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On the existence of invariant measures for Markov processes

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Abstract. Let $P\colon L_1(0,1)\to L_1(0,1)$ be a linear operator satisfying the following conditions: (a) P is positive ($f\geqslant 0 \Rightarrow Pf\geqslant 0$); (b) P is positively isometric ($f\geqslant 0 \Rightarrow \|Pf\| = \|f\|$); (c) Pf is decreasing for any decreasing $f\geqslant 0$; (d) there exists an $\varepsilon\in(0,1)$ such that

$$\operatorname{ess\,sup} P1_{[0,\varepsilon]} < 1, \quad \operatorname{ess\,sup} P1_{[\varepsilon,1]} < \infty,$$

where 1_A denotes the characteristic function of the set A. It is shown that conditions (a)–(d) imply the existence of a positive decreasing function f invariant under P ($Pf = f$, $\|f\| = 1$). From this follows immediately the existence of finite invariant measures for a certain class of point transformations. Among them the r-adic transformations are included.

The purpose of the present note is to establish a sufficient condition for the existence of finite invariant measures for a certain class of Markov processes. We shall restrict ourselves to the processes defined over the unit interval $[0,1]$ and we shall assume that any such process is given by an isometric operator.

Section 1 contains basic notations and the statement of the main theorem. Some applications to point transformations are given in Section 2. Among them a simple proof of the existence of finite invariant measures for r-adic transformations is included (cf. [8], [2], [7]).

1. Let $(L_1, \|\cdot\|)$ be the space of all integrable real-valued functions defined on the unit interval $[0,1]$. An element $f\in L_1$ is called *decreasing* if there exists a decreasing (= non-increasing) function $h\in f$. The characteristic function of a measurable set A is denoted by 1_A.

A linear operator $P\colon L_1\to L_1$ is said to be a *Markov process* if it is positive and positively isometric; that is, P satisfies the following two conditions:

(a) $Pf\geqslant 0$ for $f\geqslant 0$, $f\in L_1$,

(b) $\|Pf\| = \|f\|$ for $f\geqslant 0$, $f\in L_1$.

Given P, we define the transition probability $\Pi(x, A) = P^*1_A(x)$, where P^* denotes the operator adjoint to P. A function f satisfying $f = Pf$

is called *invariant* under P. When f is invariant, then the measure $d\mu = f dx$ satisfies

(1)
$$\mu(A) = \int_0^1 \Pi(x, A)\mu(dx),$$

for each measurable A; such a measure is also called *invariant.*

THEOREM 1. *Suppose that a Markov process P satisfies the additional two conditions:*

(c) *Pf is decreasing for any decreasing $f \geqslant 0, f \in L_1$,*

(d) *there exist an $\varepsilon > 0$ and a $\lambda < 1$ such that*

$$P1_{[0,\varepsilon]} \leqslant \lambda, \quad \operatorname{ess\,sup} P1_{[\varepsilon,1]} < \infty.$$

Then there exists a decreasing invariant function $f \in L_1$ satisfying

(2)
$$0 \leqslant f \leqslant \frac{\operatorname{ess\,sup} P1_{[\varepsilon,1]}}{\varepsilon(1-\lambda)}, \quad \|f\| = 1.$$

Proof. Let S be the set of all integrable decreasing functions which satisfy condition (2). For any $f \in S$ we have

$$1 \geqslant \int_0^x f(s)\,ds \geqslant \int_0^x f(x)\,ds = xf(x),$$

and consequently

$$f(x) \leqslant \frac{1}{x}.$$

Write $a = \operatorname{ess\,sup} P1_{[\varepsilon,1]}$. By condition (d) we obtain

$$\int_0^x Pf\,ds = \int_0^x Pf1_{[0,\varepsilon]}\,ds + \int_0^x Pf1_{[\varepsilon,1]}\,ds \leqslant \frac{\lambda a x}{\varepsilon(1-\lambda)} + \frac{1}{\varepsilon} a x = \frac{ax}{\varepsilon(1-\lambda)}.$$

Since Pf is decreasing, this implies

$$\operatorname{ess\,sup} Pf \leqslant \frac{a}{\varepsilon(1-\lambda)}.$$

From the last inequality and conditions (a), (b), (c) it follows that $P(S) \subset S$. Since S is a convex compact subset of L_1, by the Markov–Kakutani fixed point theorem there exists an invariant $f \in S$. This completes the proof.

Remark. Let $\{P_t\}_{t \in T}$ be a commutative family of Markov processes. Theorem 1 may easily be generalized in the following manner. If P_t satisfy conditions (c) and (d) with the constants ε, λ independent of t and if

$$\sup_t \operatorname{ess\,sup} P_t 1_{[\varepsilon,1]} < \infty,$$

then there exists a non-trivial ($\|f\| = 1$) positive increasing function f which is invariant with respect to every P_t:

$$f = P_t f \quad \text{for} \quad t \in T.$$

2. Denote by m the Lebesgue measure on the interval $[0, 1]$. A measurable transformation τ of $[0, 1]$ into itself is called *non-singular* if $m(A) = 0$ implies $m(\tau^{-1}(A)) = 0$. For any non-singular transformation τ we define the (Frobenius–Perron) operator

$$P_\tau f(x) = \frac{d}{dx} \int_{\tau^{-1}([0,x])} f(s)\, ds,$$

which is a Markov process. Since $P_\tau^* f(x) = f(\tau(x))$, the transition probability for P_τ is given by the formula

$$\Pi_\tau(x, A) = P_\tau^* 1_A(x) = 1_A(\tau(x)) = 1_{\tau^{-1}(A)}(x).$$

When f is invariant under P_τ, then the measure $d\mu = f dx$ satisfies the condition

$$\mu(A) = \int_0^1 \Pi_\tau(x, A)\, \mu(dx) = \int_0^1 1_{\tau^{-1}(A)}(x)\, \mu(dx) = \mu(\tau^{-1}(A)),$$

which means that μ is invariant under τ. This well-known property of the Frobenius–Perron operator enables us to use Theorem 1 in proving the existence of invariant measures for point transformations.

EXAMPLE 1. For given $r > 1$ consider the r-adic transformation

$$\tau(x) = rx \pmod 1. \tag{3}$$

A simple computation shows that the operator P_τ can be written in the form

$$P_\tau f(x) = \frac{1}{r} \sum_{k=0}^{n-1} f\left(\frac{k}{r} + \frac{x}{r}\right) + \begin{cases} \frac{1}{r} f\left(\frac{n}{r} + \frac{x}{r}\right), & 0 \leqslant x \leqslant r - n, \\ 0, & r - n < x \leqslant 1, \end{cases}$$

where n denotes the whole part of r. It is easy to see that P_τ satisfies conditions (c) and (d) with $\varepsilon = \lambda = 1/r$. This proves the existence of an absolutely continuous non-trivial invariant measure for the transformation (3).

We may generalize this result and replace (3) by a piecewise convex transformation. We say that $\varphi\colon [a, b] \to R$ is convex if it satisfies

$$\varphi(\alpha x + (1-\alpha) y) \leqslant \alpha\varphi(x) + (1-\alpha)\varphi(y)$$

for $x, y \in [a, b]$ and $0 \leqslant \alpha \leqslant 1$.

EXAMPLE 2. Let $\{[a_k, b_k]\}_{k=1}^{n,\infty}$ be an at most countable sequence of closed intervals such that

$$(4) \qquad a_1 = 0, \quad 0 \leqslant a_k < b_k \leqslant 1, \quad \sum_k (b_k - a_k) = 1.$$

We assume that, for $j \neq k$, the intersection of the corresponding open intervals $(a_j, b_j) \cap (a_k, b_k)$ is empty. Let $\varphi_k \colon [a_k, b_k] \to [0, 1]$ be a sequence of convex functions such that

$$\varphi_k(a_k) = 0, \quad \varphi_1(0) > 1, \quad \sum_k \frac{1}{\varphi_k'(a_k)} < \infty.$$

We define the function $\tau \colon [0, 1] \to [0, 1]$ by the conditions

$$(5) \qquad \tau(x) = \varphi_k(x) \quad \text{for} \quad a_k < x < b_k.$$

From (4) it follows that the function τ is defined almost everywhere on $[0, 1]$. As in the previous case a simple computation shows that the Frobenius–Perron operator corresponding to τ can be written in the form

$$P_\tau f(x) = \sum_k \psi_k'(x) f(\psi_k(x)),$$

where

$$\psi_k(x) = \begin{cases} \varphi_k^{-1}(x), & 0 \leqslant x \leqslant \varphi_k(b_k - 0), \\ b_k, & \varphi_k(b_k - 0) < x \leqslant 1. \end{cases}$$

The functions ψ_k are increasing, continuous and differentiable except on a set of an at most countable number of points. The functions ψ_k' are decreasing and $\psi_1' \leqslant 1/\varphi_k'(a_k)$. We have, moreover,

$$P_\tau 1(x) = \sum_k \psi_k'(x) \leqslant \sum_k \frac{1}{\varphi_k'(a_k)} < \infty.$$

Now, it is easy to verify that P_τ satisfies conditions (c) and (d) with $\varepsilon = b_1$ and $\lambda = 1/\varphi_1'(0)$. By Theorem 1 this implies the existence of a non-trivial absolutely continuous measure which is invariant under transformation (5).

In the case where the sequence $\{[a_k, b_k]\}$ is finite the above result was proved in [3]. Let us note that, in general, the function τ in (5) in neither an expanding map nor a local homeomorphism. Therefore, it is of interest to compare our result with the recent results of A. Avez [1], K. Krzyżewski [4], K. Krzyżewski and W. Szlenk [5] and M. Misiurewicz [6].

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