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CURVATURES OF SURFACES ASSOCIATED WITH HOLOMORPHIC FUNCTIONS

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1. Introduction. The object of this note(1) is to study various curvatures associated with holomorphic and meromorphic functions. They are interesting from several points of view. First, there is a single Gaussian curvature of the two surfaces associated with any meromorphic function, which is a real analytic function in the plane without singularities. Second, the curvatures satisfy certain extremum properties, e.g. the Gaussian curvature can only attain an interior maximum in a domain when it is zero. Finally, some theorems of differential geometry can be formulated as theorems involving arbitrary holomorphic functions.

2. Curvature formulas. We consider a holomorphic function $f(z) = u(x, y) + iv(x, y)$ and the two harmonic surfaces defined by the functions u and v in E^3. Denote by k_u the curvature of the plane curves u = constant, by K_u and M_u the Gaussian and mean curvatures of the u-surface, by k_{gu} the geodesic curvature of some specified curve on the u-surface. Corresponding notations k_v, K_v, M_v, k_{gv} are used for the function v.

We first obtain formulas for these curvatures in terms of the function f. The procedure can be illustrated with k_u, starting with the well-known formula

$$k_u = \frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{(u_x^2 + u_y^2)^{3/2}}.$$

Using the fact that u and v are conjugate harmonic functions, we find

$$k_u = -\frac{u_{xx}(u_x^2 - v_x^2) + 2u_x v_x v_{xx}}{(u_x^2 + v_x^2)^{3/2}}.$$

Then, since $df/dz = f' = u_x + iv_x$,

$$k_u = -\frac{\mathrm{Re}[(u_{xx} + iv_{xx})(u_x^2 - v_x^2 - 2iu_x v_x)]}{|f'|^3} = \frac{-\mathrm{Re}[f''\bar{f}'^2]}{|f'|^3}.$$

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By these methods we obtain the formulas given in

LEMMA 1. *If* $N = (1+|f'|^2)^{1/2}$, *then*

$$-k_u + ik_v = |f'|f''/f'^2, \tag{1}$$

$$-M_u + iM_v = |f'|^4 f''/N^3 f'^2, \tag{2}$$

$$K_u = K_v = -|f''|^2/N^4, \tag{3}$$

$$\begin{cases} k_{gu} = \sigma_s[-v_\sigma u_{\sigma\sigma}\sigma_s^2 + k_0]/N, \\ k_{gv} = \sigma_t[u_\sigma v_{\sigma\sigma}\sigma_t^2 + k_0]/N. \end{cases} \tag{4}$$

In (4) we suppose that $x = x(\sigma)$, $y = y(\sigma)$ describes a curve C in the z-plane with arc length parameter σ and curvature k_0. The curves in the u and v-surfaces whose projection is C have arc length parameters s and t respectively.

The mean curvatures and the curvatures of the level curves of u and v are related in a simple way. If γ is the angle between the normal to a harmonic surface and the vertical, then $\sin\gamma = |f'|/N$, $\cos\gamma = 1/N$, and

$$-M_u + iM_v = \sin^3\gamma(-k_u + ik_v).$$

Furthermore, the Gaussian curvatures of the conjugate harmonic surfaces are equal. We will write $K_u = K_v = K_f$, and regard K_f as the Gaussian curvature of f.

3. The Gaussian curvature. The real-valued Gaussian curvature function $K_f(z) = -|f''|^2/N^4$ is the most interesting of the curvatures. We first note that for any holomorphic function f there is another holomorphic function with exactly the same Gaussian curvature.

LEMMA 2. *Any two holomorphic functions f, g such that $g' = 1/f'$ have the same curvature $K_g(z) = K_f(z)$.*

Proof. This is easy to verify by substituting g into the curvature formula.

Some interesting pairs of functions with the same Gaussian curvature are $(z^2/2, \log z)$, $(z^n/n, z^{2-n}/(2-n))$, (e^z, e^{-z}). Since for the function $z^{2-n}/(2-n)$, $n > 2$,

$$K(z) = -(n-1)^2 r^{2n-4}/(1+r^{2n-2})^2,$$

where $r = |z|$, it is evident that if we define the curvature to be zero at poles of meromorphic functions, then the Gaussian curvature of a meromorphic function is continuous in the whole plane. In fact, we have the following

THEOREM 1. *If f is a meromorphic function, then its Gaussian curvature is a real-analytic function in the whole plane. The zeros of K_f occur* *at the zeros of f'' and (by definition) at the poles of f. The poles are characterized by the behavior of K_f; if $z = 0$ is a pole of order n, then*

$$K_f(z) \sim (\mathrm{const})r^{2n} \qquad (z \to 0).$$

Proof. To prove the first statement we use the representation

$$K_f = -(u_{xx}^2 + v_{xx}^2)/(1+u_x^2+v_x^2)^2.$$

It is clear, since u and v are analytic except at poles of f, that K_f is analytic except at poles of f. At a pole f has the form $g(z)(z-z_0)^{-n}$. By Lemma 2, $K_f = K_h$, where

$$h = \int \left[(z-z_0)^{n+1}/\big((z-z_0)g'-ng\big)\right] dz,$$

and h is analytic at z_0. Since by definition $K_f(z_0) = 0 = K_h(z_0)$ and K_h is analytic at z_0, K_f is also analytic at z_0.

To prove the second and third statements assume that f has a pole at zero, i.e. $f(z) = g(z)z^{-n}$, where g is analytic at zero and $g(0) \neq 0$. Then

$$K_f = \frac{-|g''z^{-n}-2ng'z^{-n-1}+n(n+1)gz^{-n-2}|^2}{(1+|g'z^{-n}-ngz^{-n-1}|^2)^2}.$$

Since $n > 0$, the coefficients of g are the dominating terms near zero in this formula, and we find

$$K_f \sim -(n+1)^2 r^{2n}/n^2|g(o)|^2 \qquad (z \to 0).$$

It is interesting that in spite of the great variation in the harmonic surfaces of z^{-n} near zero, K depends only upon r and tends smoothly to zero. Among the powers of z, only for $z^{\pm a}$ $(0 < a < 1)$ does the curvature become infinite at zero; for $\log z$ it is again analytic and attains the minimum -1 at zero.

The Gaussian curvature is also closely related to a condition for a family of holomorphic functions to be normal.

THEOREM 2. *If $F = \{f_n\}$ is a family of holomorphic functions in a domain D, then $F' = \{f_n'\}$ is a normal family if and only if, in each compact set C in D, K_f has a uniform bound for all $f \in F$, all $z \in C$.*

Proof. This follows immediately from the fact [1] that a necessary and sufficient condition for F' to be normal is that $|f''|/(1+|f'|^2)$ be uniformly bounded on compact subsets.

The Gaussian curvature also can be given a geometric interpretation in forms of the chordal distance in stereographic projection. If $d(z, z_0)$ is the straight line distance between the stereographically projected images of z and z_0 on the unit sphere, then

$$d(z, z_0) = \frac{2|z-z_0|}{[(1+|z|^2)(1+|z_0|^2)]^{1/2}}.$$

If we replace z and z_0 by $f'(z)$ and $f'(z+h)$, then

$$\lim_{h\to 0} \frac{d\big(f'(z), f'(z+h)\big)}{h} = 2\big(-K_f(z)\big)^{1/2}.$$

Thus K_f is essentially the square of the change of scale in the mapping from the z-plane to the Riemann sphere of f'.

4. Extremum properties of the curvatures. In [3] it was shown that if u is harmonic in some simply connected region D, and if grad $u \neq 0$, then $|k_u|$ satisfies the minimum property, i.e., it attains its minimum on ∂D and it cannot have a local minimum in the interior unless $k_u = 0$. We will show that K_f, M_u and M_v have similar properties.

THEOREM 3. *If f is holomorphic in a domain D, then the curvatures $-K_f$, $|M_u|$, $|M_v|$ cannot have any local minimum in D except zero. $(-K_f)$ attains its minimum where $f'' = 0$; $|M_u|$ and $|M_v|$ attain their minima where $f' = 0$ or where* $\mathrm{Re}(f''/f'^2) = 0$, $\mathrm{Im}(f''/f'^2) = 0$ *respectively.*

Proof. Considering the Gaussian curvature first, we see from (3) that $K_f \leqslant 0$, and

$$\log(-K_f) = 2\log|f''| - 2\log(1+|f'|^2).$$

We compute the Laplacian of $\log(-K_f)$, first observing that $\log|f''|$ is harmonic. Then if $F = |f'|$, we find

$$\Delta \log(-K_f) = \frac{-4\,[F(1+F^2)\,\Delta F + (1-F^2)\,|\mathrm{grad}\,F|^2]}{(1+F^2)^2}.$$

Now if g is any holomorphic function

$$\Delta|g| = \big|\mathrm{grad}\,|g|\big|^2/|g|,$$

and upon applying this formula to the calculation one obtains

$$\Delta \log(-K_f) = \frac{-8\,|\mathrm{grad}\,F|^2}{(1+F^2)^2} \leqslant 0.$$

Thus $\log(-K_f)$ is superharmonic except when $K_f = 0$. Hence K_f cannot have a local maximum except when $K_f = 0$.

Turning to the mean curvature we find from Lemma 1 that

$$\log|M_u| = \log|k_u| + 3\log F - \frac{3}{2}\log(1+F^2).$$

To obtain its Laplacian we observe that the last term has been computed above, the middle term is harmonic and the first term is given in [3]. We have

$$\Delta \log|M_u| = -\frac{|\mathrm{grad}\, U|^2}{U^2} - \frac{6|\mathrm{grad} F|^2}{(1+F^2)^2} \leqslant 0,$$

where $U = \mathrm{Re}(f''/f'^2)$ is harmonic. Then $\log|M_u|$ is superharmonic except where $M_u = 0$.

If we restrict f so that $f' \neq 0$, then $|M_u| = 0$ only when $U = 0$. Since U is harmonic, its level curves extend to the boundary of D. Therefore, if $f' \neq 0$, $|M_u|$ attains its minimum on the boundary.

The function e^z has local maxima for all three curvatures $|M_u|$, $|M_v|$, and $-K$.

5. Integral formulas. The most interesting integral formula of differential geometry which can be applied here is the Gauss-Bonnet theorem

$$\int_D K_f \, dA_u = 2\pi - \int_{\partial D} k_{gu}\, ds,$$

where D is a domain bounded by a smooth simple closed curve. The integrals are with respect to the area and the boundary length of the u-surface, and k_{gu} is the geodesic curvature of the boundary curve, which is imbedded in the u-surface.

If dA and $d\sigma$ represent the differentials respectively of plane area in the domain D and of arc lenght of ∂D it is easy to show that

$$(5) \qquad \begin{cases} dA_u = dA_v = N dA, \\ ds = (1+u_\sigma^2)^{1/2} d\sigma, \\ dt = (1+v\sigma^2)^{1/2} d\sigma. \end{cases}$$

For the u and v-surfaces the formula then takes the form, using (4),

$$\int_D |f''|^2/N^3 dA = -2\pi + \int_{\partial D} [-v_\sigma u_{\sigma\sigma} \sigma_s^2 + k_0]/N\, d\sigma,$$

$$\int_D |f''|^2/N^3 dA = -2\pi + \int_{\partial D} [u_\sigma v_{\sigma\sigma} \sigma_t^2 + k_0]/N\, d\sigma.$$

The two integrals on the right-hand side must be equal, and this leads to the integral formula

$$\int_{\partial D} [v_\sigma u_{\sigma\sigma} \sigma_s^2 + u_\sigma v_{\sigma\sigma} \sigma_t^2]/N\, d\sigma = 0,$$

which is again valid for any function $f = u + iv$ holomorphic in $\bar{D}$. As before, σ is arc length on ∂D while s and t are arc length parameters on the boundaries of the u and v surfaces respectively.

Beckenbach [2] has obtained an inequality which applies to non-negative functions having subharmonic logarithms. For our purposes it may be stated as follows. If p and q are two such functions, D is a closed disc in their domain of definition with boundary C, then

$$M_D(pq) \leqslant M_C(p) M_C(q),$$

where $M_D(pq)$ is the mean over D of pq, $M_C(p)$ is the mean over C of p. The functions $-1/K_f$, $1/|M_u|$, $1/|M_v|$, $1/|k_u|$, $1/|k_v|$, are all of this class Taking $q = 1$, $p = -1/K_f$ we find for example that in the unit circle

$$\iint_D \frac{dA}{K_f} \geqslant \frac{1}{2} \int_C \frac{ds}{K_f}.$$

Similar inequalities hold for the other functions listed, and for their products.

REFERENCES

[1] L. V. Ahlfors, *Complex analysis*, New York 1953.

[2] E. F. Beckenbach, *Functions having subharmonic logarithms*, Duke Mathematical Journal 8 (1941), p. 393–400.

[3] R. P. Jerrard and L. A. Rubel, *On the curvature of the level lines of a harmonic function*, Proceedings of the American Mathematical Society 14 (1963), p. 29–32.

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