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A NOTE ON INHOMOGENEOUS CONTINUOUS-STATE BRANCHING PROCESSES

In this note we prove a continuous version of Theorem 2.1 in [1], namely we state the necessary and sufficient condition for the limit of the martingale X_t/M_t $(t \to \infty)$ to be non-degenerate at zero, where $\{X_t, t \geqslant 0\}$ is a continuous-state branching process, possibly inhomogeneous, and $M_t = \mathrm{E}[X_t \mid X_0 = 1]$.

Continuous-state branching processes were introduced by Jiřina [2]. In this note we consider the case $E = R_+$, where E denotes the state space and $R_+ = [0, \infty)$. Jiřina's definition is more general than ours, namely $E = R_+^n$, $n \geqslant 1$.

Let $E = R_+$ and let $\mathscr{E} = \mathscr{F}(E)$ be the σ-field of Borel subsets of E. Let us consider the following class of functions $P_{s,t}(x, A)$:

(i) $P_{s,t}(x, A)$ is defined for $s, t \in R_+$, $s \leqslant t$, $x \in E$, $A \in \mathscr{E}$.

(ii) For fixed $s, t \in R_+$, $s \leqslant t$, and $A \in \mathscr{E}$, $P_{s,t}(x, A)$ is a measurable function in $x \in E$.

(iii) For fixed $s, t \in R_+$, $s \leqslant t$, and $x \in E$, $P_{s,t}(x, \cdot)$ is a probability measure on $\mathscr{E}$.

(iv) For each $s \in R_+$, $x \in E$ and $A \in \mathscr{E}$, $P_{s,s}(x, A) = I_A(x)$, where $I_A(x)$ is the indicator function of A, i.e., $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$.

(v) For each $s, u, t \in R_+$, $s \leqslant u \leqslant t$, $x \in E$, $A \in \mathscr{E}$, the Chapman–Kolmogorov equation is satisfied:

$$\int_0^\infty P_{s,u}(x, dy) P_{u,t}(y, A) = P_{s,t}(x, A).$$

(vi) For each $s, t \in R_+$, $s \leqslant t$, $x, y \in E$, $P_{s,t}$ satisfies

$$P_{s,t}(x+y, \cdot) = P_{s,t}(x, \cdot) * P_{s,t}(y, \cdot),$$

where $*$ denotes convolution.

A Markov process $\{X_t, t \geqslant 0\}$ with transition probabilities $P_{s,t}$ satisfying conditions (i)–(vi) is called a *continuous-state branching process*.

Denote the Laplace transform of X_t conditioned on $X_s = x$ ($x > 0$, $t \geqslant s$) by

$$\Psi_{s,t}(x, u) = \mathrm{E}[\exp(-uX_t) \mid X_s = x] = \int_0^\infty e^{-uy} P_{s,t}(x, dy).$$

Then (vi) implies the functional equation

$$\Psi_{s,t}(x+y, u) = \Psi_{s,t}(x, u)\Psi_{s,t}(y, u). \tag{1}$$

Arguing in the usual way we can write $\Psi_{s,t}(x, u)$ in the form

$$\Psi_{s,t}(x, u) = \exp(-x\psi_{s,t}(u)). \tag{2}$$

Therefore, it is easy to translate the Chapman–Kolmogorov equation in (v) into the equivalent condition

$$\psi_{s,u}(\psi_{u,t}(z)) = \psi_{s,t}(z) \quad \text{for } 0 \leqslant s \leqslant u \leqslant t \text{ and } z \geqslant 0 \tag{3}$$

which is analogous to the functional iteration property of the generating functions of a Galton–Watson process. Let $M_{s,t} = \psi'_{s,t}(0+)$ be finite, where $'$ denotes differentiation with respect to u. From [3], Lemma 2.2.6, it follows that the function $\psi_{s,t}(u)/u$ is decreasing in $u \in (0, \infty)$. Hence the function

$$H_{s,t}(u) = M_{s,t} - \frac{\psi_{s,t}(u)}{u} \quad \text{for } 0 \leqslant s \leqslant t \text{ and } u > 0$$

satisfies

$$0 \leqslant H_{s,t}(u) \leqslant M_{s,t}, \quad H_{s,t}(u)\downarrow 0 \text{ as } u \downarrow 0. \tag{4}$$

Under the condition $\psi_{s,t}(0) = 0$ we get, by (2), $\mathrm{E}[X_t \mid X_s = x] = xM_{s,t}$ and, by (3), $M_{s,t} = M_{s,u}M_{u,t}$ for $0 \leqslant s \leqslant u \leqslant t$. To avoid trivialities we exclude the degenerate case, i.e., we assume that $M_{s,t} > 0$ for every $s, t \in R_+$, $0 \leqslant s < t$, which is equivalent to the condition $P_{s,t}(x, \{0\}) < 1$ for every $s, t \in R_+$, $0 \leqslant s < t$, and for every $x > 0$.

Put $M_t = M_{0,t}$. It is easy to verify that $W_t = X_t/M_t$ is a non-negative martingale, and hence converges to a random variable W. Clearly, $0 \leqslant \mathrm{E}(W) \leqslant x$. We state the necessary and sufficient condition for $\mathrm{E}(W) > 0$.

THEOREM. *Let $\{X_t, t \geqslant 0\}$ be a continuous-state branching process, possibly inhomogeneous, and let $\delta > 0$. Then $\mathrm{E}(W) > 0$ if and only if for some $\varepsilon > 0$*

$$\sum_{\nu=1}^{\infty} M^{-1}_{(\nu-1)\delta,\nu\delta} H_{(\nu-1)\delta,\nu\delta}(\varepsilon/M_{\nu\delta}) < \infty. \tag{5}$$

Proof. Let us denote by $\tilde{\varphi}_t(u)$ and $\varphi(u)$ the Laplace transforms of W_t and W, respectively. Then

$$\tilde{\varphi}_t(u) = \mathrm{E}[\exp(-uW_t)] = \Psi_{0,t}(x, u/M_t) = \exp(-x\psi_{0,t}(u/M_t)).$$

Since $W_t \to W$, the convergence

$$\lim_{t\to\infty} \tilde{\varphi}_t(u) = \lim_{t\to\infty} \exp(-x\psi_{0,t}(u/M_t)) = \varphi(u)$$

implies

$$\lim_{t\to\infty} \psi_{0,t}(u/M_t) = B(u),$$

and hence

$$\varphi(u) = \mathrm{E}[e^{-uW}] = e^{-xB(u)}. \tag{6}$$

Now we prove the sufficiency of the Theorem. It is readily seen that there is no loss of generality in taking $\delta = 1$, which we do henceforth. To see that the random variable W is not degenerate at zero it suffices to prove that

$$\lim_{n\to\infty} \tilde{\varphi}_n(u) = \varphi(u) < 1 \quad \text{for some } u > 0.$$

Because of (6) it suffices to show that $B(u) > 0$ for some $u > 0$. Define

$$\varphi_{v,n}(u) = \psi_{v,n}(u/M_n),\ v = 0, 1, \ldots, n, \quad \psi_n(u) = \psi_{0,n}(u),$$
$$\varphi_n(u) = \varphi_{0,n}(u).$$

It follows from the definition of $H_{s,t}$ and from (3) that

$$\varphi_n(u) = M_1\, \varphi_{1,n}(u)\left(1 - \frac{H_{0,1}(\varphi_{1,n}(u))}{M_1}\right).$$

Iterating we obtain

$$\varphi_n(u) = M_{n-1}\, \varphi_{n-1,n}(u) \prod_{v=1}^{n-1} \left[1 - \frac{H_{v-1,v}(\varphi_{v,n}(u))}{M_{v-1,v}}\right].$$

Now (4) implies $\psi_{s,t}(u) \leqslant uM_{s,t}$ for $0 \leqslant s \leqslant t$ and $u \geqslant 0$. Consequently, $\varphi_{v,n}(u) \leqslant u/M_v$. Since the function $H_{s,t}$ is non-decreasing, it is clear that

$$H_{v-1,v}(\varphi_{v,n}(u)) \leqslant H_{v-1,v}(u/M_v),$$

which together with (5) implies

$$\lim_{n\to\infty} \prod_{v=1}^{n-1} \left[1 - \frac{H_{v-1,v}(\varphi_{v,n}(u))}{M_{v-1,v}}\right] > 0. \tag{7}$$

To prove that

$$\lim_{n\to\infty} \varphi_n(u) = B(u) > 0 \quad \text{for some } u > 0$$

it therefore suffices to show that the sequence $M_{n-1}\varphi_{n-1,n}(u)$ approaches a finite positive limit. But

$$M_{n-1}\varphi_{n-1,n}(u) = u\left[1-\frac{H_{n-1,n}(u/M_n)}{M_{n-1,n}}\right],$$

so that

$$\lim_{n\to\infty} M_{n-1}\varphi_{n-1,n}(u) = u$$

and the sufficiency of (5) is established.

We now prove the necessity of (5). Suppose that $\mathbf{E}(W) > 0$. Then there exist β and γ in (0, 1) such that

$$\frac{1-\varphi(u)}{u} \geqslant 2\beta \quad \text{for } 0 < u \leqslant \gamma.$$

Let

$$\lambda_n(u) = \frac{1-\tilde{\varphi}_n(u)}{u}.$$

Since $\tilde{\varphi}_n(u) \to \varphi(u)$ for all $u > 0$, we see that there exists an N_0 such that $\lambda_n(u) > \beta$ for $n \geqslant N_0$. But, for any n, $\lambda_n(u)$ is a decreasing function of $u > 0$. Thus we get $\lambda_n(u) > \beta$ for $0 < u \leqslant \gamma$, $n \geqslant N_0$. The obvious inequality

$$1-\exp\big(-x\psi_\nu(\varphi_{\nu,n}(u))\big) \leqslant x\psi_\nu(\varphi_{\nu,n}(u))$$

now yields

$$\beta < \frac{1-\tilde{\varphi}_n(u)}{u} \leqslant \frac{x\psi_\nu(\varphi_{\nu,n}(u))}{u}.$$

Formula (4) implies

$$\frac{x\psi_\nu(\varphi_{\nu,n}(u))}{u} \leqslant \frac{xM_\nu\varphi_{\nu,n}(u)}{u}.$$

Since $\varphi_{\nu,n}(u) \geqslant \beta u/xM_\nu$ and the function $H_{s,t}$ is non-decreasing, we have

$$H_{\nu-1,\nu}(\varphi_{\nu,n}(u)) \geqslant H_{\nu-1,\nu}\left(\frac{\beta u}{xM_\nu}\right). \tag{8}$$

Now the inequality

$$\varphi_n(u) \leqslant u\prod_{\nu=1}^{n-1}\left[1-\frac{H_{\nu-1,\nu}(\varphi_{\nu,n}(u))}{M_{\nu-1,\nu}}\right],$$

yields (7). Inequalities (7) and (8) imply (5), which completes the proof of the Theorem.

References

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