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On entire functions of slow growth represented by Dirichlet series

by G. S. SRIVASTAVA * (Kanpur, India)

1. Introduction. Let

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n),$$

where $s = \sigma + it$, $\lambda_{n+1} > \lambda_n$, $\lambda_1 \geqslant 0$, $\lim_{n\to\infty} \lambda_n = \infty$ and $\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0$, represent an entire function. Set

$$M(\sigma) = \operatorname*{l.u.b.}_{-\infty<t<\infty} |f(\sigma+it)|; \qquad \mu(\sigma) = \max_{n\geqslant 1} \{|a_n| \exp(\sigma\lambda_n)\}$$

and let $N(\sigma)$ denote the rank of the maximum term $\mu(\sigma)$. The Ritt order ϱ and lower order λ are defined by

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\sigma} = \begin{matrix}\varrho\\ \lambda\end{matrix}. \tag{1.1}$$

It is known that for a function of finite order [4],

$$\log M(\sigma) \sim \log \mu(\sigma). \tag{1.2}$$

For functions of zero order, the logarithmic order $\bar{\varrho}$ and lower logarithmic order $\bar{\lambda}$, [1], are defined by

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\log \sigma} = \begin{matrix}\bar{\varrho}\\ \bar{\lambda}\end{matrix}. \tag{1.3}$$

For $1 < \bar{\varrho} < \infty$, we define type and lower type, [2], by

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(\sigma)}{\sigma^{\bar{\varrho}}} = \begin{matrix}\bar{T}\\ \bar{t}\end{matrix} \tag{1.4}$$

and call $\bar{T}$ and $\bar{t}$ *logarithmic type* and *lower logarithmic type* respectively.

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We also define the logarithmic growth numbers, [3], by

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\lambda_{N(\sigma)}}{\sigma^{(\bar\varrho-1)}} = \frac{\bar\gamma}{\bar\delta}. \tag{1.5}$$

Further, let

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\mu(\sigma)}{\sigma\lambda_{N(\sigma)}} = \frac{\bar c}{\bar d}. \tag{1.6}$$

In view of (1.2) we can write

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\mu(\sigma)}{\sigma^{\bar\varrho}} = \frac{\bar T}{\bar t}. \tag{1.7}$$

It is known that, [2],

$$\limsup_{n\to\infty} \frac{\lambda_n}{\left[\frac{1}{(\bar\varrho-1)}\log|a_n|^{-\bar\varrho/\lambda_n}\right]^{(\bar\varrho-1)}} = \bar\varrho\bar T \tag{1.8}$$

and for $\lambda_n \sim \lambda_{n+1}$

$$\liminf_{n\to\infty} \frac{\lambda_n}{\left[\frac{1}{(\bar\varrho-1)}\log|a_n|^{-\bar\varrho/\lambda_n}\right]^{(\bar\varrho-1)}} \leqslant \bar\varrho\bar t. \tag{1.9}$$

Further, if $\dfrac{\log|a_n/a_{n+1}|}{(\lambda_{n+1}-\lambda_n)}$ form a non-decreasing function of n, then

$$\liminf_{n\to\infty} \frac{\lambda_n}{\left[\frac{1}{(\bar\varrho-1)}\log|a_n|^{-\bar\varrho/\lambda_n}\right]^{(\bar\varrho-1)}} \geqslant \bar\varrho\bar t. \tag{1.10}$$

In this paper we obtain some relations between the constants defined above. We derive formulae for logarithmic type, lower logarithmic type and logarithmic numbers in terms of the ratio of consecutive coefficients $\{a_n\}$. We shall always take $1 < \bar\varrho < \infty$. Symbols like σ_0, n_0 etc. denote arbitrary, non-zero finite constants which, in general, will be different on different occasions.

2. We prove the following

THEOREM 1. *The constants* $\bar T, \bar t; \bar\gamma, \bar\delta$ *as defined in* (1.4) *and* (1.5) *satisfy the relations*:

$$\bar\delta \leqslant \bar\varrho\bar t \leqslant \bar\delta[\bar\varrho-(\bar\varrho-1)(\bar\delta/\bar\gamma)^{1/(\bar\varrho-1)}] \leqslant \bar\gamma, \tag{2.1}$$

$$\bar\delta \leqslant \bar\gamma\left[\frac{\bar\varrho\bar\gamma-\bar\gamma}{\bar\varrho\bar\gamma-\bar\delta}\right]^{(\bar\varrho-1)} \leqslant \bar\varrho\bar T \leqslant \bar\gamma. \tag{2.2}$$

Proof. We know that for functions of finite order, [4],

$$(2.3)\qquad \log\mu(\sigma) = \log\mu(\sigma_0) + \int_{\sigma_0}^{\sigma} \lambda_{N(t)}\,dt, \quad \sigma \geqslant \sigma_0 > 0.$$

Let $K \geqslant 1$. Consider

$$(2.4)\qquad \log\mu(\sigma K^{1/\bar\varrho}) = \log\mu(\sigma_0) + \int_{\sigma_0}^{\sigma K^{1/\bar\varrho}} \lambda_{N(t)}\,dt$$

$$= O(1) + \int_{\sigma_0}^{\sigma} \lambda_{N(t)}\,dt + \int_{\sigma}^{\sigma K^{1/\bar\varrho}} \lambda_{N(t)}\,dt.$$

By (1.5), we have for any $\varepsilon > 0$ and for all $\sigma > \sigma_0(\varepsilon) = \sigma_0$,

$$(2.5)\qquad (\bar\delta - \varepsilon)\sigma^{(\bar\varrho-1)} < \lambda_{N(\sigma)} < (\bar\gamma + \varepsilon)\sigma^{(\bar\varrho-1)}.$$

Since $\lambda_{N(\sigma)}$ is a non-decreasing function of σ, we have

$$\log\mu(\sigma K^{1/\bar\varrho}) < O(1) + \frac{(\bar\gamma+\varepsilon)}{\bar\varrho}\sigma^{\bar\varrho} + \lambda_{N(\sigma K^{1/\bar\varrho})}\sigma(K^{1/\bar\varrho}-1)$$

for all $\sigma > \sigma_0$. Dividing by $(\sigma K^{1/\bar\varrho})$ and taking limits as $\sigma \to \infty$, we get

$$(2.6)\qquad \bar T \leqslant \frac{\bar\gamma}{K\bar\varrho} + \frac{\bar\gamma(K^{1/\bar\varrho}-1)}{K^{1/\bar\varrho}},$$

$$(2.7)\qquad \bar t \leqslant \frac{\bar\gamma}{K\bar\varrho} + \frac{\bar\delta(K^{1/\bar\varrho}-1)}{K^{1/\bar\varrho}}.$$

Taking $K = 1$ in (2.6) and $K = (\bar\gamma/\bar\delta)^{\bar\varrho/(\bar\varrho-1)}$ in (2.7), we get

$$(2.8)\qquad \bar\varrho\bar T \leqslant \bar\gamma \quad\text{ and }\quad \bar\varrho\bar t \leqslant \bar\delta[\bar\varrho - (\bar\varrho-1)(\bar\delta/\bar\gamma)^{1/(\bar\varrho-1)}].$$

Again from (2.4) and the left-hand inequality of (2.5), we get

$$(2.9)\qquad \bar T \geqslant \frac{\bar\delta}{K\bar\varrho} + \frac{\bar\gamma(K^{1/\bar\varrho}-1)}{K},$$

$$(2.10)\qquad \bar t \geqslant \frac{\bar\delta}{K\bar\varrho} + \frac{\bar\delta(K^{1/\bar\varrho-1}-1)}{K}.$$

Taking

$$K = \left[\frac{\bar\varrho\bar\gamma - \bar\delta}{\bar\varrho\bar\gamma - \bar\gamma}\right]^{\bar\varrho}$$

in (2.9) and $K = 1$ in (2.10) we get

$$(2.11)\qquad \bar\varrho\bar T \geqslant \bar\gamma\left[\frac{\bar\varrho\bar\gamma - \bar\gamma}{\bar\varrho\bar\gamma - \bar\delta}\right]^{(\bar\varrho-1)} \quad\text{ and }\quad \bar\varrho\bar t \geqslant \bar\delta.$$

It can be very easily be seen that

$$\bar{\gamma}\left[\frac{\bar{\varrho}\,\bar{\gamma}-\bar{\gamma}}{\bar{\varrho}\,\bar{\gamma}-\delta}\right]^{(\bar{\varrho}-1)} \geqslant \delta \quad \text{and} \quad \bar{\gamma} \geqslant \delta[\bar{\varrho}-(\bar{\varrho}-1)(\delta/\bar{\gamma})^{1/(\bar{\varrho}-1)}].$$

Hence combining (2.8) and (2.11) we get (2.1) and (2.2). Thus Theorem 1 follows.

Remark. We get from (2.8) and (2.11).

$$\delta \leqslant \bar{\varrho}\bar{t} \leqslant \bar{\varrho}\bar{T} \leqslant \bar{\gamma}, \tag{2.12}$$

which is already known [3].

THEOREM 2. *For the constants* $\bar{\gamma}$, δ; $\bar{c}$, $\bar{d}$ *as defined in* (1.5) *and* (1.6), *we have*

$$\frac{\delta}{\bar{\varrho}\,\bar{\gamma}} \leqslant \bar{d} \leqslant \bar{c} \leqslant \frac{\bar{\gamma}}{\bar{\varrho}\,\delta}. \tag{2.13}$$

Proof. We infer from (2.3) and (2.5) that for $\varepsilon > 0$ and for all $\sigma > \sigma_0$,

$$\log\mu(\sigma) < O(1) + \frac{(\bar{\gamma}+\varepsilon)}{\bar{\varrho}}\,\sigma^{\bar{\varrho}},$$

or,

$$\frac{\log\mu(\sigma)}{\sigma\lambda_{N(\sigma)}} < o(1) + \frac{(\bar{\gamma}+\varepsilon)}{\bar{\varrho}}\,\frac{\sigma^{(\bar{\varrho}-1)}}{\lambda_{N(\sigma)}}.$$

Taking limits as $\sigma \to \infty$, we get

$$\limsup_{\sigma\to\infty}\frac{\log\mu(\sigma)}{\sigma\lambda_{N(\sigma)}} \leqslant \frac{\bar{\gamma}}{\bar{\varrho}}\limsup_{\sigma\to\infty}\frac{\sigma^{(\bar{\varrho}-1)}}{\lambda_{N(\sigma)}}$$

or $\bar{c} \leqslant \bar{\gamma}/\bar{\varrho}\,\delta$.

Similarly, starting from (2.3) and using the left-hand inequality in (2.5), we get $\delta/\bar{\varrho}\,\bar{\gamma} \leqslant \bar{d}$. Combining the two inequalities, we get the result.

THEOREM 3. (i) *If* $0 < \bar{T} < \infty$, *then* $\log\mu(\sigma) \sim \bar{T}\sigma^{\bar{\varrho}}$ *implies that* $\lambda_{N(\sigma)} \sim \bar{\varrho}\bar{T}\sigma^{(\bar{\varrho}-1)}$ *and conversely.*

(ii) $0 < \bar{t} \leqslant \bar{T} < \infty$ *implies* $0 < \delta \leqslant \bar{\gamma} < \infty$ *and conversely.*

Proof of (i). Let $\lambda_{N(\sigma)} \sim \bar{\varrho}\bar{T}\sigma^{(\bar{\varrho}-1)}$. Then

$$\lim_{\sigma\to\infty}\ \begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\lambda_{N(\sigma)}}{\sigma^{(\bar{\varrho}-1)}} = \bar{\varrho}\bar{T},$$

or, $\bar{\gamma} = \delta = \bar{\varrho}\bar{T}$. Hence from (2.12) we get $\bar{t} = \bar{T}$ and consequently $\log\mu(\sigma) \sim \bar{T}\sigma^{\bar{\varrho}}$. Conversely, let $\log\mu(\sigma) \sim \bar{T}\sigma^{\bar{\varrho}}$. For $k = 1+\eta$, $\eta > 0$, we have

$$\begin{aligned}\lambda_{N(\sigma k^{1/\bar{\varrho}})}\,\sigma(k^{1/\bar{\varrho}}-1) &> \int_{\sigma}^{\sigma k^{1/\bar{\varrho}}} \lambda_{N(t)}\,dt\\ &= \log\mu(\sigma k^{1/\bar{\varrho}}) - \log\mu(\sigma)\\ &\sim \bar{T}[(\sigma k^{1/\bar{\varrho}})^{\bar{\varrho}} - \sigma^{\bar{\varrho}} + o(\sigma^{\bar{\varrho}})]\end{aligned}$$

or

$$\frac{\lambda_{N(\sigma k^{1/\bar{\varrho}})}}{(\sigma k^{1/\bar{\varrho}})^{\bar{\varrho}-1}} > \frac{\bar{T}\big(k-1+o(1)\big)}{(k^{1/\bar{\varrho}}-1)\,k^{(\bar{\varrho}-1)/\bar{\varrho}}},$$

or

$$\liminf_{\sigma\to\infty} \frac{\lambda_{N(\sigma k^{1/\bar{\varrho}})}}{(\sigma k^{1/\bar{\varrho}})^{\bar{\varrho}-1}} \geqslant \frac{\bar{T}(k-1)}{(k^{1/\bar{\varrho}}-1)\,k^{(\bar{\varrho}-1)/\bar{\varrho}}}.$$

Since η may be taken arbitrarily small,

$$\frac{k-1}{[k-k^{(\bar{\varrho}-1)/\bar{\varrho}}]} \sim \bar{\varrho}.$$

Hence we get $\bar{\delta} \geqslant \bar{\varrho}\bar{T}$. Similarly, taking $k = 1-\eta$, $1 > \eta > 0$, and considering the integral $\int_{\sigma k^{1/\bar{\varrho}}}^{\sigma} \lambda_{N(t)}dt$, we get $\bar{\gamma} \leqslant \bar{\varrho}\bar{T}$. Thus combining the two inequalities we get $\bar{\gamma} = \bar{\delta} = \bar{\varrho}\bar{T}$. Hence $\lambda_{N(\sigma)} \sim \bar{\varrho}\bar{T}\sigma^{(\bar{\varrho}-1)}$ and (i) follows.

Proof of (ii). We observe that, by (2.12), $\bar{t} = 0$ implies $\bar{\delta} = 0$ and $\bar{T} = \infty$ implies $\bar{\gamma} = \infty$. Conversely, suppose $\bar{\delta} = 0$. Then $\bar{t} = 0$. Indeed, $\bar{t} > 0$ gives $\bar{\gamma} > K\bar{\varrho}\bar{t}$ from (2.7), which is a contradiction since K is arbitrary in (2.7). Similarly $\bar{\gamma} = \infty$ implies $\bar{T} = \infty$. The fact that $\bar{\gamma} = \bar{\delta}$ implies $\bar{T} = \bar{t}$ and conversely follows from (i). Hence (ii) follows and this completes the proof of Theorem 3.

3. Now we shall derive formulae for logarithmic type and lower logarithmic type in terms of the ratio of consecutive coefficients. We prove

THEOREM 4. *Let* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *be an entire function of logarithmic order* $\bar{\varrho}$ $(1 < \bar{\varrho} < \infty)$ *and of logarithmic type* $\bar{T}$, *and lower logarithmic type* $\bar{t}$. *If*

(i) $$\lambda_n \sim \lambda_{n+1}$$

and

(ii) $$\sum_{m=n_0}^{n-1} \lambda_m^k(\lambda_{m+1}-\lambda_m) \sim \frac{\lambda_n^{k+1}}{k+1} \qquad (k \geqslant 0,\, n_0 \geqslant 1),$$

we have,

(3.1)

$$\liminf_{n\to\infty} \frac{\lambda_n}{\left[\dfrac{1}{(\lambda_{n+1}-\lambda_n)}\log|a_n/a_{n+1}|\right]^{(\bar{\varrho}-1)}} \leqslant \bar{\varrho}\bar{t} \leqslant \bar{\varrho}\bar{T}$$

$$\leqslant \limsup_{n\to\infty} \frac{\lambda_n}{\left[\dfrac{1}{(\lambda_{n+1}-\lambda_n)}\log|a_n/a_{n+1}|\right]^{(\bar{\varrho}-1)}}.$$

Proof. Let

$$\lim_{n\to\infty}\frac{\sup}{\inf}\frac{\left[\dfrac{1}{(\lambda_{n+1}-\lambda_n)}\log|a_n/a_{n+1}|\right]^{(\bar{\varrho}-1)}}{\lambda_n}=\frac{\bar{a}}{\bar{\beta}}.$$

Let us take $\bar{\beta}>0$, $\bar{a}<\infty$. Then for $\varepsilon>0$ and all $n\geqslant n_0$

$$(3.2)\qquad (\bar{\beta}-\varepsilon)<\frac{\left[\dfrac{1}{(\lambda_{n+1}-\lambda_n)}\log|a_n/a_{n+1}|\right]^{(\bar{\varrho}-1)}}{\lambda_n}<(\bar{a}+\varepsilon)$$

or

$$[(\bar{\beta}-\varepsilon)\lambda_m]^{1/(\bar{\varrho}-1)}(\lambda_{m+1}-\lambda_m)<\log|a_m/a_{m+1}|$$
$$<[(\bar{a}+\varepsilon)\lambda_m]^{1/(\bar{\varrho}-1)}(\lambda_{m+1}-\lambda_m)$$

for all $m\geqslant n_0$. Writing the above inequalities for $m=n_0, n_0+1, \ldots, n-1$, and adding them, we get

$$(\bar{\beta}-\varepsilon)^{1/(\bar{\varrho}-1)}\sum_{m=n_0}^{n-1}\lambda_m^{1/(\bar{\varrho}-1)}(\lambda_{m+1}-\lambda_m)<\log|a_{n_0}/a_n|$$
$$<(\bar{a}+\varepsilon)^{1/(\bar{\varrho}-1)}\sum_{m=n_0}^{n-1}\lambda_m^{1/(\bar{\varrho}-1)}(\lambda_{m+1}-\lambda_m)$$

or, using (ii), we have

$$\frac{(\bar{\beta}-\varepsilon)^{1/(\bar{\varrho}-1)}\lambda_n^{\bar{\varrho}/(\bar{\varrho}-1)}}{\bar{\varrho}/(\bar{\varrho}-1)}<\log\left|\frac{a_{n_0}}{a_n}\right|<(\bar{a}+\varepsilon)^{1/(\bar{\varrho}-1)}\frac{\lambda_n^{\bar{\varrho}/(\bar{\varrho}-1)}}{\bar{\varrho}/(\bar{\varrho}-1)}$$

for all $n>n_0$. Hence we have, on taking limits as $n\to\infty$,

$$(3.3)\qquad \bar{\beta}\leqslant\lim_{n\to\infty}\frac{\sup}{\inf}\frac{\left[\dfrac{1}{(\bar{\varrho}-1)}\log|a_n|^{-\bar{\varrho}/\lambda_n}\right]^{(\bar{\varrho}-1)}}{\lambda_n}\leqslant\bar{a}.$$

If $\bar{\beta}=0$ or $\bar{a}=\infty$, the above inequalities are obvious. If $\bar{\beta}=\infty$, then $\bar{a}=\infty$ and (3.3) follows by taking an arbitrarily large number in place of $(\bar{\beta}-\varepsilon)$ in (3.2). Similarly for the case $\bar{a}=0$. Finally we get (3.1) on combining (1.8), (1.9) and (3.3). This proves Theorem 4.

Remark. If $f(s)=\sum_{n=1}^{\infty}\exp(sn)/\exp(n^{a/(a-1)})$, $1<a<\infty$, then it can easily be seen that $f(s)$ is an entire function of logarithmic order a and inequalities (3.1) are the best possible.

Next we prove

THEOREM 5. *Let* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *be an entire function of logarithmic order* $\bar{\varrho}$ $(1 < \bar{\varrho} < \infty)$ *and logarithmic type* $\bar{T}$. *Further, let*

(i)
$$\sum_{m=n_0}^{n-1} \lambda_m^k(\lambda_{m+1}-\lambda_m) \sim \frac{\lambda_n^{k+1}}{k+1} \qquad (k \geqslant 0,\ n_0 \geqslant 1)$$

and

(ii)
$$\frac{\log|a_n/a_{n+1}|}{(\lambda_{n+1}-\lambda_n)}$$

form a non-decreasing function of n *for* $n \geqslant n_0$. *Then,*

$$\bar{\varrho}\bar{T} \leqslant \limsup_{n\to\infty} \frac{\lambda_n}{\left[\frac{1}{(\lambda_{n+1}-\lambda_n)} \log|a_n/a_{n+1}|\right]^{(\bar{\varrho}-1)}} \leqslant \left(\frac{\bar{\varrho}}{\bar{\varrho}-1}\right)^{(\bar{\varrho}-1)} \bar{\varrho}\bar{T} < e\bar{\varrho}\bar{T}. \tag{3.4}$$

Proof. Let

$$\limsup_{n\to\infty} \frac{\lambda_n}{\left[\frac{1}{(\bar{\varrho}-1)} \log|a_n|^{-\bar{\varrho}/\lambda_n}\right]^{(\bar{\varrho}-1)}} = A,$$

$$\psi(n) = \log|a_n/a_{n+1}|/(\lambda_{n+1}-\lambda_n).$$

For given $\varepsilon > 0$, we have

$$\lambda_n^{1/(\bar{\varrho}-1)}\lambda_n < (A+\varepsilon)^{1/(\bar{\varrho}-1)}\left(\frac{\bar{\varrho}}{\bar{\varrho}-1}\right)\log|a_n|^{-1} \qquad \text{for } n > n_0(\varepsilon) = n_0$$

or

$$|a_n| < \exp\left[-\frac{(\bar{\varrho}-1)}{\bar{\varrho}}(A+\varepsilon)^{-1/(\bar{\varrho}-1)}\lambda_n^{\bar{\varrho}/(\bar{\varrho}-1)}\right] \qquad \text{for } n > n_0$$

or

$$\log|a_m| + \log\left|\frac{a_{m+1}}{a_m}\right| + \ldots + \log\left|\frac{a_n}{a_{n-1}}\right| < -\frac{(\bar{\varrho}-1)}{\bar{\varrho}}(A+\varepsilon)^{-1/(\bar{\varrho}-1)}\lambda_n^{\bar{\varrho}/(\bar{\varrho}-1)}, \qquad m > n_0.$$

Since $\psi(m)$ is non-decreasing function, we have

$$\log|a_m| - (\lambda_n-\lambda_m)\psi(n-1) < -\frac{(\bar{\varrho}-1)}{\bar{\varrho}}(A+\varepsilon)^{-1/(\bar{\varrho}-1)}\lambda_n^{\bar{\varrho}/(\bar{\varrho}-1)}$$

or

$$\frac{\lambda_n}{[\psi(n-1)]^{(\bar{\varrho}-1)}} < \left(\frac{\bar{\varrho}}{\bar{\varrho}-1}\right)^{(\bar{\varrho}-1)}(A+\varepsilon)\left(\frac{\lambda_n-\lambda_m}{\lambda_n}\right)^{(\bar{\varrho}-1)}\big(1+o(1)\big).$$

Since $\psi(n-1) \leqslant \psi(n)$, taking limits as $n \to \infty$,

$$\limsup_{n\to\infty} \frac{\lambda_n}{\left[\frac{1}{(\lambda_{n+1}-\lambda_n)} \log|a_n/a_{n+1}|\right]^{(\bar{\varrho}-1)}} \leqslant \left(\frac{\bar{\varrho}}{\bar{\varrho}-1}\right)^{(\bar{\varrho}-1)} A.$$

The result follows in view of (1.8), (3.1) and the fact that $1+1/x < e^{1/x}$ for $0 < x < \infty$. This completes the proof of Theorem 5.

THEOREM 6. *Let* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *be an entire function of logarithmic order* $\bar{\varrho}$ $(1 < \bar{\varrho} < \infty)$ *having logarithmic growth numbers* $\bar{\gamma}$ *and* $\bar{\delta}$. *Further, suppose that* (i) $\lambda_n \sim \lambda_{n+1}$, (ii) $\log|a_n/a_{n+1}|/(\lambda_{n+1}-\lambda_n)$ *form an increasing function of* n. *Then*

$$\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\lambda_n}{\left[\frac{1}{(\lambda_{n+1}-\lambda_n)} \log|a_n/a_{n+1}|\right]^{(\bar{\varrho}-1)}} = \frac{\bar{\gamma}}{\bar{\delta}}. \tag{3.5}$$

Proof. By (1.5), for any $\varepsilon > 0$ and all $\sigma > \sigma_0(\varepsilon) = \sigma_0$,

$$(\bar{\delta}-\varepsilon) < \frac{\lambda_{N(\sigma)}}{\sigma^{(\bar{\varrho}-1)}} < (\bar{\gamma}+\varepsilon). \tag{3.6}$$

Since $\dfrac{\log|a_n/a_{n+1}|}{\lambda_{n+1}-\lambda_n}$ form an increasing function of n, so the n-th term is maximum for $\mathrm{Rl}\, s = \sigma$ if and only if $\lambda_n = \lambda_{N(\sigma)}$ for

$$\frac{\log|a_{n-1}/a_n|}{\lambda_n-\lambda_{n-1}} \leqslant \sigma < \frac{\log|a_n/a_{n+1}|}{(\lambda_{n+1}-\lambda_n)}. \tag{3.7}$$

Hence (3.6) and (3.7) give

$$(\bar{\delta}-\varepsilon)\left[\frac{1}{(\lambda_n-\lambda_{n-1})} \log|a_{n-1}/a_n|\right]^{(\bar{\varrho}-1)} < \lambda_n < (\bar{\gamma}+\varepsilon)\left[\frac{1}{(\lambda_{n+1}-\lambda_n)} \log|a_n/a_{n+1}|\right]^{(\bar{\varrho}-1)} \tag{3.8}$$

for all $n > n_0$. Since $\lambda_{n-1} \sim \lambda_n$, we get, on taking limits, the inequalities

$$\bar{\delta} \leqslant \liminf_{n\to\infty} \frac{\lambda_n}{\left[\frac{1}{(\lambda_{n+1}-\lambda_n)} \log|a_n/a_{n+1}|\right]^{(\bar{\varrho}-1)}}, \tag{3.9}$$

$$\bar{\gamma} \geqslant \limsup_{n\to\infty} \frac{\lambda_n}{\left[\frac{1}{(\lambda_{n+1}-\lambda_n)} \log|a_n/a_{n+1}|\right]^{(\bar{\varrho}-1)}}, \tag{3.10}$$

which obviously hold for $\bar{\delta} = 0$ and $\bar{\gamma} = \infty$. If $\bar{\delta} = \infty$, then we get (3.9) by taking an arbitrary large number in place of $(\bar{\delta}-\varepsilon)$ in (3.6). Similarly we get (3.10) for $\bar{\gamma} = 0$.

Again, from (1.5) we have $\lambda_{N(\sigma)} < (\bar{\delta}+\varepsilon)\sigma^{(\bar{\varrho}-1)}$ for a sequence of values of $\sigma = \sigma_1, \sigma_2, \ldots,$ tending to infinity. So by (3.7), corresponding to those $\{\sigma_n\}$, we get

$$\lambda_n < (\bar{\gamma}+\varepsilon)\left[\frac{\log|a_n/a_{n+1}|}{(\lambda_{n+1}-\lambda_n)}\right]^{(\bar{\varrho}-1)}$$

or

$$\frac{\lambda_n}{\left[\frac{1}{(\lambda_{n+1}-\lambda_n)}\log|a_n/a_{n+1}|\right]^{(\bar{\varrho}-1)}} < (\bar{\delta}+\varepsilon)$$

for a sequence of values of n tending to infinity. Hence

$$\liminf_{n\to\infty}\frac{\lambda_n}{\left[\frac{1}{(\lambda_{n+1}-\lambda_n)}\log|a_n/a_{n+1}|\right]^{(\bar{\varrho}-1)}} \leqslant \bar{\delta}. \tag{3.11}$$

Similarly, we get

$$\limsup_{n\to\infty}\frac{\lambda_n}{\left[\frac{1}{(\lambda_{n+1}-\lambda_n)}\log|a_n/a_{n+1}|\right]^{(\bar{\varrho}-1)}} \geqslant \bar{\gamma}. \tag{3.12}$$

Combining inequalities (3.9) to (3.12), we get (3.5) and Theorem 6 follows.

4. Applications. We give here two results which follow as direct consequences of the theorems proved before. We say that an entire function of logarithmic order $\bar{\varrho}$ $(1 < \bar{\varrho} < \infty)$, is of perfectly logarithmic linear growth if and only if $0 < \bar{t} = \bar{T} < \infty$.

Corollary 1. *Let* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *be an entire function of logarithmic order* $\bar{\varrho}$ $(1 < \bar{\varrho} < \infty)$. *Suppose*

(i) $\lambda_n \sim \lambda_{n+1}$

and

(ii) $\dfrac{\log|a_n/a_{n+1}|}{(\lambda_{n+1}-\lambda_n)}$ *form an increasing function of* n.

Then $f(s)$ *is of perfectly logarithmic linear growth if and only if*

$$\lim_{n\to\infty}\frac{\lambda_n}{\left[\frac{1}{(\lambda_{n+1}-\lambda_n)}\log|a_n/a_{n+1}|\right]^{(\bar{\varrho}-1)}} = \bar{\varrho}\bar{T}. \tag{4.1}$$

Proof. From Theorem 3 (i) it follows that $\bar{t} = \bar{T}$ implies $\bar{\gamma} = \bar{\delta} = \bar{\varrho}\bar{T}$ and conversely. This result combined with (3.5) gives (4.1) and the result follows.

COROLLARY 2. *Let* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *be an entire function of logarithmic order* $\bar{\varrho}$ $(1 < \bar{\varrho} < \infty)$ *and suppose that*

(i) $\lambda_n \sim \lambda_{n+1}$,

(ii) $\dfrac{\log|a_n/a_{n+1}|}{(\lambda_{n+1}-\lambda_n)}$ *form an increasing function of n.*

If

$$0 < \liminf_{n\to\infty} \frac{\lambda_n}{\left[\dfrac{1}{(\bar{\varrho}-1)} \log|a_n|^{-\bar{\varrho}/\lambda_n}\right]^{(\bar{\varrho}-1)}} \leqslant \limsup_{n\to\infty} \frac{\lambda_n}{\left[\dfrac{1}{(\bar{\varrho}-1)} \log|a_n|^{-\bar{\varrho}/\lambda_n}\right]^{(\bar{\varrho}-1)}} < \infty,$$

then

$$0 < \liminf_{n\to\infty} \frac{\lambda_n}{\left[\dfrac{1}{(\lambda_{n+1}-\lambda_n)} \log|a_n/a_{n+1}|\right]^{(\bar{\varrho}-1)}} \leqslant \limsup_{n\to\infty} \frac{\lambda_n}{\left[\dfrac{1}{(\lambda_{n+1}-\lambda_n)} \log\left|\dfrac{a_n}{a_{n+1}}\right|\right]^{(\bar{\varrho}-1)}} < \infty.$$

and conversely.

Proof. The result follows in view of (3.5) and Theorem 3 (ii), namely the result that $0 < \bar{t} \leqslant T < \infty$ implies $0 < \bar{\delta} \leqslant \bar{\gamma} < \infty$ and conversely.

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DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY
Kanpur (India)

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