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HAMILTONIAN CIRCUITS AND PATH COVERINGS OF VERTICES IN GRAPHS

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1. Introduction. Throughout the paper *graph* will mean a finite ordinary graph (i. e., a finite, undirected graph without loops and without multiple edges). The paper presents a new approach to the problem of the existence of Hamiltonian paths and Hamiltonian circuits in graphs. An analogous problem concerning Eulerian chains and Eulerian closed chains is treated in [1] in a chapter on chain coverings of edges. The paper of Ore [20] gives rise to investigations of path coverings of vertices in ordinary graphs. Here we give a number of sufficient conditions for an ordinary graph to have a covering of vertices by a fixed number of paths (or simple chains) and a study of interdependence of those conditions. Most of known conditions for a graph to have a Hamiltonian circuit or a Hamiltonian path are particular cases of those given below. Our weakest condition corresponds to that of Las Vergnas [16].

2. Coverings of vertices in graphs. A set of subgraphs of a graph G is said to be a *covering* of G (more precisely, a *covering of vertices* of G) if the union of subgraphs in the set contains all vertices of G. For example, a colouring of G is a covering of G by totally disconnected disjoint subgraphs and corresponds to a complete graph covering of the complementary graph $\bar{G}$ of G. In what follows, each covering will contain only disjoint connected subgraphs of a special kind and will be identified with their union. Thus a covering of vertices of G will be identified with a spanning subgraph (a partial graph [1] or a factor [11]) of G with components of a special kind. For instance, a 1-factor of G (it may be identified with a perfect matching in G) is an edge-graph covering of G; a 2-factor of G is a circuit covering of G, the covering, being a (spanning) tree, is important for applications (e. g., in telecommunication).

Throughout, $\boldsymbol{R}$ and $\boldsymbol{Z}$ will denote the set of real numbers and the set of integers, respectively. We also assume (unless otherwise stated)

(2.1) $$n > s \geqslant p \geqslant 0, \quad n+s \geqslant 3, \quad k \leqslant n-1, \quad n, s, p, k \in \boldsymbol{Z}.$$

Hence $n \geqslant 2$. The symbol G_n denotes an ordinary graph G with n vertices and we write $G = G_n$ if G contains exactly n vertices.

A *path covering* of vertices of G is a covering by disjoint paths (trivial or not). A path covering with s components ($s \geqslant 1$) is called shortly an *s-covering* or, more precisely, a *path s-covering*. Thus a 1-covering of G is a Hamiltonian path of G. A Hamiltonian circuit of G is said to be a 0-covering of G. (So one can interpret s as the number of additional edges in a circuit (without new vertices) that contains the path s-covering.) Many authors have given conditions for an ordinary graph to have a 0-covering, e. g., Dirac [7], Ore [20], Pósa [22], Erdös [8], Bondy [2], Skupień-Wojda [26], Chvátal [6], Las Vergnas [15] and [16] (see also Skupień [25]). Analogous conditions for a graph to have a 1-covering have been also found. Our aim is to give general sufficient conditions for a graph G_n (with n vertices) to have a path s-covering, $s = 0, 1, \ldots, n-1$. Each of these conditions depends on the parameter $s \geqslant 0$. These conditions (or analogous ones) were considered also for $s = -q < 0$ (cf. [1]-[6], [9], [12]-[18], [21], [23], [26]) and were proved to ensure either high connectivity or the existence of Hamiltonian chains (closed or open) with extra properties.

As modifications of the conditions due to Erdös [8] and Ore [20], pertinent to the existence of Hamiltonian chains ($s = 0; 1$), we obtain two new conditions (see (5.6) and (5.8)), both ensuring the existence of path s-coverings ($0 \leqslant s \leqslant n-1$).

3. Definitions. The symbol $:=$ will denote the equality on the strength of a definition. Let V be a finite set of elements of the same type, e. g., $V \subset \boldsymbol{R}^3$. Let $|V|$ be the cardinality of V and let $\mathscr{P}_2(V)$ be the collection of all unordered pairs $\{x, y\}$, $x, y \in V$, $x \neq y$. Then $V \cap \mathscr{P}_2(V) = \emptyset$ and, for any subset E of $\mathscr{P}_2(V)$, the ordered pair $G = \langle V, E\rangle$ is said to be a *graph* with the *vertex set* $V(G) = V$ and the *edge set* $E(G) = E$. Thus a graph may be identified with a simplicial complex of dimension $\leqslant 1$.

We define two dyadic, irreflexive and symmetric relations in a graph $G = \langle V, E\rangle$, the *incidence* and the *adjacency*, in the following manner. Two elements $u_1, u_2 \in V \cup E$ are *incident* in G if $u_i \in u_j \in E$ and $\{i, j\} = \{1, 2\}$ (u_i is a vertex and u_j is an edge of G). Two elements $u_1, u_2 \in V \cup E$ are *adjacent* in G if either $u_1, u_2 \in V$ and $\{u_1, u_2\} \in E$ or $u_1, u_2 \in E$ and there is $x \in V$ such that $u_1 \cap u_2 = \{x\}$. So u_1 and u_2 are adjacent if $u_1 \neq u_2$ and there is an element in $V \cup E$ to which u_1, u_2 are both incident.

If x and y are two different vertices of a graph, then $\{x, y\}$ will be denoted by xy or yx. The number $d(x) = d(x, G)$ of vertices adjacent in G to a vertex $x \in V(G)$ is called the *degree* of x in G. The minimal degree of vertices of G is denoted by $\delta(G)$.

The graph $\langle V, \mathscr{P}_2(V)\rangle$ is *complete* and is denoted by K or by K_n if $|V| = n$. The graph $K_0 := \langle \emptyset, \emptyset\rangle$ is called *empty*. The *complementary graph* $\bar{G}$ of $G = \langle V, E\rangle$ is defined to be the graph $\langle V, \mathscr{P}_2(V)\setminus E\rangle$. The complementary graph $\bar{K}_n$ of the complete graph K_n, $n > 0$, is said to be *totally disconnected* (possibly *trivial* if $n = 1$). A graph $H = \langle V_1, E_1\rangle$ is a *subgraph* of a graph $G = \langle V, E\rangle$, in symbols $H \subseteq G$, if $V_1 \subseteq V$ and $E_1 \subseteq E$; if $V_1 = V$, then H ($H \subseteq G$) is a *partial graph* (*spanning subgraph*) of G.

A *chain* of a graph G is an alternating sequence

$$\mu = (x_0, e_1, x_1, e_2, \ldots, e_n, x_n) = [x_0 x_1 \ldots x_n] = \mu[x_0, x_n]$$

of vertices $x_0, x_1, \ldots, x_n$ and edges $e_1, e_2, \ldots, e_n$ of G such that all edges e_i are distinct and $e_i = x_{i-1}x_i$ for $i = 1, \ldots, n$. The chain μ is *simple* (and *open*) if all vertices x_i are different; it is *closed* (or it is a *cycle*) if $x_0 = x_n$, $n \geqslant 3$; and it is a *simple closed chain* (i. e., a *simple cycle*) if it is closed and all vertices $x_0, x_1, \ldots, x_{n-1}$ (deleted x_n) are different. An *open* [*closed*] *Eulerian chain* of G is an open [closed] chain passing through all edges of G. Similarly, an *open* [*closed*] *Hamiltonian chain* is a simple chain [a cycle] containing all vertices of G.

Not only chains but also graphs which correspond to some of those chains play an important part in graph theory. These graphs are called paths and circuits. So by a *path* [a *circuit*] of G we mean a subgraph consisting of all vertices and edges of a simple [simple closed] chain of G. Thus a path is a graph isomorphic to a triangulation of a closed line interval (possibly degenerate). Hence K_1 is a *trivial* (*degenerate*) *path*. A circuit is isomorphic to a triangulation of a simple closed curve. A *Hamiltonian path* and a *Hamiltonian circuit* of a graph G are subgraphs corresponding to suitable (open or closed, respectively) Hamiltonian chains of G. A graph is *Hamiltonian* if it contains a Hamiltonian circuit (or a Hamiltonian cycle).

To avoid confusion suppose that $G^{(1)} = \langle V_1, E_1\rangle$ and $G^{(2)} = \langle V_2, E_2\rangle$ are subgraphs of a certain graph. Then $G^{(1)}$ and $G^{(2)}$ are *disjoint* (or *vertex-disjoint*) if $V_1 \cap V_2 = \emptyset$. Further, $G^{(1)} \cup G^{(2)} := \langle V_1 \cup V_2, E_1 \cup E_2\rangle$ is the graph called the *union* of $G^{(1)}$ and $G^{(2)}$.

(3.1) If $G^{(1)}$ and $G^{(2)}$ are disjoint, we write $G^{(1)} * G^{(2)}$ to denote the following graph called the *join* of $G^{(1)}$ and $G^{(2)}$:

$$G^{(1)} * G^{(2)} := \langle V_1 \cup V_2, E_1 \cup E_2 \cup E_{12}\rangle,$$

where $E_{12} := \{xy : x \in V_1 \text{ and } y \in V_2\}$ (cf. multiplication in [29], operation $\times$ in [28], and operation $*$ in [27]).

In particular, $G \cup \bar{G} = K$, $G * K_0 = G$.

4. Preliminary results. The following equivalence is obvious:

(4.1) *G_n has a path s-covering if and only if $G_n * K_s$ is Hamiltonian. More generally, G_n has a path s-covering if and only if $G_n * K_p$ has a path $(s-p)$-covering.*

To formulate conditions sufficient for a graph to have a path s-covering, we need some definitions. For a number a, $[a]$ is the greatest integer which does not exceed a, and $[^*a]$ is the least integer not less than a. Hence

$$[a] \leqslant a \leqslant [^*a], \qquad [^*a] = -[-a]. \tag{4.2}$$

Set

$$\varphi_{ns}(t) := 1 + \binom{n-t-s}{2} + (t+s)t, \tag{4.3}$$

$$\varkappa = \varkappa_{ns} := \left[\frac{n-s-1}{2}\right] \quad (\text{so } \max\{1-s, 0\} \leqslant \varkappa), \tag{4.4}$$

$$\bar{k}_s := \max\{1-s, 0, k\}, \tag{4.5}$$

(4.6)
$$\Phi_{ns}(k) := \Phi_{ns}(\bar{k}_s) := \begin{cases} \max\{\varphi_{ns}(t) \mid \bar{k}_s \leqslant t \leqslant \varkappa\} & \text{if } \bar{k}_s \leqslant \varkappa; \\ 0 & \text{if } \bar{k}_s > \varkappa,\ \bar{k}_s = k \in \boldsymbol{R}. \end{cases}$$

Now we will obtain more convenient expressions for $\Phi_{ns}(k)$. It follows from (4.3) that $\varphi_{ns}(t)$ is the square trinomial of variable t,

$$\varphi_{ns}(t) = \tfrac{3}{2}t^2 - (n-2s-\tfrac{1}{2})t + \binom{n-s}{2} + 1,$$

whence, by (4.6),

$$\Phi_{ns}(k) = \max\{\varphi_{ns}(\bar{k}_s),\ \varphi_{ns}(\varkappa)\} \quad \text{if } \bar{k}_s \leqslant \varkappa,$$

and

$$\varphi_{ns}(k) - \varphi_{ns}(\varkappa) = \tfrac{3}{2}(\varkappa - k)(a-k),$$

where

$$a = a_{ns} := \tfrac{1}{3}(2n-4s-1-3\varkappa) = \begin{cases} \dfrac{n-5s+1}{6} & \text{for odd } n+s, \\[2ex] \dfrac{n-5s+4}{6} & \text{otherwise}. \end{cases} \tag{4.7}$$

Hence, by (4.4) and (2.1), we have

$$\varkappa - a = \begin{cases} \dfrac{n-s-1}{2} - \dfrac{n-5s+1}{6} = \dfrac{n+s-2}{3} \geqslant \dfrac{1}{3} & \text{for odd } n+s, \\[2ex] \dfrac{n-s-2}{2} - \dfrac{n-5s+4}{6} = \dfrac{n+s-5}{3} \geqslant -\dfrac{1}{3} & \text{otherwise}. \end{cases}$$

Thus $[{}^{*}a] \leqslant \varkappa$ if and only if $n+s \neq 4$. Further, $a \neq \varkappa$ and $[a] \leqslant \varkappa$. Moreover, examining all the cases $n-5s \equiv i \pmod 6$ ($i = 0, 1, \ldots, 5$), we obtain

$$[{}^{*}a] = \left[a+\frac{4}{6}\right],$$

$$\left[\frac{n-5s+2}{6}\right]+1 = a < \varkappa \quad \text{if } n-5s \equiv 2 \pmod 6,$$

$$\left[\frac{n-5s+2}{6}\right] = [a] \leqslant \varkappa \quad \text{otherwise}.$$

Therefore,

(4.8) $\Phi_{ns}(k)$

$$= \Phi_{ns}(\bar{k}_s) = \begin{cases} \varphi_{ns}(\bar{k}_s) & \text{if } \bar{k}_s \leqslant \dfrac{n-5s+2}{6} \text{ (or } \bar{k}_s \leqslant a_{ns}), \\ \varphi_{ns}(\varkappa) & \text{if } \dfrac{n-5s+2}{6} < \bar{k}_s < \dfrac{n-s}{2} \text{ (or } a_{ns} \leqslant \bar{k}_s \leqslant \varkappa_{ns}), \\ 0 & \text{if } k = \bar{k}_s > \varkappa,\ k \in \boldsymbol{R}, \end{cases}$$

(4.9) $\Phi_{ns}(k) = \varphi_{ns}(\varkappa)$ if and only if $\tilde{\varkappa} \leqslant \bar{k}_s = \max\{1-s, 0, k\} \leqslant \varkappa$, where

$$(4.10) \quad \tilde{\varkappa} = \tilde{\varkappa}_{ns} := \min\{[{}^{*}a], \varkappa\} = \begin{cases} \varkappa = \dfrac{n-s-2}{2} & \text{for } n+s = 4, \\ [{}^{*}a_{ns}] = \left[a_{ns}+\dfrac{4}{6}\right] & \text{for } 4 \neq n+s \geqslant 3. \end{cases}$$

Hence

$$\left[\frac{n-5s+2}{6}\right] \leqslant \tilde{\varkappa} \leqslant \varkappa.$$

For integers $k < (n-s)/2$ we define the following multifunction:

$$(4.11) \quad T_{ns}(k) := \begin{cases} \{\bar{k}_s\} & \text{if } \bar{k}_s < a, \\ \{a, \varkappa\} & \text{if } \bar{k}_s = a \in \boldsymbol{Z}, \\ \{\varkappa\} & \text{if } \bar{k}_s > a. \end{cases}$$

Numerical values of this multifunction are not singletons only if $a = a_{ns}$ is an integer. By (4.7), it is the case if and only if $n-5s \equiv i \pmod 6$, where either $i = 2$ or $i = 5$, and then $a < \varkappa$. Observe that if $k \mapsto \tau_{ns}(k)$ is a selection of the multifunction (4.11), then $\tau_{ns}(k) = \tau_{ns}(\bar{k}_s)$ (since $\overline{(\bar{k}_s)}_s = \bar{k}_s$) and, by (4.6) and (4.8), we obtain

$$(4.12) \quad \Phi_{ns}(k) = \begin{cases} \varphi_{ns}\big(\tau_{ns}(k)\big) & \text{if } k \leqslant \varkappa,\ k \in \boldsymbol{Z}, \\ 0 & \text{if } k > \varkappa,\ k \in \boldsymbol{R}; \end{cases}$$

moreover, if $Z \ni k \leqslant \varkappa$, then

(4.12′) $\quad \Phi_{ns}(k) = \varphi_{ns}(\tau)$ and $\tau \geqslant k$ if and only if $\tau \in T_{ns}(k)$.

In the case $0 \leqslant s \leqslant 1$ and $k \leqslant 1-s$, formula (4.12) can be simplified. Indeed, by (4.5), (4.4) and (4.7),

$$\bar{k}_s = \begin{cases} 1-s = \varkappa > a & \text{if } n+s = 3 \ (0 \leqslant s \leqslant 1), \\ 1-s \leqslant a & \text{otherwise}. \end{cases}$$

Therefore, by (4.11) and (4.5), one can put $\tau_{ns}(k) = 1-s$ for $0 \leqslant s \leqslant 1$ and $k \leqslant 1-s$. From (4.12) and (4.3) it follows that

$$(4.13) \qquad \Phi_{ns}(k) = \varphi_{ns}(1-s) = \binom{n-1}{2} + 2 - s \quad \text{if } 0 \leqslant s \leqslant 1,\ k \leqslant 1-s.$$

Now, put

$$(4.14) \qquad \Phi^*_{ns}(k) := \begin{cases} \max\{\varphi_{ns}(t) + k - t \mid k \leqslant t \leqslant \varkappa\} & \text{if } k \leqslant \varkappa, \\ 0 & \text{otherwise}. \end{cases}$$

Analogously as above one obtains the following formula:

$$(4.15) \qquad \Phi^*_{ns}(k) = \max\{\varphi_{ns}(k),\ \varphi_{ns}(\varkappa) + k - \varkappa\} \quad \text{if } k \leqslant \varkappa,$$

$$= \begin{cases} \varphi_{ns}(k) & \text{if } k \leqslant \dfrac{n-5s+6}{6}, \\ \varphi_{ns}(\varkappa) + k - \varkappa & \text{if } \dfrac{n-5s+6}{6} < k \leqslant \varkappa. \end{cases}$$

In connection with the last formula note that if $a := (n-5s+6)/6$, then $[a] \leqslant \varkappa$. In fact, $a < (n-s)/2$ if and only if $n+s > 3$. In accordance with (2.1) this is not the case if and only if $\langle n, s\rangle \in \{\langle 2; 1\rangle, \langle 3; 0\rangle\}$, but then $[a] = (n-s-1)/2 = \varkappa$. On the other hand, if $a < (n-s)/2$, then $[a] < (n-s)/2$, and $[a] \leqslant \varkappa$.

5. Formulation of conditions. Let n, s, k be fixed numbers satisfying (2.1) and let G_n be a graph with n vertices. Now we shall formulate certain propositional functions (conditions) $A^{(i)}_{ns}(k)$, $i = 1, 2, \ldots, 11$, each of which associates with any specified graph G_n a sentence $A^{(i)}_{ns}(k;\ G_n)$. If the parameter k is not involved, we write $A^{(i)}_{ns}$ and $A^{(i)}_{ns}(G)$ instead of $A^{(i)}_{ns}(k)$ and $A^{(i)}_{ns}(k; G)$, respectively. A simplified version of $A^{(i)}_{ns}(k)$ will be denoted by $\tilde{A}^{(i)}_{ns}(k)$; the simplification will consist in the deletion of an unessential logical factor (logical component). Two conditions are equal if they are identical.

(5.1) $\quad A^{(1)}_{ns}(k)$ (*condition of Erdös type*): Each vertex of G_n is of degree not less than k and G_n contains at least $\Phi_{ns}(k)$ edges.

Observe that k need not be an integer if $k > \varkappa$, since then, by (4.6), $\Phi_{ns}(k)$ is 0. Thus the condition $A_{ns}^{(1)}\big((n-s)/2\big)$ is easily seen to be equivalent to $A_{ns}^{(1)}\big([^*(n-s)/2]\big)$ (cf. (4.2)).

If either $k \leqslant 0$ or $\varkappa < k \leqslant n-1$, then one of the two logical factors in $A_{ns}^{(1)}(k)$ is unessential in the sense that either it is always satisfied or can be deduced from the other. Omitting such a logical factor, one obtains a simplified version $\tilde{A}_{ns}^{(1)}(k)$ of $A_{ns}^{(1)}(k)$. We quote two examples:

(5.2) $\tilde{A}_{ns}^{(1)}\big((n-s)/2\big)$ *(condition of Dirac type)*: Each vertex of G_n is of degree $\geqslant (n-s)/2$;

(5.3) $\tilde{A}_{ns}^{(1)}(0)$ *(condition of the first Ore type)*: G_n contains at least $\Phi_{ns}(0)$ edges (cf. (4.8) and (4.13)).

The following implication holds true:

(5.4) $A_{ns}^{(1)}(k;\, G_n)$ *implies that the minimal degree* $\delta(G_n)$ *of vertices of* G_n *is not less than* $\max\{2-s,\, 0\}$.

In fact, it is trivial if either $s \geqslant 2$ or $k \geqslant 2-s$. And if $0 \leqslant s \leqslant 1$, $k \leqslant 1-s$, then, by (4.13), we have

$$\Phi_{ns}(k) = \binom{n-1}{2} + 2 - s$$

which implies (5.4).

The following equivalence is easy to prove.

(5.5) *If* $t = \bar{k}_s = \max\{1-s,\, 0,\, k\}$, *then* $A_{ns}^{(1)}(k;\, G_n) \Leftrightarrow A_{ns}^{(1)}(t;\, G_n)$.

In fact, by (4.5) and (4.6), $k \leqslant \bar{k}_s$ and $\Phi_{ns}(k) = \Phi_{ns}(\bar{k}_s)$, and so only the implication $\Rightarrow$ requires a proof. This implication, however, is evident with the possible exception for $k < \bar{k}_s$, $\bar{k}_s > 0$, but then $s = 0$ and $k < k_s = 1$, and (5.4) applies. Thus (5.5) is proved.

Thus $A_{n0}^{(1)}(1)$ can be simplified, and in view of (5.3) and (4.13), $\tilde{A}_{n0}^{(1)}(1)$ is identical with $\tilde{A}_{n0}^{(1)}(0)$:

$$\tilde{A}_{n0}^{(1)}(1) = \tilde{A}_{n0}^{(1)}(0).$$

Here are the next conditions.

(5.6) $A_{ns}^{(2)}(k)$: $\delta(G_n) = k$ and G_n contains at least $\Phi_{ns}^*(k)$ edges (see (4.14)).

(5.7) $A_{ns}^{(3)}$ *(condition of the second Ore type)*: The sum of degrees of any two non-adjacent vertices of G_n is not less than $a^{(3)}$, where

$$a^{(3)} = a_{ns}^{(3)} := \begin{cases} n-s-1 & \text{if } s \geqslant 2 \text{ and } n-s \text{ is even,} \\ n-s & \text{otherwise.} \end{cases}$$

In some cases $a^{(3)}$ can be diminished by 1. Namely, consider the following condition dependent upon a number k, $0 \leqslant k \leqslant n$:

(5.8) $A_{ns}^{(4)}(k)$: G_n contains at least $[^*k]$ $(0 \leqslant k \leqslant n)$ vertices of degree $n-1$ and the sum of degrees of any two non-adjacent vertices of G_n is not less than $a^{(4)}$, where

$$a^{(4)} = a_{ns}^{(4)}(k) := \begin{cases} n-s-1 & \text{if } s+k>1 \text{ and } n-s \text{ is even,} \\ n-s & \text{otherwise.} \end{cases}$$

Sometimes $A_{ns}^{(4)}(k)$ can be easily simplified. We quote two examples:

$$\tilde{A}_{ns}^{(4)}(0) := A_{ns}^{(3)},$$

$\tilde{A}_{ns}^{(4)}\left(\frac{n-s}{2}\right)$: G_n contains at least $\frac{n-s}{2}$ vertices of degree $n-1$.

(5.9) $A_{ns}^{(5)}$ (*condition of Pósa type*): For any $t \in \boldsymbol{Z}$ such that

$$\max\{1-s, 0\} \leqslant t \leqslant \varkappa = \left[\frac{n-s-1}{2}\right],$$

the number of vertices of G_n of degree at most t is less than $a^{(5)}(t)$, where

$$a^{(5)}(t) = a_{ns}^{(5)}(t) := \begin{cases} t+1+s = \dfrac{n+s+1}{2} & \text{if } t = \dfrac{n-s-1}{2} \in \boldsymbol{Z}, \\ t+s & \text{if } \max\{1-s, 0\} \leqslant t < \dfrac{n-s-1}{2}. \end{cases}$$

The paper of Bondy [2] suggests certain reformulations of the condition of Pósa type. Consider the following scheme of conditions dependent upon propositional functions $\omega_{ns}^{(r)}$ $(r = 6, 7, 8, 9, 10)$:

(5.10) $\Omega_n(\omega_{ns}^{(r)})$: For any arrangement of vertices x_i of G_n, if $d(x_1, G_n) \leqslant d(x_2, G_n) \leqslant \ldots \leqslant d(x_n, G_n)$, then $\omega_{ns}^{(r)}$.

Specifying $\omega_{ns}^{(r)}$ one can obtain reformulations $A_{ns}^{(r)}$ $(r = 6, 7)$ of the condition of Pósa type (5.9). To do this, consider the following conditions:

(5.11$_6$) $$d(x_i, G_n) > i-s \quad \text{if } \max\{1, s\} \leqslant i < \frac{n+s-1}{2},$$

(5.11$_7$) $$d(x_i, G_n) \leqslant i-s \Rightarrow d(x_i, G_n) \geqslant \frac{n-s-1}{2} \quad \text{if } i \in \{1, \ldots, n\},$$

(5.12) $$d(x_l, G_n) \geqslant l-s \left(= \frac{n-s+1}{2}\right) \quad \text{if } l := \frac{n+s+1}{2} \in \boldsymbol{Z}$$

($n+s$ must be odd).

If

(5.13) $\omega_{ns}^{(r)}$ is the conjunction of (5.11$_r$) and (5.12) for $r = 6, 7$,

then define

(5.14) $$A_{ns}^{(r)} := \Omega_n(\omega_{ns}^{(r)}), \quad r = 6, 7.$$

Similarly, one can formulate the next three conditions.

(5.15) $A_{ns}^{(8)}$ *(condition of Bondy type)*, $A_{ns}^{(9)}$ *(condition of Skupień-Wojda type)* and $A_{ns}^{(10)}$ *(condition of Chvátal type)*:

$$A_{ns}^{(r)} := \Omega_n(\omega_{ns}^{(r)}), \quad r = 8, 9, 10,$$

where

$\omega_{ns}^{(8)}$: if $1 \leqslant i < j \leqslant n$, $d(x_i, G_n) \leqslant i-s$, and $d(x_j, G_n) \leqslant j-s-1$, then $d(x_i, G_n) + d(x_j, G_n) \geqslant n-s$;

$\omega_{ns}^{(9)}$: if $1 \leqslant i < j \leqslant n$ and there exist $x, y \in V(G_n)$ such that $x \neq y$, $xy \notin E(G_n)$, $d(x, G_n) \leqslant d(x_i, G_n) \leqslant i-s$ and $d(y, G_n) \leqslant d(x_j, G_n) \leqslant j-s-1$, then $d(x_i, G_n) + d(x_j, G_n) \geqslant n-s$;

$\omega_{ns}^{(10)}$: for all i such that $\max\{1, s\} \leqslant i < (n+s)/2$ there is either $d(x_i, G_n) > i-s$ or $d(x_{n+s-i}, G_n) \geqslant n-i$.

(5.16) Remark. It is easily seen that for $r = 6, 7, 8, 9$ and 10 the condition $A_{ns}^{(r)}$ is equivalent to the following one:

$\Omega'_n(\omega_{ns}^{(r)})$: There is an arrangement $x_1, x_2, \ldots, x_n$ of vertices x of G_n such that $d(x_1, G_n) \leqslant d(x_2, G_n) \leqslant \ldots \leqslant d(x_n, G_n)$ and $\omega_{ns}^{(r)}$.

(5.17) $A_{ns}^{(11)}$ *(condition of Las Vergnas type)*: There is an arrangement $x_1, x_2, \ldots, x_n$ of vertices x_i of G_n such that, for all i, j, if $\max\{1, s\} \leqslant i < j \leqslant n$, $j \geqslant n+s-i$, $d(x_i, G_n) \leqslant i-s$, $d(x_j, G_n) \leqslant j-s-1$, and $x_i x_j \notin E(G_n)$, then $d(x_i, G_n) + d(x_j, G_n) \geqslant n-s$.

6. Interdependence of conditions. Now we shall prove some relations between the above conditions. A few implications are obvious and some others are already proved for $s = 0$ or $s = 1$. For instance, Ore [20] proved that, for $s = 0, 1$, the condition of the first Ore type (5.3) implies the condition of the second Ore type (5.7); Erdös proved [8] that his condition $A_{n0}^{(1)}(k)$ (see (5.1)) with $s = 0$ implies the condition of Pósa $A_{n0}^{(5)}$ (see (5.9)) with $s = 0$. It is known (see Kronk [14] and Sachs [24]) that for $s = 0$ the condition of the second Ore type (5.7) is stronger (essentially stronger for $n \geqslant 5$) than the condition of Pósa type (5.9). Bondy observed [2] that the condition of Pósa (5.9) with $s = 0$ is equivalent to the condition $A_{n0}^{(7)}$ (see (5.14)) with $s = 0$. He formulated a new condition as an improvement of that one. The condition $A_{n0}^{(8)}$ (see (5.15)) with $s = 0$, found by Nash-Williams and Bondy (cf. Berge [1], p. 199), is just a refinement of the original condition of Bondy [2]. The original condition of Skupień and Wojda, formulated (with errata ([1])) in [26] for $s \leqslant 0$, was proved to be equivalent for $s = 0$ to that of Bondy. Las Vergnas [16] formulated his condition $A_{ns}^{(11)}$ (see (5.17)) for $s \leqslant 0$ as a refinement for $s = 0$ of the condition of Bondy type $A_{n0}^{(8)}$ (see (5.15)) with $s = 0$. Chvátal [6] proved

([1]) The error was kindly pointed out to the authors by J. C. Bermond.

ingeniously that his condition $A^{(10)}_{n0}$ is weaker than that of Bondy type (clearly for $s = 0$ only).

LEMMA 1. *Each condition of Erdös type* (5.1) *implies a certain condition* (5.6); *namely,*

(6.1) *if* $k_1 = \delta(G)$ *and* $G = G_n$, *then* $A^{(1)}_{ns}(k; G)$ *implies* $A^{(2)}_{ns}(k_1; G)$.

Proof. Let $G = G_n$ satisfy $A^{(1)}_{ns}(k)$. If $\delta(G) = k_1 > \varkappa$, then, by (4.14), we have $\Phi^*_{ns}(k_1) = 0$. Hence G clearly satisfies $A^{(2)}_{ns}(k_1)$. And if $\delta(G) = k_1 \leqslant \varkappa$, then $k \leqslant k_1$ and, owing to (5.4), we have $\max\{2-s, 0, k\} \leqslant k_1 \leqslant \varkappa$. Therefore, by the definitions of $\Phi_{ns}(k)$ and $\Phi^*_{ns}(k)$, we obtain

$$\Phi_{ns}(k) \geqslant \Phi_{ns}(k_1) \geqslant \Phi^*_{ns}(k_1),$$

which completes the proof.

Remark. *If* $k < \max\{2-s, 0\}$, *then the condition* $A^{(2)}_{ns}(k)$ *is self--contradictory.*

In fact, there is no graph G with $\delta(G) = k < 0$, so suppose that $\delta(G_n) = k$ and $0 \leqslant k < \max\{2-s, 0\}$. Hence

$$|E(G_n)| \leqslant k + \binom{n-1}{2},$$

and two cases are possible:

I. $k = 0$ and $0 \leqslant s \leqslant 1$;

II. $k = 1$ and $s = 0$.

Owing to (4.15) and (4.3), in both cases

$$\Phi^*_{ns}(k) = \varphi_{ns}(k) = 1 + \binom{n-k-s}{2} + (k+s)k,$$

whence

$$|E(G_n)| \leqslant k + \binom{n-1}{2} < \Phi^*_{ns}(k).$$

Hence also in the case $0 \leqslant k < \max\{2-s, 0\}$ no G_n satisfies the condition $A^{(2)}_{ns}(k)$.

Now we prove the following proposition:

(6.2) *If* $G = G_n$ *and either* $k < \max\{2-s, 0, a_{ns}+1\}$ *or* $\varkappa_{ns} \leqslant k < n$, *then* $A^{(2)}_{ns}(k; G)$ *implies* $A^{(1)}_{ns}(k; G)$.

Proof. By the Remark, implication (6.2) is true if $k < \max\{2-s, 0\}$. And if either $\max\{1-s, 0\} \leqslant k \leqslant a_{ns}$ or $\varkappa_{ns} \leqslant k \leqslant n-1$, then, by (4.5) and (4.4), we have $\bar{k}_s = k$. Hence, by (4.8), (4.14) and (4.15), we obtain $\Phi_{ns}(k) = \Phi^*_{ns}(k)$. This completes the proof.

LEMMA 2. *Each condition* (5.6) *implies condition of Pósa type* (5.9), *i. e.,*

(6.3) *if* $G = G_n$, *then* $A^{(2)}_{ns}(k; G)$ *implies* $A^{(5)}_{ns}(G)$.

Proof. It suffices to consider the case $0 \leqslant k \leqslant \varkappa$. Assume that $A^{(2)}_{ns}(k; G)$ is true for a graph $G = G_n$ and that there is an integer t, $\max\{1-s, 0, k\} \leqslant t \leqslant \varkappa$, such that G contains at least $t+s$ vertices of degree $\leqslant t$. By $A^{(2)}_{ns}(k; G)$, we can assume that at least one of them is of degree k. Therefore G contains at most $(t+s)t+k-t$ edges incident to our $t+s$ vertices of degree $\leqslant t$. The number of the remaining edges of G does not exceed $\binom{n-t-s}{2}$. Thus, by (4.3) and (4.14), G contains at most $\Phi^*_{ns}(k)-1$ edges, a contradiction to $A^{(2)}_{ns}(k; G)$. Therefore G satisfies $A^{(5)}_{ns}$. (As a matter of fact, G satisfies a condition stronger than $A^{(5)}_{ns}$.)

LEMMA 3. *Condition of the second Ore type* (5.7) *is equivalent to* (5.8) *for* $k = 0$, *i. e.*,

$$\text{(6.4)} \qquad \text{if } G = G_n, \text{ then } A^{(3)}_{ns}(G) \Leftrightarrow A^{(4)}_{ns}(0; G).$$

The proof is obvious.

LEMMA 4. *Each condition* (5.8) *implies condition of Pósa type* (5.9), *i. e.*,

$$\text{(6.5)} \qquad \text{if } G = G_n, \text{ then } A^{(4)}_{ns}(k; G) \text{ implies } A^{(5)}_{ns}(G).$$

Proof. Assume that $A^{(4)}_{ns}(k; G)$ is true for a certain $G = G_n$ and that there is an integer t, $\max\{1-s, 0\} \leqslant t \leqslant \varkappa = [(n-s-1)/2]$, such that G contains at least $t+s$ vertices of degree $\leqslant t$. Let them be the vertices $x_1, x_2, \ldots, x_{t+s}$. Since $2t < a^{(4)}_{ns}(k)$, all these vertices are mutually adjacent (if $t+s \geqslant 2$), and each of them is adjacent to at least $[^*k]$ other vertices. Hence each of these vertices is of degree $\geqslant t+s-1+k$ (also in the case $t+s < 2$) and, by the assumption, of degree $\leqslant t$, i. e.,

$$\text{(6.6)} \qquad t+s-1+k \leqslant d(x_i, G) \leqslant t, \quad i = 1, \ldots, t+s.$$

This is a contradiction if $s+k > 1$. Now, it suffices to consider two cases: either $s = 0$ and $0 \leqslant k \leqslant 1$ or $s = 1$ and $k = 0$.

Let the remaining vertices of G be x_i, $i = t+s+1, \ldots, n$. If $s = 0$ and $0 \leqslant k \leqslant 1$, then, by (6.6), they contain at most t vertices (1 vertex if $k = 1$) each of which is adjacent to a vertex from $x_1, x_2, \ldots, x_{t+s} = x_t$. Since $2t < n$, the vertices $x_1, x_2, \ldots, x_t$, together with those adjacent to them, do not exhaust all the vertices of G. Therefore there is a vertex of G, say x_r, different from and non-adjacent to $x_1, x_2, \ldots, x_t$. Then

$$d(x_1, G) + d(x_r, G) \leqslant t + (n-t-1) < n,$$

contrary to (5.8) with $s = 0$ and $0 \leqslant k \leqslant 1$. And if $s = 1$ and $k = 0$, then, by (6.6), $d(x_i, G) = t$ for $i = 1, 2, \ldots, t+1$ and $x_i x_j \notin E(G)$ for $i \leqslant t+1 < j$. Thus

$$d(x_1, G) + d(x_n, G) \leqslant t + (n-t-2) < n-1,$$

a contradiction to (5.8) for $s = 1$ and $k = 0$. This completes the proof of (6.5).

Now consider the condition

(6.7) $V(G_n) = \{x_1, x_2, \ldots, x_n\}$ and $d(x_1, G_n) \leqslant d(x_2, G_n) \leqslant \ldots \leqslant d(x_n, G_n)$.

In what follows we assume (without any loss of generality) that

(6.8) if a graph $G = G_n$ satisfies one of the conditions $A_{ns}^{(r)}$ $(r = 5, 6, \ldots, 10)$, then (6.7) and $\omega_{ns}^{(r)}$ hold true.

Thus in order to prove the implication $A_{ns}^{(r)} \Rightarrow A_{ns}^{(\varrho)}$ for $5 \leqslant r, \varrho \leqslant 10$ and $\varrho \geqslant 6$, it suffices to show that any graph G_n satisfying $A_{ns}^{(r)}$ satisfies also $\omega_{ns}^{(\varrho)}$ (cf. (5.16)).

LEMMA 5. *Condition of Pósa type* (5.9) *and conditions* (5.14) *are equivalent, that is,*

(6.9) $$\text{if } G = G_n, \text{ then } A_{ns}^{(5)}(G) \Leftrightarrow A_{ns}^{(6)}(G) \Leftrightarrow A_{ns}^{(7)}(G).$$

Proof consists of three parts.

1. Let $G = G_n$ satisfy $A_{ns}^{(5)}$. Then from (6.8), (6.7) and (5.9) it follows the conjunction

$$d(x_{t+s}, G) > t \quad \text{if } \max\{1-s, 0\} \leqslant t < \frac{n-s-1}{2},$$

$$d(x_{t+1+s}, G) \geqslant t+1 \quad \text{if } t = t_0 : = \frac{n-s-1}{2} \in \boldsymbol{Z}.$$

Hence, putting $t = i-s$ and $t_0+1+s = l$, we obtain $\omega_{ns}^{(6)}$ (see (5.13)). Therefore the graph G_n satisfies $A_{ns}^{(6)}$. So $A_{ns}^{(5)} \Rightarrow A_{ns}^{(6)}$.

2. Let $G = G_n$ satisfy $A_{ns}^{(6)}$. Then G satisfies (6.7) and $\omega_{ns}^{(6)}$. Hence, owing to definition (5.13) of $\omega_{ns}^{(6)}$, G satisfies conditions (5.11_6) and (5.12). To prove that G satisfies $\omega_{ns}^{(7)}$ (see (5.13)), put

$$i_0 : = \left[{}^*\frac{n+s-1}{2}\right].$$

Hence, by (2.1), $i_0 \geqslant 1$. If $i_0 = 1$, then obviously either $s = 0$ and $n = 3$ or $s = 1$ and $n = 2$, and then (5.11_6) is unessential. Therefore, by (5.13), we have

$$\omega_{ns}^{(6)} \Leftrightarrow (5.12) \Leftrightarrow [d(x_2, G_n) \geqslant 2-s].$$

Hence, by virtue of (6.7), we obtain (5.11_7) and $\omega_{ns}^{(7)}$. Analogously, if $s = n-1$, then (5.11_6) and (5.11_7) are both true. So $\omega_{ns}^{(7)}$ holds true. And if $i_0 \geqslant 2$ and $s < n-1$, then

$$\max\{1, s\} \leqslant i_0-1 < \frac{n+s-1}{2}.$$

Hence, by (5.11_6), we have

$$d(x_{i_0-1}, G) \geqslant i_0 - s \geqslant \frac{n-s-1}{2}.$$

Therefore, by condition (6.7), if $i \geqslant i_0$ ($\geqslant (n+s-1)/2$), then $d(x_i, G) \geqslant (n-s-1)/2$. Thus, clearly, $(5.11_6) \Rightarrow (5.11_7)$. So G satisfies $\omega_{ns}^{(7)}$.

Hence the implication $A_{ns}^{(6)} \Rightarrow A_{ns}^{(7)}$ holds true in all cases.

3. Let $G = G_n$ satisfy $A_{ns}^{(7)}$. Therefore, by (6.8) and definition (5.13) of $\omega_{ns}^{(7)}$, G satisfies (5.11_7), (5.12) and (6.7). Now we easily deduce that G satisfies $A_{ns}^{(5)}$. So $A_{ns}^{(7)} \Rightarrow A_{ns}^{(5)}$.

LEMMA 6. *Condition of Pósa type* (5.9) *is stronger than condition of Bondy type* (5.15), *i. e., by* (6.9),

(6.10) *if* $G = G_n$, *then* $A_{ns}^{(7)}(G)$ *implies* $A_{ns}^{(8)}(G)$.

Proof. Suppose that a certain $G = G_n$ satisfies $A_{ns}^{(7)}$ (see (5.14)). Therefore G satisfies (6.7) and $\omega_{ns}^{(7)}$. Hence, by (5.13), G satisfies conditions (5.11_7) and (5.12). Now assume that integers i, j satisfy the conditions

$$1 \leqslant i < j \leqslant n, \quad d(x_i, G) \leqslant i-s, \quad d(x_j, G) \leqslant j-s-1.$$

Hence, by (5.11_7), we have

$$i-s \geqslant d(x_i, G) \geqslant \frac{n-s-1}{2}.$$

So $i \geqslant (n+s-1)/2$, and therefore

$$j \geqslant i+1 \geqslant \frac{n+s+1}{2}.$$

Therefore, by (6.7) and (5.12), we have $d(x_j, G) \geqslant (n-s+1)/2$ if $n-s$ is odd, otherwise

$$d(x_j, G) \geqslant d(x_i, G) \geqslant \frac{n-s}{2}.$$

Thus $d(x_i, G) + d(x_j, G) \geqslant n-s$. So $G = G_n$ satisfies $\omega_{ns}^{(8)}$ (see (5.15)) and implication (6.10) is true.

LEMMA 7. *Condition* $A_{ns}^{(9)}$ *is equivalent to condition of Bondy type, i. e.,*

(6.11) *if* $G = G_n$, *then* $A_{ns}^{(9)}(G) \Leftrightarrow A_{ns}^{(8)}(G)$.

Proof. By (5.15), it clearly suffices to show that $A_{ns}^{(9)}$ implies $A_{ns}^{(8)}$. To construct an indirect proof of this implication, suppose that a graph $G = G_n$ satisfies $A_{ns}^{(9)}$ but does not satisfy $A_{ns}^{(8)}$. Hence, by (6.8), the graph G satisfies (6.7) and $\omega_{ns}^{(9)}$, but $\omega_{ns}^{(8)}$ (see (5.15)) is not satisfied. So there are integers i, j such that

$$1 \leqslant i < j \leqslant n, \quad d(x_i, G) \leqslant i-s,$$
$$d(x_j, G) \leqslant j-s-1 \quad \text{and} \quad d(x_i, G) + d(x_j, G) < n-s.$$

Hence G is not complete. Suppose that $x \in V(G)$ and $d(x, G) \leqslant d(x_i, G)$. Then, by $\omega_{ns}^{(9)}$ (see (5.15)), x is adjacent to any other vertex belonging to W, where

$$W = \{y \in V(G)\colon\ d(y, G) \leqslant d(x_j, G)\}.$$

Moreover, $x \in W$ and, by (6.7), $W \supseteq \{x_1, x_2, \ldots, x_j\}$. Therefore

$$i-s \geqslant d(x_i, G) \geqslant d(x, G) \geqslant |W|-1 \geqslant j-1 \geqslant i.$$

It is a contradiction if $s > 0$.

Now let $s = 0$. Then $|W| = j = i+1$, and the subgraph G_W of G induced by W is clearly complete. However, since G is not complete, we have $|W| = i+1 < n$, and the vertices $x_1, x_2, \ldots, x_i$ are all of degree i in G_W and in G. Therefore $x_i x_n \notin E(G)$ and $d(x_n, G) \leqslant n-1-i < n-s-1$ (for $s = 0$). However,

$$d(x_i, G)+d(x_n, G) \leqslant (i-s)+(n-1-i) < n-s,$$

contrary to $\omega_{ns}^{(9)}$. This completes the proof.

LEMMA 8. *Condition of Bondy type* (5.15) *implies condition of Chvátal type* (5.15), *i. e.*,

(6.12) *if* $G = G_n$, *then* $A_{ns}^{(8)}(G)$ *implies* $A_{ns}^{(10)}(G)$.

Proof. Suppose the contrary. Then a certain graph $G = G_n$ satisfies $A_{ns}^{(8)}$ (i. e., (6.7) and $\omega_{ns}^{(8)}$), but, despite of $\omega_{ns}^{(10)}$ (see (5.15)), there is an integer i, $\max\{1, s\} \leqslant i < (n+s)/2$, such that $d(x_i, G) \leqslant i-s$ and $d(x_j, G) \leqslant j-s-1$, where $j = n+s-i$. Therefore $1 \leqslant i < j \leqslant n$. Moreover,

$$d(x_i, G)+d(x_j, G) \leqslant (i-s)+(n-i-1) < n-s,$$

contrary to $\omega_{ns}^{(8)}$ (see (5.15)). The proof is complete.

LEMMA 9. *Condition of Chvátal type* (5.15) *implies condition of Las Vergnas type* (5.17), *i. e.*,

(6.13) *if* $G = G_n$, *then* $A_{ns}^{(10)}(G)$ *implies* $A_{ns}^{(11)}(G)$.

Proof. Assume that G $(= G_n)$ satisfies $A_{ns}^{(10)}$ (see (5.15)), but does not satisfy condition $A_{ns}^{(11)}$ of Las Vergnas type. Hence, by (6.8), the arrangement of vertices of G is defined by (6.7) and G satisfies $\omega_{ns}^{(10)}$. Moreover, in accordance with the negation of (5.17), there are integers i, j such that $\max\{1, s\} \leqslant i < j \leqslant n$, $j \geqslant n+s-i$, $d(x_i, G) \leqslant i-s$, $d(x_j, G) \leqslant j-s-1$, $x_i x_j \notin E(G)$, and $d(x_i, G)+d(x_j, G) < n-s$. Observe that, by $\omega_{ns}^{(10)}$, if $i_0 := \max\{1, s\}$, then

(6.14) $$d(x_{i_0}, G) > 0.$$

In order to simplify the notation, put $d(x_i, G) = h-s$. Hence, by the assumption, $h \leqslant i$ and $i \geqslant i_0$, whence, owing to (6.7) and (6.14), we have $h-s > 0$. So $h \geqslant s+1 \geqslant \max\{1, s\}$. On the other hand,

$$h = d(x_i, G)+s \leqslant \frac{1}{2}\big(d(x_i, G)+d(x_j, G)\big)+s < \frac{1}{2}(n-s)+s = \frac{n+s}{2}.$$

Moreover,

$$d(x_h, G) \leqslant d(x_i, G) \quad (= h-s),$$

since $h \leqslant i$. So, by $\omega_{ns}^{(10)}$ and the properties of i and j,

$$d(x_{n+s-h}, G) \geqslant n-h = n-s-d(x_i, G) > d(x_j, G).$$

Therefore $\max\{n+s-i, i+1\} \leqslant j < n+s-h$. Hence if $j = n+s-g$, then

$$\max\{1, s\} \leqslant h < g \leqslant \min\{i, n+s-i-1\} < \frac{n+s}{2}.$$

Now we obtain

$$d(x_g, G) \leqslant d(x_i, G) = h-s < g-s$$

and

$$d(x_{n+s-g}, G) = d(x_j, G) \leqslant j-s-1 = n-g-1,$$

contrary to $\omega_{ns}^{(10)}$. This completes the proof.

Lemmas 1, 2, ..., 9 imply the following corollary:

COROLLARY. *Condition* (5.17) *of Las Vergnas type is the weakest among all conditions* $A_{ns}^{(r)}(k)$, *i. e.*,

(6.15) *if* $G = G_n$, *then* $A_{ns}^{(r)}(k; G)$ *implies* $A_{ns}^{(11)}(G)$ *for* $r = 1, 2, \ldots, 10$, *where* n, s, k *satisfy* (2.1) *except for* $r = 4$ *when* $k \in [0, n]$.

Note that the implication converse to (6.15) can be true for certain values of r, n, s, k (in particular, for small $n-s$).

7. Relations between conditions of Section 5 and operation $*$. Recall (2.1) and note that the existence of a path s-covering of G_n is equivalent to the existence of a path $(s-p)$-covering of $G_n * K_p$ (cf. (4.1)). So it seems interesting to investigate relations between conditions $A_{ns}^{(r)}(k)$ and the operation $*$. We will investigate relations between conditions $A_{ns}^{(r)}(k; G)$ and $A_{n+p,s-p}^{(r)}(k+p; G*K_p)$ for $G = G_n$.

In the proofs of the first two lemmas below we will make use of the following observations which easily follow from definitions (4.3), (4.4), (4.7), and (4.10) of $\varphi_{ns}(t)$, $\varkappa_{ns}$, a_{ns}, and $\tilde{\varkappa}_{ns}$, respectively. Namely, one has

$$(7.1) \qquad \begin{cases} \varphi_{n+p,s-p}(t+p) = \varphi_{ns}(t) + np + \binom{p}{2}, \\ \varkappa_{n+p,s-p} = \varkappa_{ns} + p, \\ a_{n+p,s-p} = a_{ns} + p, \\ \tilde{\varkappa}_{n+p,s-p} = \tilde{\varkappa}_{ns} + p. \end{cases}$$

In what follows we will also use the following easy remarks about the graph $G_n * K_p$:

$$(7.2)\qquad \begin{cases} d(x, G_n * K_p) \geqslant p & \text{for all } x \in V(G_n * K_p), \\ d(x, G_n * K_p) = \begin{cases} p + d(x, G_n) & \text{if } x \in V(G_n), \\ p + n - 1 & \text{if } x \in V(K_p), \end{cases} \\ |E(G_n * K_p)| = |E(G_n)| + np + \binom{p}{2}. \end{cases}$$

LEMMA 10_1. *Conditions of Erdös type* (5.1) *have the properties*

$$(7.3)\qquad \begin{aligned} A^{(1)}_{ns}(k;\, G_n) &\Leftrightarrow A^{(1)}_{ns}(\bar{k}_s;\, G_n) \\ &\Leftrightarrow A^{(1)}_{n+p,s-p}(\bar{k}_s + p;\, G_n * K_p) \\ &\Leftarrow A^{(1)}_{n+p,s-p}\big(\overline{(k+p)}_{s-p};\, G_n * K_p\big) \Leftrightarrow A^{(1)}_{n+p,s-p}(k+p;\, G_n * K_p) \end{aligned}$$

and the converse of the one-way implication in (7.3) *need not hold only if*

$$(7.4)\qquad \Phi_{n+p,s-p}\big(\overline{(k+p)}_{s-p}\big) > \Phi_{n+p,s-p}(\bar{k}_s + p),$$

which is equivalent (*under assumption* (2.1)) *to the condition*

$$(7.4')\qquad 1-s < 0 < p, \quad k < 0, \quad n+s \geqslant 6\max\{1, s-p, s+k\} - j$$

with $j = 0$ *for odd* $n+s$, *and* $j = 3$ *otherwise.*

The second equivalence in (7.3) *is equivalent to*

$$(7.5)\qquad A^{(1)}_{ns}(k_1;\, G_n) \Leftrightarrow A^{(1)}_{n+p,s-p}(k_1 + p;\, G_n * K_p)$$

for each k_1 *satisfying the condition*

$$(7.6)\qquad \max\{1-s, 0\} \leqslant k_1 \leqslant n-1 \quad \textit{and} \quad k_1 \in \mathbf{Z} \textit{ unless } k_1 > \varkappa_{ns}.$$

Proof. The first and the last equivalences in (7.3) are true by (5.5). The one-way implication in (7.3) also holds true. In fact, by (4.5),

$$(7.7)\qquad \begin{aligned} \overline{(k+p)}_{s-p} &= \max\{1-s+p, 0, k+p\} \\ &\leqslant \max\{1-s+p, p, k+p\} = \bar{k}_s + p, \end{aligned}$$

and

$$(7.8)\qquad \overline{(k+p)}_{s-p} < \bar{k}_s + p$$

if and only if

$$(7.8')\qquad 1-s < 0 < p, \quad k < 0,$$

whence $\bar{k}_s + p = p$.

Hence, by (7.2), for each $x \in V(G_n * K_p)$,

$$d(x, G_n * K_p) \geqslant \overline{(k+p)}_{s-p} \Rightarrow d(x, G_n * K_p) \geqslant \bar{k}_s + p.$$

By (4.6), the function $\Phi_{n+p,s-p}(\cdot)$ is decreasing. Hence, by (7.7),

$$|E(G_n * K_p)| \geqslant \Phi_{n+p,s-p}\big(\overline{(k+p)}_{s-p}\big) \Rightarrow |E(G_n * K_p)| \geqslant \Phi_{n+p,s-p}(\bar{k}_s + p).$$

Now, the law of multiplying implications side by side, applied to the last two implications above, yields the one-way implication stated in (7.3) (cf. formulation (5.1) of condition $A_{ns}^{(1)}(k)$ of Erdös type). Further, owing to (7.7), it is easily seen that the converse of that implication need not hold only if (7.4) holds true. It is the case only if (7.8) or (7.8′) hold true, and then, by (7.7), (4.4), and (7.1), we have

$$\max\{1-s+p,\, 0\} \leqslant \overline{(k+p)}_{s-p} < \bar{k}_s + p = p \leqslant \varkappa_{ns} + p = \varkappa_{n+p,s-p}.$$

Therefore, by definition (4.6) of $\Phi_{ns}(k)$, and by (4.8) and (4.9), inequality (7.4) is equivalent to the conjunction of (7.8) and (7.9), where

$$\overline{(k+p)}_{s-p} < \tilde{\varkappa}_{n+p,s-p}. \tag{7.9}$$

For $s > 1$ (ensured by (7.8) or (7.8′)), owing to (4.10) and (2.1), inequality (7.9) is equivalent to the condition

$$\overline{(k+p)}_{s-p} < [{}^{*}a_{n+p,s-p}] = \left[a_{n+p,s-p} + \frac{4}{6}\right],$$

i.e., $\overline{(k+p)}_{s-p} < a_{n+p,s-p}$, since $\overline{(k+p)}_{s-p} \in \boldsymbol{Z}$. Hence, by (7.7) and by definition (4.7) of a_{ns}, for $s > 1$ inequality (7.9) is equivalent to the condition

$$n > \max\{6-s,\, 5s-6p,\, 5s+6k\} - j_1 \tag{7.9′}$$

with $j_1 = 1$ for odd $n+s$, and $j_1 = 4$ otherwise.

Thus

$$\begin{aligned}(7.4) &\Leftrightarrow (7.8) \text{ and } (7.9)\\ &\Leftrightarrow (7.8') \text{ and } (7.9')\\ &\Leftrightarrow (7.4').\end{aligned}$$

Now it remains to prove (7.5) for each k_1 satisfying (7.6). In fact, if k satisfies (2.1), then, by (4.5), $k_1 := \bar{k}_s$ satisfies (7.6), and conversely, for any k_1 satisfying (7.6), we have $\overline{(k_1)}_s = k_1$. Therefore the implication (7.6) $\Rightarrow$ (7.5) is equivalent to the second equivalence stated in (7.3) (and yet remaining to be proved).

Assume that $k = k_1$ satisfies (7.6). Hence, by (7.2) and (5.1), we infer that the first factors of the conditions standing in equivalence (7.5) are mutually equivalent. Now, to prove (7.5), it clearly suffices to show the following equivalence concerning the number of edges in G_n and $G_n * K_p$:

$$|E(G_n)| \geqslant \Phi_{ns}(k) \Leftrightarrow |E(G_n * K_p)| \geqslant \Phi_{n+p,s-p}(k+p). \tag{7.10}$$

This equivalence is obvious if $k > \varkappa = \varkappa_{ns}$. In fact, in that case $k+p > \varkappa_{n+p,s-p}$ (see (7.1)) and therefore, by (4.6), we have

$$\Phi_{ns}(k) = 0 = \Phi_{n+p,s-p}(k+p).$$

Assume now that

$$\max\{1-s, 0\} \leqslant k \leqslant \varkappa_{ns}, \quad k \in \boldsymbol{Z}. \tag{7.11}$$

By (7.2), for the proof of (7.10) it suffices to show the following equality:

$$\Phi_{n+p,s-p}(k+p) = \Phi_{ns}(k) + np + \binom{p}{2} \tag{7.12}$$

$$(\text{for } \max\{1-s, 0\} \leqslant k \leqslant \varkappa_{ns}).$$

Let τ belong to the set $T_{ns}(k) \subset \{k_s, a_{ns}, \varkappa_{ns}\}$ defined in (4.11). Then, by (4.12′) and (7.1), the right-hand side of (7.12) equals

$$\varphi_{ns}(\tau) + np + \binom{p}{2} = \varphi_{n+p,s-p}(\tau+p).$$

However, it can easily be proved that

$$\tau \in T_{ns}(k) \Leftrightarrow \tau + p \in T_{n+p,s-p}(k+p)$$

for k satisfying (7.11). Hence, by (4.12′) we have (7.12). Thus the proof of Lemma 10_1 is complete.

Analogously, using (4.14) and (4.15), we can prove that

$$\Phi^*_{n+p,s-p}(k+p) = \begin{cases} \Phi^*_{ns}(k) = 0 & \text{if } k > \varkappa_{ns}, \\ \Phi^*_{ns}(k) + np + \binom{p}{2} & \text{if } k \leqslant \varkappa_{ns}. \end{cases}$$

Hence, by (7.2), we have the following lemma concerning $A^{(2)}_{ns}(k)$ (see (5.6)):

LEMMA 10_2. *Condition* (5.6) *is compatible with the operation* $*$ *on graphs, that is,*

$$A^{(2)}_{ns}(k;\ G_n) \Leftrightarrow A^{(2)}_{n+p,s-p}(k+p;\ G_n * K_p). \tag{7.13}$$

(Note that both sides of equivalence (7.13) are self-contradictory if either $k \notin \boldsymbol{Z}$ or $k < \max\{2-s, 0\}$ (cf. the Remark following Lemma 1)).

Lemma 10_3. *Condition* (5.7) *of the second Ore type has the property*

$$A^{(3)}_{n+s,s-p}(G_n * K_p) \text{ implies } A^{(3)}_{ns}(G_n), \tag{7.14}$$

and the converse implication need not hold only if

$$n-s \text{ is even, } s \geqslant 2, \text{ and } s-1 \leqslant p \leqslant s. \tag{7.15}$$

Proof. The proof is based on the following property of $a^{(3)}_{ns}$ (see (5.7)):

$$a^{(3)}_{n+p,s-p} \geqslant a^{(3)}_{ns} + 2p,$$

where the equality does not hold if and only if (7.15) is true. Since, moreover, any two vertices non-adjacent in $G_n * K_p$ belong to $V(G_n)$, Lemma 10_3 is clearly true.

LEMMA 10_r ($r \geqslant 4$). *Condition* $A_{ns}^{(r)}$ (*with* $r \geqslant 4$) *is compatible with the operation* $*$ *on graphs, that is,*

$$(7.16_r) \qquad A_{ns}^{(r)}(k; G_n) \Leftrightarrow A_{n+p,s-p}^{(r)}(k+p; G_n * K_p), \qquad r = 4, 5, \ldots, 11.$$

So modification (5.8) of the condition of the second Ore type, condition of Pósa type (5.9) and its reformulations (5.14), as well as conditions $A_{ns}^{(r)}$ of all remaining types, $8 \leqslant r \leqslant 11$, i. e., of Bondy, Skupień-Wojda, Chvátal, and Las Vergnas types (see (5.15) and (5.17)), are compatible with the operation of join $*$.

Easy proofs of (7.16_r) for $r = 4, 5, \ldots, 10$ are left to the reader; we will only prove equivalence (7.16_{11}). Of importance in the sequel is only the implication

$$(7.17) \qquad A_{ns}^{(11)}(G_n) \Rightarrow A_{n+p,s-p}^{(11)}(G_n * K_p)$$

concerning condition (5.17) of Las Vergnas type.

Proof of (7.16_{11}). There is nothing to prove if either $s = 0$ or $p = 0$. Assume therefore that $s \geqslant 1$ and $p \geqslant 1$ ($s \geqslant p$). To prove (7.17), suppose that G_n satisfies (5.17). Let $y_1, \ldots, y_n$ be an arrangement of vertices of G_n ensured by (5.17). Let $K_p \cap G_n = K_0$ and $V(K_p) = \{y_{n+1}, \ldots, y_{n+p}\}$. We will prove that $y_1, \ldots, y_n, \ldots, y_{n+p}$ is an arrangement which meets the requirements of $A_{n+p,s-p}^{(11)}(G_n * K_p)$. So let i, j be any integers such that

$$(7.18) \qquad \begin{gathered} \max\{1, s-p\} \leqslant i < j \leqslant n+p, \quad j \geqslant n+s-i, \\ d(y_i, G_n * K_p) \leqslant i-s+p, \quad d(y_j, G_n * K_p) \leqslant j-s-1+p, \\ y_i y_j \notin E(G_n * K_p). \end{gathered}$$

Thus $y_i, y_j \in V(G_n)$. So $n \geqslant j \geqslant n+s-i$, and $i \geqslant s$. Therefore, in view of (7.18) and (7.2),

$$\begin{gathered} \max\{1, s\} \leqslant i < j \leqslant n, \quad j \geqslant n+s-i, \\ d(y_i, G_n) \leqslant i-s, \quad d(y_j, G_n) \leqslant j-s-1, \quad \text{and} \quad y_i y_j \notin E(G_n). \end{gathered}$$

Hence, by $A_{ns}^{(11)}$, it follows that $d(y_i, G_n) + d(y_j, G_n) \geqslant n-s$, whence

$$d(y_i, G_n * K_p) + d(y_j, G_n * K_p) \geqslant (n+p) - (s-p).$$

This completes the proof of implication (7.17)

To prove the converse implication, let $y_1, \ldots, y_{n+p}$ be an arrangement of vertices of $G_n * K_p$ satisfying the requirements of $A_{n+p,s-p}^{(11)}(G_n * K_p)$. So for all i, j condition (7.18) implies

$$d(y_i, G_n * K_p) + d(y_j, G_n * K_p) \geqslant n-s+2p.$$

Assume that

$$(7.19)\quad V(G_n) = \{y_{\nu_1}, \ldots, y_{\nu_n}\}, \quad 1 \leqslant \nu_1 < \ldots < \nu_n \leqslant n+p, \text{ and } x_i = y_{\nu_i} \text{ for } i = 1, \ldots, n.$$

Hence

$$(7.20)\qquad i \leqslant \nu_i \quad \text{for } i = 1, 2, \ldots, n.$$

Suppose that i, j are any integers satisfying the condition

$$\max\{1, s\} \leqslant i < j \leqslant n, \quad j \geqslant n+s-i,$$

$$d(x_i, G_n) \leqslant i-s, \quad d(x_j, G_n) \leqslant j-s-1, \quad \text{and} \quad x_i x \notin E(G_n).$$

It suffices to prove that

$$(7.21)\qquad d(x_i, G_n) + d(x_j, G_n) \geqslant n-s.$$

Observe that, by the above supposition and in view of (7.19), (7.20) and (7.2), we have

$$\max\{1, s-p\} \leqslant i \leqslant \nu_i < \nu_j \leqslant n+p, \quad \nu_j \geqslant j \geqslant n+s-i \geqslant n+s-\nu_i,$$

$$d(y_{\nu_i}, G_n * K_p) = d(x_i, G_n) + p \leqslant \nu_i - s + p,$$

$$d(y_{\nu_j}, G_n * K_p) = d(x_j, G_n) + p \leqslant \nu_j - s - 1 + p, \quad \text{and} \quad y_{\nu_i} y_{\nu_j} \notin E(G_n * K_p).$$

So condition (7.18) is satisfied with i, j replaced by ν_i and ν_j, respectively. Therefore

$$d(y_{\nu_i}, G_n * K_p) + d(y_{\nu_j}, G_n * K_p) \geqslant n-s+2p,$$

whence, owing to (7.19) and (7.2), we have (7.21). This completes the proof.

8. Main result and concluding remarks. Now we are able to prove the main result.

THEOREM. *Each of the conditions $A_{ns}^{(r)}(k)$, $1 \leqslant r \leqslant 11$, is sufficient for a graph $G = G_n$ to have a path s-covering of vertices.*

In the proof we refer to the following theorem:

THEOREM (Las Vergnas [16]). $A_{n0}^{(11)}$ *(see* (5.17) *with $s = 0$ and $r = 11$) is a sufficient condition for a graph G_n to be Hamiltonian.*

No other quotations are needed. In fact, implications (6.15) and (7.17) with $p = s$ yield

$$(8.1)\qquad A_{ns}^{(r)}(k; G_n) \Rightarrow A_{n+s,0}^{(11)}(G_n * K_s) \quad \text{for } r = 1, 2, \ldots, 11.$$

Now the theorem of Las Vergnas ensures the existence of a Hamiltonian circuit in $G_n * K_s$. So G_n clearly contains a path s-covering (cf. (4.1)).

For $r = 1, 3, 5, 8, 9, 10$ the theorems analogous to that of Las Vergnas are also known. Except for $r = 3$, those theorems, Lemmas 10_r, and equivalence (4.1) clearly ensure the sufficiency of the separate conditions $A_{ns}^{(r)}(k)$, $r = 1, 5, 8, 9, 10$. In fact, in any such case, owing to Lemmas 10_r, we have

(8.2) $A_{ns}^{(r)}(k; G_n) \Rightarrow A_{n+s,0}^{(r)}(k+s; G_n * K_s), \qquad r \neq 3,\ k = \bar{k}_s$ for $r = 1$.

It is the case also for the condition of the second Ore type, i. e., for $r = 3$, if either $n-s$ is odd or $s < 2$ (see Lemma 10_3).

It seems worthy to notice, in connection with Section 7 of this paper, that Ghouila-Houri [10] (see also [19]) appears to be the first who made use of the operation of join $*$ in order to obtain and to prove a sufficient condition for the existence of a Hamiltonian path. His results [10] pertain to the existence of unidirected paths and unidirected circuits in digraphs.

Investigations on path coverings of vertices in ordinary graphs were originated by Ore [20]. His results imply the following sufficient condition for a graph $G = G_n$ to have a path s-covering:

$$\forall x, y \in V(G_n):\ xy \notin E(G_n) \Rightarrow d(x, G_n) + d(y, G_n) \geqslant n - s.$$

It is easily seen that condition $A_{ns}^{(3)}$ (see (5.7)) is a slight refinement of that one.

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