ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XIX, 1 (1987), pp. 117-123

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ON THE DEGREE OF ASYMMETRY IN THE TRAVELLING SALESMAN PROBLEM

The distance matrix of the travelling salesman problem is changed by suitable equivalence transformations in such a way that it becomes pseudosymmetric. This fact is exploited algorithmically in order to obtain "good" approximation tours for the asymmetric travelling salesman problem. A simple example shows that the given bounds are tight.

1. Introduction. The travelling salesman problem, a well-known combinatorial optimization problem, can be described as follows: Consider n towns. Find a closed guided tour which visits every town exactly once and the total length of which is minimum. Denote the length of a tour T by $V(T)$. Then the travelling salesman problem is uniquely defined if a distance matrix $C = (c_{ij})_{i,j=1,\ldots,n}$ is given, where the value of c_{ij} denotes the "distance" from town i to town j.

Several authors ([1], [3], [4] and [5]) solve this problem by approximation, i.e. they find a tour T_{appr} which approximates an optimum tour T_{opt} in the sense that the inequality

$$V(T_{\text{appr}}) \leqslant \lambda V(T_{\text{opt}}) \tag{1}$$

holds with $\lambda > 1$. Here, λ depend on the number of cities n and also on an algorithm.

Without any assumptions on the distance matrix C the computation of a tour T_{appr} satisfying (1) is as difficult as the computation of a tour T_{opt} (cf. [10]). If the matrix C fulfils the triangle inequality

$$c_{ij} \leqslant c_{ik} + c_{kj} \quad \text{for all } i, j, k \text{ with } i \neq j,\ i \neq k \text{ and } j \neq k,$$

then there exist values λ and corresponding polynomial-time algorithms yielding a tour T_{appr} which satisfies (1).

It is well known that asymmetric instances of the problem (i.e. those in which the distance matrix C is asymmetric) have a greater degree of difficulty with respect to the construction of approximate tours than symmetric ones. The purpose of this paper consists in somewhat reducing this difficulty.

2. An equivalence transformation. Let us define $\underline{c}_i$ and $\bar{c}_i$ as follows

$$\underline{c}_i := \sum_{\substack{j=1 \\ j \neq i}}^{n} c_{ij} \quad \text{and} \quad \bar{c}_i := \sum_{\substack{j=1 \\ j \neq i}}^{n} c_{ji}, \quad i = 1, \ldots, n.$$

Definition. A matrix C is called *pseudosymmetric*, if

$$\underline{c}_i = \bar{c}_i \quad \text{for} \quad i = 1, \ldots, n.$$

Now, the following transformation is considered:

$$c'_{ij} := c_{ij} - u_i + u_j, \quad i, j = 1, \ldots, n. \tag{2}$$

The transformation (2) does not change the length of a tour and if the matrix C fulfils the triangle inequality, then also matrix C' does. Moreover, we have

LEMMA 1. *Every quadratic matrix C can be transformed into a pseudosymmetric matrix C^* by transformations of the form* (2).

Proof. Let

$$\varphi(u) := \sum_{\substack{i,j=1 \\ i \neq j}}^{n} (c_{ij} - u_i + u_j - c_{ji} + u_j - u_i)^2.$$

If $\varphi(u^*) \leqslant \varphi(u)$ holds for all vectors u, then the matrix C^* is pseudosymmetric, where

$$c^*_{ij} := c_{ij} - u^*_i + u^*_j.$$

Now let us prove this proposition. In order to achieve a contradiction, assume that C^* is not pseudosymmetric. Without loss of generality, let $\underline{c}^*_1 \neq \bar{c}^*_1$ and define

$$\psi(u_1) := \sum_{i=2}^{n} (c^*_{i1} - c^*_{1i} + 2u_1)^2 = \sum_{i=2}^{n} (c^*_{i1} + c^*_{1i})^2 + 4u_1(\bar{c}^*_1 - \underline{c}^*_1) + 4u_1^2.$$

With $u'_1 = -\frac{1}{2}(\bar{c}^*_1 - \underline{c}^*_1)$ we obtain

$$\psi(u'_1) < \psi(0).$$

This is a contradiction with the optimality of u^*.

The function φ is convex and quadratic, thus u^* is a solution vector of the system

$$\nabla\varphi(u) = 0,$$

which may be written in the form

$$\begin{bmatrix} n-1 & & -1 \\ & \ddots & \\ -1 & & n-1 \end{bmatrix} u = \begin{bmatrix} \frac{1}{2}(\underline{c}_1 - \bar{c}_1) \\ \frac{1}{2}(\underline{c}_n - \bar{c}_n) \end{bmatrix}. \quad \blacksquare$$

Let

$$\varrho^* := \max\{c^*_{ij}/c^*_{ji}:\ i, j = 1, \ldots, n,\ i \neq j,\ c^*_{ji} \neq 0\}.$$

We have

LEMMA 2. *If the matrix C^* is pseudosymmetric and satisfies the triangle inequality, then the following relations hold:*

(a) $\varrho^* \leqslant n-1$,

(b) $c^*_{ij} \geqslant 0,\ i, j = 1, \ldots, n,\ i \neq j$,

(c) *if $c^*_{ij} = 0$ for a certain pair of indices i, j with $i \neq j$, then the initial problem is equivalent to a problem with $n-1$ towns.*

Proof. (a) By the triangle inequality we have $c^*_{ij} \leqslant c^*_{ik} + c^*_{kj}$ for all i, j, k with $i \neq j$, $i \neq k$ and $j \neq k$. The summation over all indices j such that $j \neq i$ and $j \neq k$ yields

$$\underline{c}^*_i - c^*_{ik} \leqslant (n-2)\, c^*_{ik} + \bar{c}^*_k - c^*_{ki}$$

and, consequently,

$$c^*_{ki} \leqslant (n-1)\, c^*_{ik} + \bar{c}^*_k - \underline{c}^*_i.$$

The matrix C^* can be modified by a suitable permutation of rows and columns in such a way that $\underline{c}^*_1 \leqslant \ldots \leqslant \underline{c}^*_n$. Thus

$$c^*_{ki} \leqslant (n-1)\, c^*_{ik} \quad \text{for all } k < i.$$

By transposing the matrix C^* and using the equality $\underline{c}^*_i = \bar{c}^*_i$, we get

$$c^*_{ki} \leqslant (n-1)\, c^*_{ik} \quad \text{for all } k > i.$$

Thus part (a) is proved.

(b) Obviously, $c^*_{ij} \leqslant c^*_{im} + c^*_{mk} + c^*_{kj}$. Putting $k = i$, we have

$$0 \leqslant c^*_{im} + c^*_{mi} = nc^*_{im},$$

which was to be proved.

(c) If $c^*_{ij} = 0$ then $c^*_{ji} = 0$ follows from (a). Furthermore, it is easy to show that

$$c^*_{ik} = c^*_{jk} \quad \text{and} \quad c^*_{ki} = c^*_{kj} \quad \text{for all } k \text{ with } k \neq i \text{ and } k \neq j.$$

Now we delete the row i and the column j of matrix C^*, i.e. we integrate the towns i and j into one. ■

Let us define the distance matrix D as follows:

$$d_{ij} := 1 \quad \text{for all } i, j \text{ with } i < j, \tag{3}$$

$$d_{ij} := 0 \quad \text{for all } i, j \text{ with } i \geqslant j.$$

Obviously, matrix D fulfils the triangle inequality.

Remarks. 1. If the triangle inequality does not hold, then (a) or (b) may or may not be true, as the following example shows:

$$C := \begin{bmatrix} * & 1 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}, \quad \text{then} \quad C^* = \frac{1}{6}\begin{bmatrix} * & 4 & -1 \\ 2 & * & 1 \\ 1 & -1 & * \end{bmatrix}.$$

2. If T is a tour, then we denote by $\bar{T}$ the tour which goes in the opposite direction. For the special case where $V(T) = V(\bar{T})$ for all tours T, it was recognized already in [11] that there exists a vector u^* so that C^* is symmetric. The transformation presented here yields the same result for this particular case.

3. Let T be an arbitrary tour and let μ_1 be defined by the relation $V(T) = \mu_1 V(\bar{T})$. Then there exists a u such that $c'_{ij} = \mu_1 c'_{ji}$ holds for all pairs i, j where the path $i \to j$ is contained in T. Consequently, $\varrho^* \geqslant \mu_1$.

If now $V(T) \leqslant \mu_2 V(\bar{T})$ for all tours T and some μ_2, then $\varrho^* \leqslant \mu_2$ may not be true in general, as the following example demonstrates:

Let

$$C := \begin{bmatrix} * & 2 & 1 \\ 1 & * & 1 \\ 1 & 1 & * \end{bmatrix}.$$

Consequently,

$$C^* = \frac{1}{6}\begin{bmatrix} * & 10 & 5 \\ 8 & * & 7 \\ 7 & 5 & * \end{bmatrix}.$$

For this example we have $\mu_2 = 4/3$, but $\varrho^* = 7/5$.

These investigations can be used to find "good" approximate tours for the asymmetric travelling salesman problem.

3. The asymmetric case. Christofides [3] presented an algorithm for which $\lambda_s = 1.5$ holds if the distance matrix C is symmetric and the triangle inequality is fulfilled. If the matrix C fulfils the triangle inequality and if it is not necessarily symmetric but if

$$c_{ij} \leqslant \varrho c_{ji} \tag{4}$$

holds for all i, j with $i \neq j$, then in [4] this algorithm was modified so that

$$\lambda_s = 1.5\varrho.$$

The example given by (3) shows that ϱ may not be bounded even if the triangle inequality and the non-negativity requirement are met. The indicated transformation (2) therefore appears to be advantageous since $\varrho^* \leqslant \varrho$ and $\varrho^* \leqslant n-1$, as was shown in Lemma 1. The procedure from [4] will be here substantially generalized.

Algorithm

S1: Calculate $F := C + C^T$.

S2: Determine with respect to matrix F an approximation tour T_{appr}.

The matrix F is obviously symmetric and fulfils the triangle inequality if C does so. An arbitrary algorithm for constructing a tour T_{appr} can be applied which possesses an estimation constant λ_s if F fulfils the triangle inequality. Now the question arises as to the value of the constant λ_{as} with respect to the matrix C.

LEMMA 3. *Let T_{opt}^F be an optimal tour with respect to the matrix F and let T_{appr} be a tour with respect to matrix F with*

$$V_F(T_{appr}) \leqslant \lambda_s V_F(T_{opt}^F).$$

Let T_{opt} be an optimal tour with respect to matrix C, let $V_C(\bar{T}_{opt}) = \xi V_C(T_{opt})$ and let

$$r \cdot \min\{V_C(T_{appr}),\ V_C(\bar{T}_{appr})\} = \max\{V_C(T_{appr}),\ V_C(\bar{T}_{appr})\}.$$

Then the inequality

$$\min\{V_C(T_{appr}),\ V_C(\bar{T}_{appr})\} \leqslant \frac{1+\xi}{1+r} \lambda_s V_C(T_{opt})$$

holds.

Proof. Clearly,

$$\begin{aligned}(1+r)\min\{V_C(T_{appr}),\ V_C(\bar{T}_{appr})\} &= V_F(T_{appr}) \leqslant \lambda_s V_F(T_{opt}^F)\\ &\leqslant \lambda_s V_F(T_{opt}) = \lambda_s\big(V_C(T_{opt}) + V_C(\bar{T}_{opt})\big) = \lambda_s(1+\xi) V_C(T_{opt}),\end{aligned}$$

from which the statement can be derived. ∎

The values ξ and r in Lemma 3 are unknown, in general. However, $\xi \leqslant \varrho^*$ and $r \geqslant 1$ always hold. By putting $\xi = \varrho^*$ and $r = 1$, we obtain

$$\lambda_{as} = \tfrac{1}{2}(\varrho^* + 1)\lambda_s.$$

The algorithm described above has the following two advantages:

1. $\frac{1}{2}(\varrho^* + 1) \leqslant \varrho^*$ always holds. Therefore, the estimation obtained here is at least as good as that in [4].

2. No special algorithms are needed for the asymmetric problems.

4. Difficulties in the asymmetric case. It is known that the asymmetric problem can be solved more easily than the symmetric one, but the asymmetric problem has a greater degree of difficulty with respect to the construction of approximate tours than symmetric ones. The following lemma confirms this fact.

LEMMA 4. *Let C be a matrix fulfilling the triangle inequality, and let T_{opt} be an optimal tour. Then there always exists a tour T, $T \neq T_{opt}$, such that:*

(a) $V(T) \leqslant 2V(T_{opt})$,

(b) $V(T) \leqslant (1+2/n)\,V(T_{\text{opt}})$ *if* C *is symmetric,*

(c) *the inequalities* (a) *and* (b) *cannot be improved.*

Proof. The inequality (b) is proved in [9]. The example given by (3) shows that (a) is tight.

Now, let us prove (a). Without loss of generality we may assume that $T_{\text{opt}} = (1, 2, \ldots, n)$ and $T = (1, 3, 2, 4, 5, \ldots, n)$ with

$$c_{23}+c_{32} = \min\{c_{1n}+c_{n1},\ c_{i,i+1}+c_{i+1,i}\colon\ i = 1, \ldots, n-1\}.$$

Then

$$V(T)-V(T_{\text{opt}}) = c_{13}+c_{32}+c_{24}-c_{12}-c_{23}-c_{34}$$

holds. From $c_{13} \leqslant c_{12}+c_{23}$ and $c_{24} \leqslant c_{23}+c_{34}$ it follows that

$$V(T)-V(T_{\text{opt}}) \leqslant c_{23}+c_{32} \leqslant \frac{1}{n}\big(V(T_{\text{opt}})+V(\bar{T}_{\text{opt}})\big)$$

$$\leqslant \frac{1}{n}\big(V(T_{\text{opt}})+(n-1)\,V(T_{\text{opt}})\big) = V(T_{\text{opt}}).$$

Thus the statement is proved. ∎

5. Concluding remarks. To every distance matrix C an equivalent pseudosymmetric distance matrix C^* is assigned. The value ϱ^* can be interpreted as the measure of the violation of symmetry. If the distance matrix C fulfils the triangle inequality, then $\varrho^* \leqslant n-1$. For many practical problems $\varrho^* \approx 2$ has been obtained. For these problems,

$$\lambda_{\text{as}} \approx 1.5\lambda_{\text{s}}.$$

If C is not symmetric then in [7] C is changed into a matrix $\tilde{C}$ such that $\tilde{C}$ is symmetric. The main difficulty in this approach is the following: Unfortunately, not all matrices $\tilde{C}$ do fulfil the triangle inequality.

The author is indebted to the referee for valuable suggestions and to Dr. B. Legler and Dr. B. Luderer for their valuable remarks.

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Received on 1984.07.12;
revised version on 1984.11.29