations of the element u by the recurrent formula

$$a_0(u) = u, \quad a_{n+1}(u) = a(a_n(u)) \quad \text{if } a_n(u) \in \Delta, \quad n = 1, 2, \dots$$

We state the following

LEMMA 1 (see [4]). *If Assumption H_2 is fulfilled and there exists a $c \in \Delta$ such that $a(c) \leq c$, then all iterations $a_n(c)$, $n = 0, 1, \dots$, of the element c exist and*

$$a_{n+1}(c) \leq a_n(c) \leq c, \quad n = 0, 1, \dots, \text{ and } a_n(c) \searrow 0.$$

COROLLARY 1. From Lemma 1 and Assumption H_2 we have

$$a_n(u) \leq a_n(c) \leq c, \quad u \in [0, c], \quad n = 0, 1, \dots$$

ASSUMPTION H'_2 . $u = 0$ is the only solution in Δ of the equation $u = a_k(u)$, $k \geq 1$, where $a_k(u)$ is definite in the definition.

LEMMA 2. If Assumption H_2 (except 5°) is fulfilled and there exist $q \in \Delta$ and $b \in \Delta$ such that

$$q + a(b) \leq b,$$

then the equation

$$u = a_k(u) + q, \quad k \geq 1,$$

has the solution $u = m(k, b, q) \leq b$, which has the properties:

$$1^\circ \quad m(k, b, q) = \lim_{n \rightarrow \infty} b_n(k, b, q), \quad \text{where } b_0(k, b, q) = b, \quad b_{n+1}(k, b, q) = a_k(b_n(k, b, q)) + q, \quad n = 0, 1, \dots,$$

$$2^\circ \quad \text{if } p \leq b \text{ and } p \leq a_k(p) + q, \text{ then } p \leq m(k, b, q).$$

LEMMA 3. If assumptions H_2 (except 5°) and H'_2 are fulfilled and there exist $q_n, b_s \in \Delta$, $q_{n+1} \leq q_n$, $b_{s+1} \leq b_s$, $n, s = 0, 1, \dots$, such that

$$q_n + a(b_s) \leq b_s \quad \text{for any } n, s = 0, 1, \dots,$$

then the equation

$$u = a_k(u) + q_n, \quad k \geq 1$$

has a solution $u = m(k, b_s, q_n) \leq b_s$ such that

$$m(k, b_s, q_{n+1}) \leq m(k, b_s, q_n), \quad s, n = 0, 1, \dots,$$

$$m(k, b_{s+1}, q_n) \leq m(k, b_s, q_n), \quad s, n = 0, 1, \dots$$

Moreover, if $q_n \searrow q$ and $b_s \searrow b$, then

$$m(k, b_s, q_n) \searrow m(k, b_s, q), \quad s = 0, 1, \dots,$$

$$m(k, b_s, q_n) \searrow m(k, b, q_n), \quad n = 0, 1, \dots,$$

and consequently, if $q = 0$, then $m(k, b_s, q_n) \searrow 0$.

Lemmas 2 and 3 can be proved in the same way as Lemma 1 [4], and 3 [2].

ASSUMPTION H_3 . The function $A(u, v)$ is defined for $u, v \in \Delta$ and has the following properties:

$$1^\circ \quad A(\Delta \times \Delta) \subset G;$$

$$2^\circ \quad \text{if } u, \bar{u}, v, \bar{v} \in \Delta \text{ and } u \leq \bar{u}, v \leq \bar{v}, \text{ then } A(u, v) \leq A(\bar{u}, \bar{v});$$

$$3^\circ \quad \text{if } u_n, v_n \in \Delta, n = 0, 1, \dots, \text{ and } u_n \searrow u, v_n \searrow v, \text{ then } A(u_n, v_n) \searrow A(u, v);$$

$$4^\circ \quad u = 0 \text{ is the only solution in } \Delta \text{ of the equation } u = A(u, u);$$

$$5^\circ \quad u = 0 \text{ is the only solution in } \Delta \text{ of the equation } u = \beta_k(0, A, u, u)$$

where $\beta_1(q, A, u, v) = q + A(u, v)$, $\beta_i(q, A, u, v) = q + \mathcal{A}(u, \beta_{i-1}(q, A, u, v))$, $q, u, v \in \Delta$, $i = 2, 3, \dots, k$.

LEMMA 4. If Assumption H_3 (except 4° - 5°) is fulfilled, then

1° if $u, \bar{u}, v, \bar{v}, q, \bar{q} \in \Delta$, and $\bar{A}(\Delta \times \Delta) \subset G$, and $u \leq \bar{u}$, $v \leq \bar{v}$, $q \leq \bar{q}$, $A(u, v) \leq \bar{A}(u, v)$, then

$$\beta_i(q, A, u, v) \leq \beta_i(\bar{q}, \bar{A}, \bar{u}, \bar{v}), \quad i = 1, 2, \dots, k,$$

2° if $u_n, v_n, q_n \in \Delta$, $n = 0, 1, \dots$, and $u_n \searrow u$, $v_n \searrow v$, $q_n \searrow q$, then

$$\beta_i(q_n, A, u_n, v_n) \searrow \beta_i(q, A, u, v), \quad i = 1, 2, \dots, k.$$

Moreover, if there exist $q, b \in \Delta$ such that

$$q + A(b, b) \leq b,$$

then

$$\beta_i(q, A, u, v) \leq b, \quad u, v \in [0, b], \quad i = 1, 2, \dots, k,$$

and

$$\beta_i(q, A, u, b) \leq \beta_{i-1}(q, A, u, b), \quad u \in [0, b], \quad i = 2, 3, \dots, k.$$

The proof of Lemma 4 is obvious.

LEMMA 5. If the assumptions of Lemma 4 are fulfilled, then for any $v \leq b$ there exists a solution $m_v(k, A, b, q) \leq b$ of the equation

$$(6) \quad u = \beta_k(q, A, u, v), \quad k \geq 1.$$

Moreover, if $w \leq b$ and $w \leq \beta_k(q, A, w, v)$, then

$$w \leq m_v(k, A, b, q) \leq b.$$

Proof. Put

$$u_0 = b, \quad u_{n+1} = \beta_k(q, A, u_n, v), \quad n = 0, 1, \dots,$$

we have

$$u_n \searrow m_v(k, A, b, q) \leq b, \quad w \leq u_n, \quad n = 0, 1, \dots,$$

whence we get the assertion of Lemma 5.

LEMMA 6. If Assumption H_3 (except 4°) is fulfilled and there exist $q, b \in \Delta$ such that $q + A(b, b) \leq b$ and $q_{n+1} \leq q_n \leq q$, $n = 0, 1, \dots$, $q_n \searrow 0$ and $m_{q_{n+1}}(k, A, b, q_{n+1})$ is a solution of the equation

$$u = \beta_k(q_{n+1}, A, u, m_n(k, A, b, q_n)), \quad m_0(k, A, b, q_0) = b, \quad n = 0, 1, \dots,$$

then $m_n(k, A, b, q_n) \searrow 0$.

Moreover, if $w_{n+1} \leq \beta_k(q_{n+1}, A, w_{n+1}, w_n)$ and $w_n \leq b$, then $w_n \leq m_n(k, A, b, q_n)$, and consequently, $w_n \searrow 0$ (if $w_{n+1} \leq w_n$).

Proof. Because $m_0(k, A, b, q_0) = b$, it follows from Lemma 5 that there exists an $m_n(k, A, b, q_n)$, and we obtain by induction

$$m_{n+1}(k, A, b, q_{n+1}) \leq m_n(k, A, b, q_n), \quad n = 0, 1, \dots,$$

$$w_n \leq m_n(k, A, b, q_n), \quad n = 0, 1, \dots$$

Hence, if $n \rightarrow \infty$, we get the assertion of Lemma 6.

Remark 1. If the assumption $q + A(b, b) \leq b$ is replaced by $\beta_k(q, A, b, b) \leq b$, then Lemmas 5 and 6 remain true.

DEFINITION (see [2]). $m(k, A, q)$ is called the *maximal solution* of equation (6) if it satisfies this equation and for any solution $u(k, A, q)$ of (6) the inequality $u(k, A, q) \leq m(k, A, q)$ holds true.

LEMMA 7. If Assumption H_3 (except 4°) is fulfilled, $\Delta = G$ and if for any $v, q \in G$ there exist maximal solutions $m(k, A, q)$, $m_v(k, A, q)$ of the equations

$$u = \beta_k(q, A, u, u), \quad u = \beta_k(q, A, u, v)$$

and if $d \leq \beta_k(q, A, d, v)$, $d \in G$, then $d \leq m_v(k, A, q)$.

Moreover, if $q_n \searrow 0$ and $m_{n+1}(k, A, q_{n+1})$ is the maximal solution of the equation

$$u = \beta_k(q_{n+1}, A, u, m_n(k, A, q_n)), \quad m_0(k, A, q_0) = m(k, A, q_0),$$

$$n = 0, 1, \dots,$$

and

$$w_{n+1} \leq \beta_k(q_{n+1}, A, w_{n+1}, w_n), \quad w_0 \leq m_0(k, A, q_0), \quad n = 0, 1, \dots,$$

then

$$m_n(k, A, q_n) \searrow 0 \quad \text{and} \quad w_n \leq m_n(k, A, q_n), \quad n = 0, 1, \dots,$$

and consequently, $w_n \searrow 0$ (if $w_{n+1} \leq w_n$).

Proof. Let u_0 be a solution of the equation

$$u = \beta_k(q + d, A, u, v);$$

then $u_0 \geq \beta_k(q, A, u_0, v)$ and $u_0 \geq d$.

Putting

$$u_{n+1} = \beta_k(q, A, u_n, v), \quad n = 0, 1, \dots,$$

we obtain by induction

$$u_{n+1} \leq u_n, \quad d \leq u_n, \quad n = 0, 1, \dots$$

If $n \rightarrow \infty$, we obtain

$$u_n \searrow m_v(k, A, q), \quad d \leq m_v(k, A, q).$$

Now we can prove that

$$m_{n+1}(k, A, q_{n+1}) \leq m_n(k, A, q_n), \quad n = 0, 1, \dots,$$

and if $n \rightarrow \infty$, we obtain $m_n(k, A, q_n) \searrow 0$.

The next assertions of Lemma 7 are obvious.

2. Definition of the space R . We introduce

ASSUMPTION H_4 (see [4], [2]). R is an abstract space such that

1° for some sequences $\{x_n\}, x_n \in R, n = 0, 1, \dots$, there exists a uniquely determined limit $\lim_{n \rightarrow \infty} x_n = x, x \in R$; $\lim_{n \rightarrow \infty} x_n$ is invariant with respect to the change of a finite number of elements of $\{x_n\}$ (the relation $\lim_{n \rightarrow \infty} x_n = x$ will also be written as $x_n \rightarrow x$);

2° if $x_n = s \in R, n = 0, 1, \dots$, then $\lim_{n \rightarrow \infty} x_n = s$;

3° the function $\varrho(x, y)$ is defined on the product $R \times R$ and has the following properties:

(a) $\varrho(x, y) \in G$,

(b) $\varrho(x, y) = 0$ iff $x = y$,

(c) for any $x, y, z \in R$

$$\varrho(x, y) \leq \varrho(x, z) + \varrho(y, z);$$

4° for any $x^* \in R$ and $b \in G$ the sphere

$$S(x^*, b) = [x : x \in R, \varrho(x, x^*) \leq b]$$

is a closed set;

5° the space R is complete in the following sense: if $c_n \in G, n = 0, 1, \dots$, $c_n \searrow 0$ and for $x_n \in R, n = 0, 1, \dots$, the Cauchy condition

$$\varrho(x_n, x_{n+m}) \leq c_n, \quad n, m = 0, 1, \dots,$$

is satisfied, then there exists a limit $y \in R$ of the sequence $\{x_n\}$.

ASSUMPTION H_5 (see [2], [4]). 1° $f : S(x^*, b) \rightarrow R, x^* \in R, b \in \Delta$;

2° for any $x, y \in S(x^*, b)$

$$\varrho(f(x), f(y)) \leq a(\varrho(x, y)),$$

where the function $a(u)$ satisfies Assumption H_2 , and $b + b \stackrel{\text{df}}{=} 2b \in \Delta$;

3° there exists a $q \in \Delta$ such that

$$\varrho(x^*, f(x^*)) \leq q \quad \text{and} \quad q + a(b) \leq b.$$

ASSUMPTION H_6 . 1° $f_n : S(x^*, b) \times S(x^*, b) \rightarrow R, n = 0, 1, \dots, x^* \in R, b \in \Delta$;

2° for any $x, y, s, t \in S(x^*, b)$, $n = 0, 1, \dots$, we have

$$\varrho(f_n(x, y), f_n(s, t)) \leq A_n(\varrho(x, s), \varrho(y, t)),$$

where the function $A_n(u, v)$ satisfies 1°-3° of Assumption H_3 , $2b \in \Delta$;

3° for any $u, v \in [0, 2b]$ we have $A_n(u, v) \leq A(u, v)$, where the function $A(u, v)$ satisfies Assumption H_3 ;

4° there exists a $q \in \Delta$ such that for any $n = 0, 1, \dots$,

$$\varrho(x^*, f_n(x^*, x^*)) \leq q \quad \text{and} \quad q + A(b, b) \leq b.$$

ASSUMPTION H_7 . 1° $f_n: R \times R \rightarrow R$, $n = 0, 1, \dots$;

2° for any $x, y, s, t \in R$, $n = 0, 1, \dots$, we have

$$\varrho(f_n(x, y), f_n(s, t)) \leq A_n(\varrho(x, s), \varrho(y, t)),$$

where the function $A_n(u, v)$ satisfies 1°-3° of Assumption H_3 with $\Delta = G$;

3° for any $u, v \in G$ we have $A_n(u, v) \leq A(u, v)$, where the function $A(u, v)$ satisfies Assumption H_3 with $\Delta = G$;

4° for any $q \in G$ there exists a maximal solution of the equation

$$u = \beta_k(q, A, u, u),$$

where $\beta_k(q, A, u, v)$ is defined in Assumption H_3 .

3. Generalized-approximate iterations, the local theorem. We formulate the following

THEOREM 1 (see [2], [1], [4]). *If Assumption H_5 is fulfilled, then there exists in $S(x^*, b)$ a unique solution \bar{x} of equation (1) and $\lim_{n \rightarrow \infty} x_n = \bar{x}$, where $x_0 = x^*$, $x_{n+1} = f(x_n)$, $n = 0, 1, \dots$*

Moreover,

$$\varrho(x_n, \bar{x}) \leq a_n(b), \quad n = 0, 1, \dots,$$

where $a_0(b) = b$, $a_{n+1}(b) = a(a_n(b))$, $n = 0, 1, \dots$

LEMMA 8. *If Assumption H_6 is fulfilled, then*

$$\varrho(f_{n,i}(x, y), x^*) \leq b \quad \text{for } x, y \in S(x^*, b), i = 1, 2, \dots, k, n = 0, 1, \dots$$

Proof. Indeed, for $i = 1$, $n = 0, 1, \dots$, we have

$$\begin{aligned} \varrho(f_{n,1}(x, y), x^*) &\leq \varrho(f_n(x, y), f_n(x^*, x^*)) + \varrho(f_n(x^*, x^*), x^*) \\ &\leq A_n(\varrho(x, x^*), \varrho(y, x^*)) + q \leq A(b, b) + q \leq b. \end{aligned}$$

Further, if we suppose that

$$\varrho(f_{n,r}(x, y), x^*) \leq b,$$

then

$$\begin{aligned} \varrho(f_{n,r+1}(x, y), x^*) &\leq \varrho(f_n(x, f_{n,r}(x, y)), f_n(x^*, x^*)) + \varrho(f_n(x^*, x^*), x^*) \\ &\leq A_n(\varrho(x, x^*), \varrho(f_{n,r}(x, y), x^*)) + q \leq A(b, b) + q \leq b. \end{aligned}$$

Now we can formulate the following

THEOREM 2. *If*

1° $f: S(x^*, b) \rightarrow R$,

2° Assumption H_6 is fulfilled,

3° $\varrho(f_n(x, x), f(x)) \leq \varepsilon_n(x) \searrow 0$, $x \in S(x^*, b)$, $\varepsilon_0(x) \leq q$, then there exists in $S(x^*, b)$ a unique solution \bar{x} of equation (1). The sequence $\{y_n\}$ is well-defined by relation (2), and if $\varrho(y_n, \bar{x}) \leq b$, then $y_n \rightarrow \bar{x}$.

Moreover,

$$(7) \quad \varrho(y_n, \bar{x}) \leq m_n(k, A, b, \varepsilon_n(\bar{x})), \quad n = 0, 1, \dots,$$

where $m_n(k, A, c, d)$ is defined in Lemma 6, and

$$(8) \quad \varrho(y_n, \bar{x}) \leq m_{\varrho(y_{n-1}, \bar{x})}(k, A_n, b, \varepsilon_n(\bar{x})),$$

where $m_v(k, B, c, d)$ is defined in Lemma 5.

Proof. In paper [2] it is proved that Assumption H_5 with $a(u) = A(u, u)$ holds true, and consequently there exists a unique solution \bar{x} of equation (1) (see Theorem 1).

Now we prove that y_n exists and $y_n \in S(x^*, b)$, $n = 0, 1, \dots$. Put

$$y_n^{s+1} = f_{n,k}(y_n^s, v), \quad y_n^0 = x^*, \quad v \in S(x^*, b), \quad k \geq 1, \quad s = 0, 1, \dots$$

Because $v, y_n^0 \in S(x^*, b)$; by Lemma 8 we state, $y_n^s \in S(x^*, b)$, $s = 0, 1, \dots$, i.e.

$$\varrho(y_n^s, x^*) \leq b, \quad s = 0, 1, \dots$$

Further, we obtain by induction

$$\varrho(f_{n,k}(x, y), f_{n,k}(z, w)) \leq \beta_k(0, A, \varrho(x, z), \varrho(y, w)), \quad x, y, z, w \in S(x^*, b).$$

Set

$$\gamma_0(b) = b, \quad \gamma_i(b) = \beta_k(0, A, \gamma_{i-1}(b), 0), \quad i = 1, 2, \dots;$$

then $\gamma_s(b) \searrow 0$.

Now we can prove by induction

$$\varrho(y_n^s, y_n^{s+r}) \leq \gamma_s(b), \quad s, r = 0, 1, \dots$$

Hence, if $r \rightarrow \infty$, we obtain $y_n^r \rightarrow y_n$, where y_n is a unique solution of the equation

$$y = f_{n,k}(y, v), \quad v \in S(x^*, b).$$

and $y_n \in S(x^*, b)$.

Now we show that for $x, y \in S(x^*, b)$

$$\varrho(f_{n,k}(x, y), \bar{x}) \leq \beta_k(\varepsilon_n(\bar{x}), A_n, \varrho(x, \bar{x}), \varrho(y, \bar{x})), \quad n = 0, 1, \dots$$

In fact, we have

$$\begin{aligned} \varrho(f_{n,1}(x, y), \bar{x}) &\leq \varrho(f_n(x, y), f_n(\bar{x}, \bar{x})) + \varrho(f_n(\bar{x}, \bar{x}), f(\bar{x})) \\ &\leq A_n(\varrho(x, \bar{x}), \varrho(y, \bar{x})) + \varepsilon_n(\bar{x}) \\ &= \beta_1(\varepsilon_n(\bar{x}), A_n, \varrho(x, \bar{x}), \varrho(y, \bar{x})), \end{aligned}$$

and if

$$\varrho(f_{n,r}(x, y), \bar{x}) \leq \beta_r(\varepsilon_n(\bar{x}), A_n, \varrho(x, \bar{x}), \varrho(y, \bar{x})),$$

then

$$\begin{aligned} \varrho(f_{n,r+1}(x, y), \bar{x}) &\leq \varrho(f_n(x, f_{n,r}(x, y)), f_n(\bar{x}, \bar{x})) + \varrho(f_n(\bar{x}, \bar{x}), f(\bar{x})) \\ &\leq A_n(\varrho(x, \bar{x}), \varrho(f_{n,r}(x, y), \bar{x})) + \varepsilon_n(\bar{x}) \\ &\leq \beta_{r+1}(\varepsilon_n(\bar{x}), A, \varrho(x, \bar{x}), \varrho(y, \bar{x})). \end{aligned}$$

Now we see that

$$\begin{aligned} \varrho(y_n, \bar{x}) = \varrho(f_{n,k}(y_n, y_{n-1}), \bar{x}) &\leq \beta_k(\varepsilon_n(\bar{x}), A_n, \varrho(y_n, \bar{x}), \varrho(y_{n-1}, \bar{x})) \\ &\leq \beta_k(\varepsilon_n(\bar{x}), A_n, \varrho(y_n, \bar{x}), \varrho(y_{n-1}, \bar{x})). \end{aligned}$$

Putting $\varrho(y_n, \bar{x}) = d_n \leq b$, we have

$$d_n \leq \beta_k(\varepsilon_n(\bar{x}), A, d_n, d_{n-1}), \quad n = 1, 2, \dots$$

By Lemma 6

$$\varrho(y_n, \bar{x}) \leq m_n(k, A, b, \varepsilon_n(\bar{x})) \searrow 0,$$

and, consequently, $y_n \rightarrow \bar{x}$.

We have estimation (8) from Lemma 5.

Remark 2. Theorem 2 [2] shows that by Assumption H_5 and additional conditions, the sequence $\{\tilde{x}_n\}$ (defined in that assumption of this theorem) is convergent to \bar{x} , and the estimation

$$\varrho(\tilde{x}_n, \bar{x}) \leq z_n(\varepsilon^*), \quad n = 0, 1, \dots,$$

holds true, where $\varepsilon_n^* \searrow 0$, $\varepsilon_0^* \leq q$, $z_0(\varepsilon^*) = b$, $z_{n+1}(\varepsilon^*) = \varepsilon_n^* + a(z_n(\varepsilon^*))$, $n = 0, 1, \dots$

If the assumptions of Theorem 2 are fulfilled, and $a(u) = A(u, b)$, $u \in [0, b]$, then

$$\varrho(y_n, \bar{x}) \leq m_n(k, A, b, \varepsilon_n(\bar{x})) \leq z_n(\varepsilon), \quad n = 0, 1, \dots \quad (\varepsilon_n \stackrel{\text{def}}{=} \varepsilon_n(\bar{x})).$$

In fact, we have $m_0(k, A, b, \varepsilon_0(\bar{x})) = b = z_0(\varepsilon)$, and if

$$m_s(k, A, b, \varepsilon_s(\bar{x})) \leq z_s(\varepsilon),$$

then

$$\begin{aligned} m_{s+1}(k, A, b, \varepsilon_{s+1}(\bar{x})) &= \beta_k(\varepsilon_{s+1}(\bar{x}), A, m_{s+1}(k, A, b, \varepsilon_{s+1}(\bar{x})), m_s(k, A, b, \varepsilon_s(\bar{x}))) \\ &\leq \beta_k(\varepsilon_s(\bar{x}), A, m_s(k, A, b, \varepsilon_s(\bar{x})), m_s(k, A, b, \varepsilon_s(\bar{x}))) \\ &\leq \beta_k(\varepsilon_s(\bar{x}), A, z_s(\varepsilon), z_s(\varepsilon)) \leq \varepsilon_s(\bar{x}) + A(z_s(b), b) = z_{s+1}(\varepsilon). \end{aligned}$$

Remark 3. If

$$f_{n,i}(y, z) = f_i(z), \quad i = 1, 2, \dots, k,$$

where

$$\begin{aligned} f_1(x) &= f(x), & x \in S(x^*, b), \\ f_i(x) &= f(f_{i-1}(x)), & x \in S(x^*, b), \quad i = 2, 3, \dots, k, \end{aligned}$$

then $\varepsilon_n(x) \equiv 0$, $n = 0, 1, \dots$. Moreover, if $A(u, v) = a(v)$, $u, v \in [0, b]$, then for the resulting sequence $\{y_n\}$ we have

$$y_n \rightarrow \bar{x}, \quad y_n = x_{n,k}, \quad k \geq 1, n = 0, 1, \dots,$$

and

$$\varrho(y_n, \bar{x}) \leq m_n(k, A, b, 0) = a_{n,k}(b), \quad k \geq 1, n = 0, 1, \dots,$$

where $\{x_n\}$, $\{a_n(b)\}$ are defined in Theorem 1. If we put

$$d_n(b) = a_{n,k}(b), \quad k \geq 1, n = 0, 1, \dots,$$

then Theorem 2 gives the identical estimations as those pointed out in Theorem 1.

Remark 4. If Assumptions H_5 and H'_5 are fulfilled, then assumption 3° in Theorem 2 can be replaced by

$$\varrho(f_{n,k}(y_n, y_{n-1}), f_k(y)) \leq \varepsilon_{n-1} \searrow 0, \quad \varepsilon_0 \leq q, \quad k \geq 1, n = 1, 2, \dots,$$

and if $\varrho(y_n, \bar{x}) \leq b$, then

$$\varrho(y_n, \bar{x}) \leq m(k, b, \varepsilon_n) \searrow 0, \quad \text{i.e. } y_n \rightarrow \bar{x},$$

where $m(k, b, q)$ is defined in Lemma 2.

Indeed, we have

$$\varrho(y_n, \bar{x}) \leq \varrho(y_n, f_k(y_n)) + \varrho(f_k(y_n), f(\bar{x})) \leq \varepsilon_n + a_k(\varrho(y_n, \bar{x})),$$

and the next assertions of the remark are obvious.

4. Generalized-approximate iterations, the non-local theorem. Now we can formulate a theorem having a non-local character.

THEOREM 3. If

1° $f: R \rightarrow R$,

2° Assumption H_7 is fulfilled,

3° $\varrho(f_n(x, x), f(x)) \leq \varepsilon_n(x) \searrow 0$, $\varrho(f_n(x^*, x^*), x^*) \leq \varepsilon_0(x)$, then there exists a unique solution \bar{x} of equation (1).

The sequence $\{y_n\}$ is well-defined by relation (2), where x^* is an arbitrarily fixed element of R , and $y_n \rightarrow \bar{x}$.

Moreover,

$$\varrho(y_n, \bar{x}) \leq m_n(k, A, \varepsilon_n(\bar{x})), \quad \varrho(y_n, \bar{x}) \leq m_{\varrho(y_{n-1}, \bar{x})}(k, A_n, \varepsilon_n(\bar{x})),$$

where $m_n(k, A, q)$ and $m_v(k, A, q)$ are defined in Lemma 7.

Proof. We will prove that equation (1) has in R a unique solution. If x is a solution of equation (1), then

$$(9) \quad \varrho(x, x^*) \leq \beta_k(\varepsilon_n(x) + \bar{q}, A, \varrho(x, x^*), \varrho(x, x^*)),$$

where $\varrho(f_n(x^*, x^*), x^*) \leq \bar{q} \leq \varepsilon_0(x)$.

Indeed, we have

$$\begin{aligned} \varrho(x, x^*) &\leq \varrho(f(x), f_n(x, x)) + \varrho(f_n(x, x), f_n(x^*, x^*)) + \varrho(f_n(x^*, x^*), x^*) \\ &\leq \varepsilon_n(x) + A_n(\varrho(x, x^*), \varrho(x, x^*)) + \bar{q} \\ &\leq \beta_1(\varepsilon_n(x) + \bar{q}, A, \varrho(x, x^*), \varrho(x, x^*)), \end{aligned}$$

and if

$$\varrho(x, x^*) \leq \beta_r(\varepsilon_n(x) + \bar{q}, A, \varrho(x, x^*), \varrho(x, x^*)),$$

then

$$\begin{aligned} \varrho(x, x^*) &\leq \varepsilon_n(x) + \bar{q} + A(\varrho(x, x^*), \varrho(x, x^*)) \\ &\leq \varepsilon_n(x) + \bar{q} + A(\varrho(x, x^*), \beta_r(\varepsilon_n(x) + \bar{q}, A, \varrho(x, x^*), \varrho(x, x^*))) \\ &= \beta_{r+1}(\varepsilon_n(x) + \bar{q}, A, \varrho(x, x^*), \varrho(x, x^*)). \end{aligned}$$

Now, if $n \rightarrow \infty$, then from (9) and Lemma 7 we have

$$\begin{aligned} \varrho(x, x^*) &\leq \beta_k(\bar{q}, A, \varrho(x, x^*), \varrho(x, x^*)), \\ \varrho(x, x^*) &\leq m(k, A, \bar{q}) \leq m(k, A, \varepsilon_0(x)), \end{aligned}$$

where

$$\begin{aligned} m(k, A, \varepsilon_0(x)) &= \beta_k(\varepsilon_0(x), A, m(k, A, \varepsilon_0(x)), m(k, A, \varepsilon_0(x))) \\ &\geq \beta_k(\bar{q}, A, m(k, A, \varepsilon_0(x)), m(k, A, \varepsilon_0(x))). \end{aligned}$$

This means that all solutions of equation (1) are in the sphere $S(x^*, m(k, A, \varepsilon_0(x)))$. But in this sphere the assumptions of Theorem 1 (with $a(u) = A(u, u)$) are fulfilled, and therefore there exists only one solution of equation (1) in the space R . The sequence $\{y_n\}$ is well-defined by relation (2) (see Theorem 2).

Putting $u_n = \varrho(y_n, \bar{x})$, $n = 0, 1, \dots$, we have

$$\begin{aligned} u_0 &= \varrho(y_0, \bar{x}) = \varrho(x^*, \bar{x}) \leq m(k, A, \varepsilon_0(\bar{x})) \stackrel{\text{df}}{=} m_0(k, A, \varepsilon_0(\bar{x})), \\ \varrho(y_n, \bar{x}) &= \varrho(f_{n,k}(y_n, y_{n-1}), \bar{x}) \leq \beta_k(\varepsilon_n(\bar{x}), A, \varrho(y_n, \bar{x}), \varrho(y_{n-1}, \bar{x})), \\ u_n &\leq \beta_k(\varepsilon_n(\bar{x}), A, u_n, u_{n-1}), \quad n = 1, 2, \dots \end{aligned}$$

From Lemma 7 we obtain

$$\varrho(y_n, \bar{x}) \leq m_n(k, A, \varepsilon_n(\bar{x})) \searrow 0,$$

and consequently $y_n \rightarrow \bar{x}$.

Remark 5. If

$$f_{n,i}(x, y) = f_i(x, y),$$

then we get Theorem 5.1 [1], and if

$$f_{n,k}(x, y) = f_n(x, y)$$

we have Theorem 8 [2].

References

- [1] Н. С. Курпель, *Проекционно-итеративные методы решения операторных уравнений*, Киев 1968.
- [2] M. Kwapisz, *On the convergence of approximate iterations for an abstract equation*, Ann. Polon. Math. 22 (1969); p. 73-87.
- [3] J. M. Ortega and W. C. Rheinboldt, *On a class of approximate iterative processes*, Arch. Rat. Mech. Anal. 23, No. 5 (1967), p. 352-365.
- [4] T. Ważewski, *Sur un procédé de prouver la convergence des approximations successives sans utilisation des séries de comparaison*, Bull. Acad. Sci., sér. sci. math. astr. et phys., 8, No. 1 (1960), p. 45-52.

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