COLLOQUIUM MATHEMATICUM

VOL. LII 1987 FASC. 1

ON SOME SPACES OF LINEAR FUNCTIONALS ON THE ALGEBRAS $A_p(G)$ FOR LOCALLY COMPACT GROUPS

BY

E. E. GRANIRER (VANCOUVER, BRITISH COLUMBIA)

Introduction. Let G be a locally compact group, $1 < p < \infty$ and $A_p(G)$ be the Banach algebra of functions f on G which can be represented as

$$f = \sum u_n * v_n^{\vee},$$

where

$$u_n \in L^{p'}(G), \quad v_n \in L_p(G) \quad (1/p + 1/p' = 1) \quad \text{with} \quad \sum \|u_n\|_{p'} \|v_n\|_p < \infty,$$

with norm as the infimum of the last expression over all such representations of f (see the following notations). Let $PM_p(G)$ be the dual Banach space of A_p.

We study in this paper containment and density properties of some Banach subalgebras (subspaces) of PM_p, some of which correspond, in case $p = 2$ and G is abelian, to $C_0(\hat{G})$, $AP(\hat{G})$ $[WAP(\hat{G})]$ $\{UC(\hat{G})\}$, $C(\hat{G})$, $L^\infty(\hat{G})$, the usual algebras of continuous bounded functions on $\hat{G}$: which tend to 0 at ∞, are [weakly] almost periodic, {uniformly continuous} or measurable (mod. a.e. equivalence). These algebras (subspaces) will be denoted for arbitrary G and $1 < p < \infty$ by $PF_p(G)$, $AP_p(\hat{G})$, $W_p(\hat{G})$, $UC_p(\hat{G})$, $C_p(\hat{G})$, $PM_p(G)$.

The results obtained in this paper improve results obtained by Dunkl and Ramirez [4]–[6] and by this author [9]–[11] and recent results of Ching Chou [3] (all obtained for $p = 2$). We point out that C^* algebra method which worked in some cases for $p = 2$ do not usually work for $p \neq 2$ (see for example Theorems 16, 17).

A combination of results obtained in this paper yields the following

THEOREM. *For arbitrary G and $1 < p < \infty$:*

$$PF_p \subset AP_p + \mathcal{M}_p \subset W_p \cap UC_p \subset UC_p \subset C_p \subset PM_p$$

(see the following notations).

We are unable to prove that $W_p \subset UC_p$ even for discrete nonamenable G and $p = 2$, even though $W_p \subset UC_p$ holds for amenable G.

We improve hereby, in Theorems 16, 17, results of ours ([9]–[11]) and weaker versions of recent results of Ching Chou [3], Corollary 3.7 and Theorem 3.8 (obtained in [3] for $p = 2$, using somewhat difficult C^* algebra methods) namely:

THEOREM 16. *Let G be second countable, $1 < p < \infty$. If for some norm separable subspace $X \subset PM_p$ and some open neighborhood V of the unit e*

$$UC_p(\hat{G}) \subset \operatorname{norm cl}\{W_p(\hat{G}) + X + L_V\}$$

where $L_V = \{T \in PM_p;\ \operatorname{supp} T \subset G \sim V\}$ then G is discrete(1).

If G is discrete then moreover $UC_p \subset AP_p$.

THEOREM 17. *Let G be second countable, $1 < p < \infty$, $K \subset A_p(G)$ convex, $T_n \subset PM_p$ and*

$$A = \{w^* \operatorname{cl} K\} \cap \{\psi \in PM_p^*;\ \psi(u \cdot \varphi) = u(e)\psi(\varphi) \text{ for } u \in A_p,\ \varphi \in PM_p \text{ and } \psi(T_n) = 0 \text{ for } n \geqslant 1\}.$$

If A is norm separable or has w^* exposed points then G is discrete(1).

We would like to thank hereby Ching Chou for his kindness in sending us a preprint of his paper.

In Section 1 of this paper we define and show the existence of topological invariant means on PM_p improving thereby a result of Renaud [18] (for $p = 2$). We show that the invariant mean is necessarily unique on $W_p(\hat{G})$ and use this fact in the proof of Theorems 16, 17.

If G is abelian, $PM_p(G)$ can be identified with an algebra of bounded measurable functions on $\hat{G}$ (see for example [16], p. 148). Elements of $L^\infty(\hat{G})$ which belong to PM_p are hard to characterize though. Diverse properties of elements of PM_p, for abelian G, are obtained in [14] and [20] and others.

For nonabelian G, PM_p is identified with a nonabelian algebra of operators on L^p (which commute with convolution from the right). The situation is more complicated in this case and theorems, which are easy for the abelian case and $p = 2$, require a much more involved proof in the general case (see for example Theorem 13).

Definitions and notations. Let G be a locally compact group with a fixed left Haar measure $\lambda = dx$ and $L^p(G)$, $1 \leqslant p \leqslant \infty$, the usual function spaces with norm $\|f\|_p = \left(\int |f|^p d\lambda\right)^{1/p}$ if $1 \leqslant p < \infty$ and $\operatorname{ess\,sup}|f| = \|f\|_\infty$.

Let $C_{00}(G)$, $C_0(G)$, $UC(G)$, $C(G)$ denote the spaces of complex continuous functions on G with compact support, which tend to 0 at ∞, are two-sided uniformly continuous, or continuous functions resp. with $\|\ \|_\infty$ norm. $AP(G)$ $[WAP(G)]$ will denote the [weakly] almost periodic bounded continuous functions on G. $M(G) = C_0(G)^*$ denotes the bounded Borel complex measures on G with convolution as multiplication, see [15], vol. I,

(1) Both these results have been definitively improved in our paper [21], Theorems 27 and 28, pp. 172–173.

and with variation norm. If f, g are functions on G let $f^{\vee}(x) = f(x^{-1})$, $f^{\sim}(x) = \overline{f(x^{-1})}$, $f * g(x) = \int g(y^{-1}x) f(y) dy$ whenever this makes sense.

If $1 < p < \infty$ then $A_p(G)$ will denote the set of functions on G which have a representation $f = \sum_1^{\infty} v_n * u_n^{\vee}$, an absolutely and uniformly convergent sum where $u_n \in L^p$, $v_n \in L^q$ (with $p^{-1} + q^{-1} = 1$), $\sum \|u_n\|_p \|v_n\|_q < \infty$, with the inf. over the last expression, over all representations of f as above, being the norm of f in A_p (denoted by $\|f\|_{A_p}$). We refer the reader to [12], [13] for properties of the regular tauberian Banach algebras A_p. $PM_p(G) = A_p^*$ will denote the Banach space dual of A_p. We note that $M(G) \subset PM_p$ with $\langle \mu, u \rangle = \int u \, d\mu$ for u in A_p.

Denote $B_p^M = \{u \in C(G);\ uv \in A_p$ for all $v \in A_p\}$ with the norm $\|u\|_M = \sup \{\|uv\|_{A_p};\ v \in A_p,\ \|v\|_{A_p} = 1\}$. Clearly, if $u \in A_p$, then $\|u\|_M \leqslant \|u\|_{A_p}$. Our B_p^M is different from B_p used in [13].

Note the module action of B_p^M on PM_p and PM_p^* its Banach space dual: $\langle u \cdot \varphi, v \rangle = \langle \varphi, uv \rangle$, $\langle u \cdot \psi, \varphi \rangle = \langle \psi, u \cdot \varphi \rangle$ for $u \in B_p^M$, $v \in A_p$, $\varphi \in PM_p$, $\psi \in PM_p^*$. We define then the operators $t'_u\colon A_p \to A_p$, $t_u = (t'_u)^*\colon PM_p \to PM_p$, $T_u = t_u^*\colon PM_p^* \to PM_p^*$ by $t'_u v = uv$, $t_u \varphi = u \cdot \varphi$, $T_u \psi = u \cdot \psi$. Clearly, $\|u \cdot \varphi\| \leqslant \|u\|_M \|\varphi\|$.

If Y is a space of functionals on X then $\sigma(X, Y)$ will denote the weakest topology on X which makes all y in Y continuous. If τ is a topology on X and $K \subset X$ then τ-cl$K$ is the τ-closure of K in X.

1. Invariant means on $PM_p(G)$. In this section we show the existence of invariant means on $PM_p(G)$ for any $1 < p < \infty$. The results improve results of Renaud [18] and of ours [9].

Definition. Denote

$$M_p = \{\psi \in PM_p^*(G);\ \|\psi\| = \psi(I) = 1\}$$

(the set of "means" on $PM_p(G)$),

$$S_B^p = \{u \in B_p^M;\ \|u\|_M = u(e) = 1\}$$

and

$$S_A^p = \{u \in A_p;\ \|u\|_{A_p} = u(e) = 1\}.$$

Note that $S_A^p \subset S_B^p$ since $\|u\|_\infty \leqslant \|u\|_M \leqslant \|u\|_{A_p}$ for $u \in A_p \subset B_p^M$. If $u \in B_p^M$, $\psi \in PM_p^*$, $\varphi \in PM_p$ let $(u \cdot \psi)(\varphi) = \psi(u \cdot \varphi)$.

Remark. If $u \in B_p^M$ and $\psi \in PM_p^*$ and $\varphi \in PM_p$ then

$$\|u \cdot \varphi\| \leqslant \|u\|_M \|\varphi\| \quad \text{and} \quad \|u \cdot \psi\| \leqslant \|u\|_M \|\psi\|.$$

Proposition 1. *S_A^p and S_B^p are convex sets and abelian semigroups (under pointwise multiplication) and $u \cdot M_p \subset M_p$ for all $u \in S_B^p$.*

Proof. If $u, v \in S_B^p$, $0 \leqslant \alpha \leqslant 1$, then

$$[\alpha u+(1-\alpha)v](e) = 1 \leqslant \|\alpha u+(1-\alpha)v\|_M \leqslant 1$$

since $|u(x)| \leqslant \|u\|_M$ for all $u \in B_p^M$ and $x \in G$. Similarly for S_A^p.

Let now $u \in S_B^p$ and $\psi \in M_p$. Then

$$1 = \|\psi\| = \|u\|_M \|\psi\| \geqslant \|u\cdot\psi\| \geqslant (u\cdot\psi)(I) = \psi(u\cdot I) = \psi(u(e)I) = \psi(I) = \|\psi\|.$$

Thus $\|u\cdot\psi\| = (u\cdot\psi)(I) = 1$. The rest is immediate.

Remarks. 1. If Q: $A_p \to PM_p^* = A_p^{**}$ is the canonical map then

$$Q[S_A^p] \subset M_p \quad \text{since} \quad (Qu)(I) = \langle I, u\rangle = u(e) = 1 = \|u\|_{A_p} = \|Qu\|.$$

2. Let $V = V^{-1}$ be such that $e \in V$ and $\lambda(V) < \infty$. Then

$$1_V * 1_{\tilde{V}}(x) = \int 1_V(x^{-1}y)1_V(y)dy = \lambda(xV \cap V) \leqslant \lambda(V).$$

Define $\varphi_V(x) = \lambda(V)^{-1}[1_V * 1_{\tilde{V}}(x)]$. Then

$$0 \leqslant \varphi_V(x) \leqslant 1 = \varphi_V(e) \leqslant \|\varphi_V\|_{A_p} \leqslant \lambda(V)^{-1}\|1_V\|_p\|1_V\|_{p'} = \lambda(V)^{-1}\lambda(V) = 1.$$

Hence, for each such V, $\varphi_V \in S_A^p$. (Thus $M_p \neq \emptyset$.) Furthermore

$$\{x;\ \varphi_V(x) \neq 0\} \subset \{x;\ xV \cap V \neq \emptyset\} \subset V^2.$$

3. One can easily show that $S_B^p S_A^p \subset S_A^p$ and, for $\psi \in M_p$, $S_A^p\cdot\psi = \psi$ implies $S_B^p\cdot\psi = \psi$.

PROPOSITION 2. *There exists some $\psi \in M_p$ such that $u\cdot\psi = \psi$ for each u in S_B^p. (We write in this case $S_B^p\cdot\psi = \psi$.)*

Proof. $M_p = \{\psi \in PM_p^*;\ \|\psi\| \leqslant 1 = \psi(I)\}$ is clearly a w^* compact convex set and S_M^p is a commutative semigroup (under pointwise multiplication) which acts as a semigroup of w^* continuous affine operators on PM_p^* by $T_u(\psi) = u\cdot\psi$. In fact, $T_u(T_v\psi) = (uv)\cdot\psi = T_{uv}(\psi)$ where $(uv)(x) = u(x)v(x)$ for all x, and if t_u: $PM_p \to PM_p$ is defined by $t_u\varphi = u\cdot\varphi$ then $T_u = t_u^*$: hence the w^* continuity. By Proposition 1, $T_u M_p \subset M_p$ if $u \in S_B^p$.

The Markov–Kakutani fixed point theorem ([7], p. 456) will imply now that there is some $\psi_0 \in M_p^*$ such that $T_u\psi_0 = u\cdot\psi_0 = \psi_0$ for all u in S_B^p.

Remark. Let $K_a = w^*$ closure of $\{aS_A^p\}$ for fixed $a \in S_B^p$, or any other w^* compact convex $\{T_u;\ u \in S_B^p\}$-invariant subset of M_p. Then there exists some ψ in K_a such that $u\cdot\psi = \psi$ for all u in S_B^p.

PROPOSITION 3. *Let $\psi \in M_p$ be such that $S_B^p\cdot\psi = \psi$. If $u \in B_p^M$ is such that $u = 1$ $[u = 0]$ on some neighborhood V of e then $u\cdot\psi = \psi$ $[u\cdot\psi = 0]$.*

Proof. Assume that $u = 1$ on V. Let U be open such that $U = U^{-1}$ and $U^2 \subset V$. The function $\varphi = \varphi_U = \lambda(U)^{-1}1_U * 1_{\tilde{U}}$ of Remark 2 above satisfies $\varphi \in A_p$, $\varphi(e) = 1 = \|\varphi\|_{A_p}$ so $\varphi \in S_B^p$ and $\varphi = 0$ off U^2. Hence $u(x)\varphi(x) = \varphi(x)$ for all x. Thus $u\cdot\psi = u\cdot(\varphi\cdot\psi) = (u\varphi)\cdot\psi = \varphi\cdot\psi = \psi$, which proves the first part. Assume now that $u \in B_p^M$ and $u = 0$ on V. Then $1-u \in B_p^M$ and $1-u = 1$ on V. Hence $\psi = (1-u)\cdot\psi = \psi - u\cdot\psi$, i.e. $u\cdot\psi = 0$.

Remark. This proposition expresses the fact that any S_B^g-invariant ψ in M_p has "support" included in every neighborhood of e.

PROPOSITION 4. *Let $\psi \in M_p$ be such that $S_B^g \psi = \psi$. Then for each $u \in B_p^M$, $u \cdot \psi = u(e)\psi$.*

Proof. Assume at first that $v \in A_p(G)$ is such that $v(e) = 0$. The set $\{e\}$ is a set of spectral synthesis for the algebra $A_p(G)$ (see [13], p. 91, Theorem B, with $H = \{e\}$). Hence there exists a sequence $v_n \in A_p$ such that $v_n = 0$ on some neighborhood V_n of e, v_n has compact support and $\|v_n - v\|_{A_p} \to 0$. By Proposition 3, $v_n \cdot \psi = 0$. But $\|v \cdot \psi\| = \|(v_n - v) \cdot \psi\| \leqslant \|v_n - v\|_{A_p} \|\psi\| \to 0$. Thus $v \cdot \psi = 0$ for any v in A_p such that $v(e) = 0$.

Let now $u \in A_p$ be such that $u(e) = 1$. Choose $v \in A_p$ such that $v = 1$ on some neighborhood V of e. Then $(u-v) \cdot \psi = 0$ by the above. Thus $u \cdot \psi = v \cdot \psi = \psi = u(e)\psi$ if $u(e) = 1$ (hence clearly for any $u \in A_p$).

Let $u \in B_p^M$ be arbitrary. Choose $v \in A_p$ such that $v(e) = 1$. Then $uv \in A_p$ and $(uv)(e) = u(e)$. Hence $u \cdot \psi = u \cdot (v \cdot \psi) = (uv) \cdot \psi = (uv)(e)\psi = u(e)\psi$, which finishes this proof.

Definition. Let $TIM_p(\hat{G}) = \{\psi \in M_p;\ u \cdot \psi = u(e)\psi \text{ for all } u \in B_p^M(G)\}$ be the set of topological invariant means on $PM_p(G)$.

THEOREM 5. *For all G and $1 < p < \infty$, $TIM_p(\hat{G}) \neq \emptyset$. Moreover, for any convex w^* compact $\{T_u;\ u \in S_B^g\}$-invariant subset K of M_p, $K \cap TIM_p(\hat{G}) \neq \emptyset$.*

Proof. Use the remark after Proposition 2 and Proposition 4.

Remark. That $TIM_2(\hat{G}) \neq \emptyset$ is a result of Renaud [18], p. 287, for an easier proof see [9], p. 376.

We note that the proof for $p \neq 2$ is necessarily more complicated since the C^* algebra techniques are not available anymore. We note that we even needed the fact that single point sets are sets of spectral synthesis in order to prove that $TIM_p(\hat{G}) \neq \emptyset$, even though this is not needed in the proof for $p = 2$ given in [9], p. 376 (it is used though in [18], p. 287).

2. p-weakly almost periodic and p-uniformly continuous functionals on $\hat{G}$.

Definition. We denote by $W_p(\hat{G})$ $[AP_p(\hat{G})]$ the linear space of all $\varphi \in PM_p$ for which the operator $u \to u \cdot \varphi$ from $A_p(G)$ to $PM_p(G)$ is weakly compact [compact].

Remarks. If $\varphi \in W_p(\hat{G})$ $[AP_p(\hat{G})]$ and $u \in B_p^M$ then $u \cdot \varphi \in W_p(\hat{G})$ $[AP_p(\hat{G})]$ as readily checked. Propositions 6, 7 below improve results of Dunkl–Ramirez [6], p. 505.

PROPOSITION 6. *$W_p(\hat{G})$ and $AP_p(\hat{G})$ are norm closed $B_p^M(G)$-submodules of $PM_p(G)$ and $I \in AP_p(\hat{G}) \subset W_p(\hat{G})$.*

Proof. It is routine to check that both are linear spaces. Assume that $\varphi_n \in W_p(\hat{G})$ $[AP_p(\hat{G})]$ and $\|\varphi_n - \varphi\|_{PM_p} \to 0$ for some $\varphi \in PM_p$. Then $\sup\{\|v \cdot (\varphi_n - \varphi)\|,\ \|v\|_{A_p} \leqslant 1\} \to 0$ thus $\varphi_n \to \varphi$ in the operator norm (from A_p

to PM_p). However uniform limits of weakly compact [compact] operators are weakly compact [compact] [7], p. 483.

Note that $u \cdot I = u(e)I$ thus $u \to u \cdot I$ has one-dimensional range. The rest is immediate.

Remark. $AP_2(\hat{G})$ $[W_2(\hat{G})]$ coincides in the abelian case with the [weakly] almost periodic continuous functions on $\hat{G}$ as shown by Dunkl and Ramirez [6], p. 503.

Remarks. $M(G)$ can be considered as a subspace of $PM_p(G) = A_p(G)^*$ by

$$\langle \mu, g \rangle = \int g \, d\mu, \quad \text{for} \quad g \in A_p.$$

Then

$$|\langle \mu, g \rangle| \leqslant \|g\|_\infty |\mu|(G) \leqslant \|g\|_{A_p} \|\mu\|_{M(G)}.$$

Hence $\|\mu\|_{PM_p} \leqslant \|\mu\|_{M(G)}$. We also note the module action of $A_p(G)$ on $M(G)$ (in fact of $B_p^M(G)$): If $u \in B_p^M$, $v \in A_p(G)$ and $\mu \in M(G)$ then $\langle v, u \cdot \mu \rangle = \langle uv, \mu \rangle = \int v(u \, d\mu)$. Thus $u \cdot \mu \in M(G)$ is just the measure $u \, d\mu$.

Proposition 7. *$M(G) \subset W_p(\hat{G})$, hence $\| \ \|_{PM_p}$-closure of $[M(G)] \subset W_p(\hat{G})$.*

Proof. We follow basically the proof of Dunkl–Ramirez [6], p. 505, given there for the case $p = 2$. It is enough to show that for any probability measure $\mu \in M(G)$ one has $\mu \in W_p(\hat{G})$. Define the map S: $H = L^2(G, d\mu) \to PM_p(G)$ by: For $v \in A_p$ and $f \in H$, $\langle v, Sf \rangle = \int vf \, d\mu$. Then

$$|\langle v, Sf \rangle| \leqslant \|v\|_H \|f\|_H \leqslant \|v\|_\infty \|f\|_H \leqslant \|v\|_{A_p} \|f\|_H$$

since $\mu \geqslant 0$ and $\mu(G) = 1$. Thus Sf is a continuous linear functional on A_p with $\|Sf\|_{PM_p} \leqslant \|f\|_H$. S is a continuous linear operator and hence by [7], p. 422, is weakly continuous.

Let now $\{f_n\} \subset A_p(G)$, $\|f_n\|_{A_p} \leqslant 1$. Then $\|f_n\|_H \leqslant 1$ and since the unit ball of H is weakly sequentially compact there is a subsequence $f_{n_k} \to h$, weakly, for some $h \in H$. But then $Sf_{n_k} \to Sh$ in the weak topology of PM_p. To finish the proof of the theorem we will show that for $f \in A_p(G)$, $Sf = f \cdot \mu$ (module action of A_p on PM_p). In fact, if $v \in A_p$ then $\langle v, Sf \rangle = \int v(f \, d\mu) = \langle v, f \cdot \mu \rangle$ by the remark preceding this proposition. We have shown that $\{f \cdot \mu, \|f\|_{A_p} \leqslant 1\}$ is a relatively weakly sequentially (hence relatively weakly) compact subset of PM_p and thus $M(G) \subset W_p(\hat{G})$. The fact that $W_p(\hat{G})$ is closed implies the rest.

Proposition 8. *$l^1(G) \subset AP_p(\hat{G})$.*

Proof. Clearly, $l^1(G) \subset M(G)$ and if $\mu \in l^1(G)$ then

$$\|\mu\|_{M(G)} = \sum_{x \in G} |\mu(x)| \quad \text{and} \quad \|\mu\|_{PM_p} \leqslant \|\mu\|_{M(G)}$$

by the above. If $u \in A_p$ then

$$u \cdot \mu = u\,d\mu \in l^1(G) \quad \text{and} \quad \|u \cdot \mu\|_{PM_p} \leqslant \|u\|_{A_p} \|\mu\|_{PM_p}$$

by the definition of the module action. Thus the norm of the operator $u \to u \cdot \mu$ is dominated by $\|\mu\|_{PM_p}$.

Any $\mu \in l^1(G)$ is an $\|\ \|_{M(G)}$ norm (and a fortiori $\|\ \|_{PM_p}$) limit of finite linear combinations $\sum_1^n \alpha_i \delta_{x_i}$ of point masses at x_i. Since norm limits of compact operators are compact, it is enough to prove that δ_a is a compact operator for all $a \in G$. Now $\{u \cdot \delta_a;\ \|u\|_{A_p} \leqslant 1\} \subset \{\alpha\delta_a;\ |\alpha| \leqslant 1\}$ and the last is a 1-dimensional bounded closed (hence compact) set. This finishes the proof.

Remark. It is proved in Dukl–Ramirez [5], p. 529, that if G is a compact group such that $\{\alpha \in \hat{G};\ \dim \alpha = l\}$ is a finite set for each $l = 1, 2, 3, \dots$ (such as $SU(2)$) then $PF_2(G) = PM_2$-norm closure of $\{L^1(G)\} \subset AP_2(\hat{G})$. This is in marked contrast with the abelian case where $PF_2(G) = C_0(\hat{G})$ and $C_0(\hat{G}) \cap AP_2(\hat{G}) = \{0\}$.

PROPOSITION 9. *There exists a unique* $\psi \in W_p(\hat{G})^*$ *such that* $\psi(I) = 1$ *and* $\psi(u \cdot \varphi) = u(e)\psi(\varphi)$ *for all* $u \in B_p^M$.

Proof. Any $\psi_0 \in TIM_p(\hat{G})$ restricted to $W_p(\hat{G})$ satisfies this condition. Moreover $\|\psi_0\| = 1$. Keep now $\psi_0 \in TIM_p(\hat{G}) = [A_p(G)]^{**}$ fixed and let $u_\alpha \in A_p$, $\|u_\alpha\|_{A_p} \leqslant 1$ be such that for all $\varphi \in PM_p$, $\varphi(u_\alpha) \to \psi_0(\varphi)$. Then $u_\alpha(e) = I(u_\alpha) \to \psi_0(I) = 1$.

Let $\psi_1 \in W_p(\hat{G})^*$ be such that $\psi_1(I) = 1$ and $\psi_1(u \cdot \varphi) = u(e)\psi_1(\varphi)$ for $u \in B_p^M$. Let $c = \psi_0(\varphi_0)$ where $\varphi_0 \in W_p(\hat{G})$. Then for each $u \in A_p$ one has

$$(t_{u_\alpha}\varphi_0)(u) = (t_u\varphi_0)(u_\alpha) \to \psi_0(t_u\varphi_0) = u(e)\psi_0(\varphi_0) = u(e)c = (cI)(u).$$

Thus $t_{u_\alpha}\varphi_0 \to cI$ in the w^* topology of PM_p. Since $\varphi_0 \in W_p(\hat{G})$ there is a subnet $t_{u_{\alpha_\nu}}\varphi_0 \to \varphi \in PM_p$ weakly (and a fortiori w^*) in PM_p. Thus $\varphi = cI$ and in particular we have $\psi_1(t_{u_{\alpha_\nu}}\varphi_0) \to \psi_1(cI) = c = \psi_0(\varphi_0)$. However $\psi_1(t_{u_{\alpha_\nu}}\varphi_0) = u_{\alpha_\nu}(e)\psi_1(\varphi_0) \to \psi_1(\varphi_0)$. Thus $\psi_1(\varphi_0) = \psi_0(\varphi_0)$ and since φ_0 is arbitrary, ψ_1 coincides with the restriction of the fixed $\psi_0 \in TIM_p(\hat{G})$ to $W_p(\hat{G})$ which finishes this proof.

PROPOSITION 10. *If* $\psi_0 \in W_p(\hat{G})^*$ *is the unique invariant mean of Proposition 9 then* $\psi_0(\mu) = \mu\{e\}$ *for each* $\mu \in M(G)$.

Proof. Let $0 \leqslant \mu \in M(G)$. Choose, by the regularity of μ, open sets with compact closure $V_n = V_n^{-1} \subset G$ such that $V_{n+1}^2 \subset V_n$ and $\mu(V_n) \to \mu\{e\}$. Let $v_n = \varphi_{V_n}$ be as in the Remarks before Proposition 2. Then $v_n(x) \to 1_V(x)$ for all x, where $V = \bigcap V_n$. But $1_V = 1_e$ a.e. μ. Thus $v_n \to 1_e$ a.e. μ; hence, for all $v \in A_p$, $vv_n \to v(e)1_e$ a.e. μ. Let $v \in A_p$. Then

$$\langle v_n \cdot v, v\rangle - \langle \mu\{e\}\delta_e, v\rangle = \int (vv_n - v(e)1_e)\,d\mu \to 0.$$

Thus $v_n \cdot \mu \to \mu\{e\}\delta_e$ in $w^* = \sigma(PM_p, A_p)$. But $M(G) \subset W_p(\hat{G})$. Hence $\{v\cdot\mu;\ \|v\|_{A_p} \leqslant 1\}$ is weakly relatively compact and it follows routinely that $v_n \cdot \mu \to \mu\{e\}\delta_e$ weakly $= \sigma(PM_p, PM_p^*)$. Thus

$$\psi_0(\mu) = \psi_0(v_n\cdot\mu) \to \psi_0(\mu\{e\}\delta_e) = \mu\{e\}\psi(I) = \mu\{e\}.$$

Remark. If G is not discrete then $\psi_0(\mu) = \mu\{e\} = 0$ for any $\mu = f\,dx$ ($f \in L^1(G)$). If $PF_p(G)$ denotes the norm closure of $L^1(G)$ in $W_p(\hat{G})$ it follows that $\psi_0(\varphi) = 0$ for all $\varphi \in PF_p$. If G is abelian nondiscrete and $p = 2$ then $PF_p(G) = C_0(\hat{G})$ and any invariant mean ψ_0 on $L^\infty(\hat{G})$ will satisfy $\psi_0(C_0(\hat{G})) = 0$ since $\hat{G}$ is not compact. We can state the

Corollary 11. *$\psi_0[PF_p] = 0$ whenever G is not discrete.*

Definition. Denote

$$C_p(\hat{G}) = \{\varphi \in PM_p;\ \varphi\varphi_1 + \varphi_2\varphi \in PF_p \text{ if } \varphi_1, \varphi_2 \in PF_p\},$$

$$UC_p(\hat{G}) = \overline{\{A_p(G)\cdot PM_p(G)\}},$$

$$\mathcal{M}_p(\hat{G}) = \overline{M(G)}, \quad \mathcal{M}_p^d(\hat{G}) = \overline{l^1(G)} \quad \text{and} \quad PF_p(G) = \overline{L^1(G)}$$

where bar means here closure in $\|\ \|_{PM_p}$-norm.

Remarks. (a) $\{A_p\cdot PM_p\} = \{u\cdot\varphi;\ u \in A_p,\ \varphi \in PM_p\}$ is a linear space as known and readily shown (see for example [9], p. 373).

(b) As known and readily seen UC_p coincides with the PM_p norm closure of the set of $T \in PM_p$ with compact support (for definition of support see [12], p. 101 and 117).

(c) If $p = 2$ and G is abelian then $C_2(\hat{G})$, $[UC_2(\hat{G}) = UC(\hat{G})]$, $\{PF_2(G) = C_0(\hat{G})\}$ is the space of bounded continuous [uniformly continuous] {continuous vanishing at ∞} functions on $\hat{G}$. $\mathcal{M}_2(\hat{G})$ is just the sup norm closure of $B_2(\hat{G}) = B(\hat{G})$, a subalgebra of $W_2(\hat{G})$.

Proposition 12. *$UC_p(\hat{G})$, $\mathcal{M}_p(\hat{G})$ and $\mathcal{M}_p^d(\hat{G})$ are closed subalgebras and B_p^M-submodules of PM_p, $\mathcal{M}_p(\hat{G}) \subset UC_p(\hat{G}) \cap W_p(\hat{G})$ and $\mathcal{M}_p^d(\hat{G}) \subset UC_p(\hat{G}) \cap AP_p(\hat{G})$.*

Proof. We show at first that UC_p is an algebra. Let at first $f_1, f_2 \in C_{00}(G)$ have compact supports A_1, A_2 resp. Then $f_1 * f_2(x) = 0$ if $x \notin A_1 A_2$. Let $v \in A_p \cap C_{00}(G)$ have support disjoint from $A_1 A_2$. Then

$$\langle f_1 * f_2, v\rangle = \int (f_1 * f_2) v\, d\lambda = 0 \quad (\lambda \text{ is left Haar measure}).$$

By [12], Corollary on p. 120 and p. 101 we have $\operatorname{supp}(f_1 * f_2) \subset \overline{A_1 A_2}$ where $\operatorname{supp}(f_1 * f_2)$ denotes here the support of $f_1 * f_2$ as an element of PM_p. Let now S, $T \in PM_p$ have compact supports A, B and A_1, B_1 be open sets with compact closures such that $A \subset A_1$, $B \subset B_1$. Let s_α, t_β be nets in C_{00}, with $\operatorname{supp} s_\alpha \subset A_1$, $\operatorname{supp} t_\beta \subset B_1$, such that $|||s_\alpha|||_p \leqslant |||S|||_p$ and $|||t_\beta|||_p \leqslant |||T|||_p$ and, for all $f \in L^p$, $\|(s_\alpha - S)f\|_p \to 0$ and $\|(t_\beta - T)f\|_p \to 0$. Such nets can be found by [12], p. 117, Proposition 9. An immediate consequence of the Corollary on p. 120 and of the remarks on p. 101 both of [12] implies that if W, $W_\alpha \in PM_p$, $\operatorname{supp} W_\alpha \subset E$ (E closed) and $W_\alpha \to W$ in $\sigma(PM_p, A_p)$ then $\operatorname{supp} W \subset E$. But $\langle k * u, v \rangle = \langle k, v * u^\vee \rangle$ if $k \in L^1$, $v \in L^{p'}$, $u \in L^p$ ([13], p. 153). Hence for fixed α, $s_\alpha * t_\beta \to s_\alpha T$ $\sigma(PM_p, A_p)$ and strongly on L^p. Thus $\operatorname{supp} s_\alpha T \subset \overline{A_1 B_1}$. If we let now α vary then we get that $\operatorname{supp} ST \subset \overline{A_1 B_1}$ which readily implies that UC_p is an algebra. Any $T \in UC_p$ is a norm limit of elements $u \cdot S$ where $u \in A_p \cap C_{00}$ and $S \in PM_p$. If $v \in B_p^M$ then $v \cdot T$ is again a norm limit of such elements. If $\mu \in M(G)$ then $v \cdot \mu$ is just $v d\mu$ for $v \in B_p^M$.

As for the inclusion we note that if μ_K denotes the restriction of the measure μ to the compact $K \subset G$ then for suitable K_n we have $\|\mu_{K_n} - \mu\|_{PM_p} \leqslant \|\mu_{K_n} - \mu\|_{M(G)} \to 0$ and clearly $\mu_{K_n} \in UC_p(\hat{G})$. Propositions 6 and 7 imply $\mathcal{M}_p(\hat{G}) \subset W_p(\hat{G})$ while Proposition 8 implies that $\mathcal{M}_p^d(\hat{G}) \subset AP_p(\hat{G})$. Since $M(G)$ and $l^1(G)$ are B_p^M-submodules so will $\mathcal{M}_p(\hat{G})$ and $\mathcal{M}_p^d(\hat{G})$.

Remark. It is not known for nonamenable G, whether $W_p(\hat{G}) \subset UC_p(\hat{G})$ (even for $p = 2$). For amenable G this inclusion holds true (Proposition 14).

If $p = 2$ then $\mathcal{M}_2(\hat{G})$ is a C^*-algebra (since $\mu^* \in M(G)$ if $\mu \in M(G)$) which coincides with the $\| \ \|_\infty$-closure of $B(\hat{G})$, in case G is abelian.

The following improves our result in [10], p. 64.

THEOREM 13. *$UC_p(\hat{G}) \subset C_p(\hat{G})$ for any locally compact group G, $C_p(\hat{G})$ is a closed subalgebra and $A_p(G)$ a submodule of PM_p.*

Proof. Let $T \in UC_p$, $\varphi \in PF_p$. One has to show that $T\varphi$ and φT both belong to PF_p. Now $A_p \cap C_{00}$ is norm dense in A_p and $A_p \cdot PM_p$ is norm dense in UC_p. There is hence a sequence $u_n \in A_p \cap C_{00}$ with $\|u_n \cdot T - T\|_{PM_p} \to 0$. But $\operatorname{supp}(u_n \cdot T) \subset (\operatorname{supp} u_n) \cap \operatorname{supp} T$ (see [12], p. 101 or p. 117, also Corollary on p. 120 of [12]).

We can hence assume that $\operatorname{supp} T$ is compact. Let V be a neighborhood of e with compact closure and let $\Omega = V(\operatorname{supp} T)$. Then, by [12], Proposition 9, p. 117, there is a net $w_\alpha \in C_{00}(G)$ with $\operatorname{supp} w_\alpha \subset \Omega$ such that for each f in L^p $\|w_\alpha * f - Tf\|_p \to 0$ (ultra weak $= \sigma(PM_p, A_p)$ by [12], p. 116; compare with [1], p. 91).

For $h \in L^1(G)$ let $\varrho(h)$: $L^p \to L^p$ be defined by

$$(\varrho h)(f) = h * f.$$

Fix $g \in C_{00}$. We show that $\varrho(w_\alpha * g)$ and $\varrho(g * w_\alpha)$ are PM_p-norm Cauchy sequences. In fact if $K = \operatorname{supp} g$, for each $v \in A_p$ with $\|v\|_{A_p} \leqslant 1$ one has

$$|\langle \varrho[(w_\alpha - w_\beta) * g], v \rangle| = \Big| \int_{\Omega K} [(w_\alpha - w_\beta) * g](x) v(x) dx \Big| \leqslant \|(w_\alpha - w_\beta) * g\|_p \|v\|_\infty C$$
$$\leqslant C \|(w_\alpha - w_\beta) * g\|_p$$

(where $C = \lambda(\Omega \cdot K)^{1/p'}$ and λ is Haar measure) by the Hölder inequality. But

$\|(w_\alpha - w_\beta) * g\|_p \to 0$ with α, β, since $g \in L_p$ and the above estimate is independent of $v \in A_p$ with $\|v\|_{A_p} \leqslant 1$. Thus $\varrho(w_\alpha * g)$ is a PM_p norm Cauchy net, and there is some $T_1 \in PM_p$ such that $\varrho(w_\alpha * g) \to T_1$ in PM_p norm. However $\varrho(w_\alpha * g) \to (T)(\varrho g)$ strongly on L^p. Thus $T_1 = (T)(\varrho g)$ and $\varrho(w_\alpha * g) \to (T)(\varrho g)$ in PM_p norm. But $\varrho(w_\alpha * g) \in \varrho(C_{00}) \subset PF_p$. Thus $T[\varrho(C_{00})] \subset PF_p$. But $\varrho(C_{00})$ is dense in PF_p. Thus $T(PF_p) \subset PF_p$.

We still have to show that $[\varrho(C_{00})]\,T \subset PF_p$.

As well known, if v, w, $g \in C_{00}$ and $h^\sim(x) = \overline{h(x^{-1})}$, $h^* = \frac{1}{\Delta} h^\sim$ then

$$\langle v, g * w\rangle = \left\langle \frac{1}{\Delta}\left(\frac{1}{g}\right)^\sim * v, w\right\rangle = \left\langle v^* * \left[\frac{1}{\Delta}(\bar{g})^\sim\right]^*, w\right\rangle = \langle v^* * \bar{g}, w\rangle$$
$$= \langle v^*, w * g^\sim\rangle.$$

Hence, if $v \in A_p \cap C_{00}$ then for the net w_α and g_0 chosen above we have

$$|\langle v, g_0 * (w_\alpha - w_\beta)\rangle| = |\langle v^*, (w_\alpha - w_\beta) * g_0^\sim\rangle|$$
$$= \left| \int_{\Omega K^{-1}} ((w_\alpha - w_\beta) * g_0^\sim)(x) \frac{1}{\Delta(x)} \overline{v(x^{-1})}\, dx \right| \leqslant \|(w_\alpha - w_\beta) * g^\sim\|_p \|v\|_\infty \cdot C$$

where $C = \|\Delta^{-1} \chi_{\Omega K^{-1}}\|_{p'}$ does not depend on $v \in C_{00}$. But $\|v\|_\infty \leqslant \|v\|_{A_p}$ and $\|(w_\alpha - w_\infty) * g^\sim\|_p \to 0$ with α, β, since $g^\sim \in C_{00} \subset L^p$. This shows that $\varrho(g * w_\alpha)$ is a PM_p-norm Cauchy sequence in $\varrho[C_{00}]$ which readily implies that $[PF_p]\,T \subset PF_p$. Note that $A_p \cdot C_p \subset UC_p \subset C_p$ by the first part.

COROLLARY. *For arbitrary* G

$$PF_p \subset \mathcal{M}_p(\hat{G}) \subset W_p(\hat{G}) \cap UC_p(\hat{G}) \subset UC_p(\hat{G}) \subset C_p(\hat{G}) \subset PM_p$$
$$\text{and} \quad \mathcal{M}_p^d(\hat{G}) \subset AP_p(\hat{G}):$$

The following improves our result in [9], p. 374:

PROPOSITION 14. *Let G be an amenable locally compact group. Then*

$$W_p(\hat{G}) \subset UC_p(\hat{G}) = \operatorname{norm\,cl}[A_p(G) \cdot PM_p(G)] = A_p(G) \cdot PM_p(G).$$

Proof. As is well known $A_p(G)$ has an approximate identity e_α with $\|e_\alpha\| \leqslant 1$. Let $\varphi \in W_p(\hat{G})$. Then $e_\alpha \cdot \varphi \to \varphi$ in the w^* topology of $PM_p(G)$. But by the definition of $W_p(\hat{G})$ there exists a subnet e_{α_β} such that $e_{\alpha_\beta} \cdot \varphi \to \varphi'$ weakly (i.e. in $\sigma(PM_p, PM_p^*)$).

Thus $\varphi = \varphi'$ and since the weak and norm closures of $A_p \cdot PM_p$ are the same, one has $W_p(\hat{G}) \subset UC_p(\hat{G})$.

As for the equality $UC_p(\hat{G}) = A_p(G) \cdot PM_p(G)$ we note the following: $A_p(G)$ is a Banach algebra with a bounded left approximate unit and $PM_p(G)$ is a (left) Banach $A_p(G)$-module as in [15], p. 263, (32.14). By Cohen's factorization theorem ([15], (32.22), p. 268) $A_p(G) \cdot PM_p(G)$ is norm closed which finishes this proof.

Remark. If G is the discrete free group on 2 generators then $A_2(G)\cdot PM_2(G)$ is not closed. This is due to A. Figà-Talamanca (see [10], p. 69). Question: Is $W_p(\hat{G}) \subset UC_p(\hat{G})$ for arbitrary G? **(P 1311)**

COROLLARY. *Let G be an amenable locally compact group. Then for all* $1 < p < \infty$

$$PF_p(G) \subset AP_p(\hat{G}) + PF_p(G) \subset W_p(\hat{G}) \subset UC_p(\hat{G}) \subset C_p(\hat{G}) \subset PM_p(\hat{G}).$$

As remarked before $PF_2(G) \subset AP_2(\hat{G})$ if $G = SU(2)$ in marked difference from the abelian case.

PROPOSITION 15. (a) *Let G be discrete. Then*

$$PF_p(G) = \mathcal{M}_p^d(\hat{G}) = UC_p(\hat{G}) = C_p(\hat{G}) \subset AP_p(\hat{G}) \subset W_p(\hat{G}).$$

(b) *If G is discrete and amenable then the inclusions in* (a) *become equalities. Hence all the above spaces are algebras in this case.*

(c) *If G is compact then $UC_p = C_p = PM_p$.*

Proof. (a) $\delta_e \in PF_p$ hence $PF_p = \mathcal{M}_p^d(\hat{G})$ has identity. The definition of $C_p(\hat{G})$ together with Theorem 13 imply now that $PF_p = UC_p = C_p$. Proposition 8 yields that $\mathcal{M}_p^d(\hat{G}) \subset AP_p(\hat{G}) \subset W_p(\hat{G})$.

(b) Proposition 14 shows that $W_p(\hat{G}) \subset UC_p(\hat{G})$ which together with (a) finishes the proof.

(c) $1 \in A_p(G)$ in this case and $PM_p = 1\cdot PM_p \subset UC_p \subset C_p$ by Theorem 13.

Remarks. If $V \subset G$ is open, let L_V be the w^* closed space of all $\varphi \in PM_p$ such that $\operatorname{supp}\varphi \subset G \sim V$ (the complement of V in G) (see [12], p. 119).

We improve now our Theorem 4 (a) in [10], p. 66, for the case that G is second countable and a result of C. Chou [3] (see remarks after Theorem 17):

THEOREM 16. *Let G be second countable. If for some norm separable subspace $X \subset PM_p(G)$ and some open neighborhood of the unit V*

$$(*) \qquad UC_p(\hat{G}) \subset \operatorname{norm\,cl}\{W_p(\hat{G}) + X + L_V\}$$

then G is discrete. If G is discrete then $UC_p(G) \subset AP_p(G)$ even, by Proposition 15.

Proof. We can and shall assume that $\bar{V}$ is compact. Let $v \in S_A^p$ be such that $\operatorname{supp} v = \operatorname{cl}\{x;\, v(x) \neq 0\} \subset V$. If $\varphi \in L_V$ then $\operatorname{supp} v\cdot\varphi \subset \operatorname{supp} v \cap \operatorname{supp}\varphi = \emptyset$ ([12], p. 118). Thus we have $v\cdot\varphi = 0$ by [12], p. 101. Thus if $v^\perp = \{\varphi \in PM_p;\, v\cdot\varphi = 0\}$ then $L_V \subset v^\perp$. Hence $(*)$ implies that

$$UC_p \subset \operatorname{norm\,cl}\{W_p(\hat{G}) + X + v^\perp\}.$$

$L^p(G)$ is separable, hence so is $A_p(G)$. Let $\{u_n\}$ be dense in S_A^p and denote

$K = vS_A^p \subset S_A^p$. Any $\varphi \in v^\perp$ will satisfy $\varphi(K) = 0$. Hence any ψ in $w^* \operatorname{cl} K$ will satisfy $\psi(\varphi) = 0$ for all φ in $v^\perp$ (K is identified with its canonical image in PM_p^*).

Fix now some $\psi_0 \in \{w^* \operatorname{cl} K\} \cap TIM_p(\hat{G})$ (see the remark after Proposition 2) and let $\{\varphi_n\}$ be norm dense in X. Let $\psi_0(\varphi_n) = \alpha_n$ and consider the w^* compact convex set

$$A = \{w^* \operatorname{cl} K\} \cap \{\psi \in PM_p^*;\ (t'_{u_n} - I')^{**} \psi = 0,\ \psi(\varphi_n) = \alpha_n,\ n \geqslant 1\}$$

where I': $A_p \to A_p$ is the identity and $t'_a(u) = au$ for a, u in A_p. Since $(t'_{u_n})^* = t_{u_n}$, it follows that any ψ in A satisfies $u \cdot \psi = \psi$ for all $u \in S_A^p$. If $w \in S_B^p$ and $u \in S_A^p$ then $w \cdot \psi = (wu) \cdot \psi = \psi$. Thus $A \subset TIM_p(\hat{G})$ by Proposition 4. Clearly $\psi_0 \in A$. We claim that $A = \{\psi_0\}$. In fact let $\psi_1 \in A$. Then $\psi_1 = \psi_0$ on X and $\psi_1(v^\perp) = 0$ since $\psi_1 \in w^* \operatorname{cl} K$. By Proposition 9 $\psi_1 = \psi_0$ on $W_p(\hat{G})$. Our assumption implies now that $\psi_1 = \psi_0$ on $UC_p(G)$. If $\varphi \in PM_p$, choose $u \in S_A^p$. Then $\psi_1(\varphi) = \psi_1(u \cdot \varphi) = \psi_0(u \cdot \varphi) = \psi_0(\varphi)$ since $u \cdot \varphi \in UC_p$. Thus $\psi_1 = \psi_0$.

We apply now Corollary 1.3 on p. 21 of [8] and get that there exists a sequence $v_n \in K \subset S_A^p$ such that for all $\varphi \in PM_p$, $\varphi(v_n) \to \psi_0(\varphi)$. Thus v_n is a weak Cauchy sequence in

$$A_p^E(G) = \{u \in A_p(G);\ \operatorname{supp} u \subset E\} \quad \text{with} \quad E = \bar{V}.$$

But by Lemma 18 it follows that $A_p^E(G)$ is weakly sequentially complete for any compact $E \subset G$.

There is hence some $v_0 \in A_p(G)$ such that $\varphi(v_0) = \psi_0(\varphi)$ for all $\varphi \in PM_p$. Thus $\varphi(wv_0) = \varphi(v_0)$ for all φ in PM_p and all w in S_B^p, hence $wv_0 = v_0$. By choosing $w \in S_A^p$ with support in a small neighborhood U of e we get that $v_0(x) = 0$ if $x \neq e$. However $v_0(e) = I(v_0) = \psi_0(I) = 1$. Since $v_0 \in A_p$ is continuous, G is discrete.

The alert reader will have noticed that the above proof will yield a proof for the following

THEOREM 17. *Let G be second countable, $K \subset A_p$ convex, $T_n \in PM_p$ and*

$$A = \{w^* \operatorname{cl} K\} \cap \{\psi \in TIM(\hat{G});\ \psi(T_n) = 0 \text{ for } n \geqslant 1\}.$$

If $A \neq \emptyset$ is norm separable or if A has w^* exposed points then G is discrete.

Proof. By Lemma 3 on p. 23 of [8] we can assume, by possibly adding a countable set to the T_n's, that A has w^* exposed points and by Corollary 1.3, p. 21, of [8] any such ψ_0 is necessarily a weak* sequential limit o elements v_n of K. If $v_0 \in S_A^p$ has compact support then

$$\varphi(v_0 v_n) = (v_0 \cdot \varphi) v_n \to \psi_0(v_0 \cdot \varphi) = \psi_0(\varphi) \quad \text{for all } \varphi \text{ in } PM_p.$$

Thus $v_0 v_n \to \psi_0$ in the w^* topology of PM_p^* and $v_0 v_n \in A_p^E$ with compac $E = \operatorname{supp} v_0$. The rest is shown as in Theorem 16.

Remarks. Theorem 17 is an improvement of a weaker version of Theorem 3.8 of [3]. Chou displays there, for any given $A \neq \emptyset$ as above, a linear isometry Λ of $(l^\infty)^*$ into $PM_2(G)^* = VN(G)^*$ such that $\Lambda(\beta N \sim N) \subset A$, whenever G is nondiscrete (and $p = 2$). It follows by Chou's result that for such A even $\operatorname{card} A \geqslant 2^c$, while we are only able to show here that A is not separable. Our result is however true for all $1 < p < \infty$ and the heavy C^*-algebra machinery developed by Chou in Chapter II and used in Chapter III of [3], to prove his result, is not available in our case anymore. A better result than our Theorem 16 is obtained by Chou in [3], Corollary 3.7 again for the case $p = 2$ and arbitrary G, using C^*-algebra machinery (see footnote on p. 120).

To complete the proof of Theorems 16, 17 we bring here a lemma, whose proof improves a proof due to M. Cowling. The proof is very gratefully acknowledged.

LEMMA 18. *For any compact* $E \in G$, $A_p^E(G) = \{v \in A_p(G);\ \operatorname{supp} v \subset E\}$ *is weakly sequentially complete.*

Proof. If G is discrete, E is finite and A_p^E is finite dimensional. We can hence assume that G is not discrete.

Assume at first that G is second countable. Given $f \in L^p$, $g \in L^{p'}$. Let

$$P(f \otimes g) = g * f^\vee = \int f(y) g(xy)\, dy \qquad \text{where} \qquad f^\vee(y) = f(y^{-1})$$

(see [12], p. 97). Consider $f \otimes g$ as a function on $G \times G$. For $f \in C(G)$ let $(Mf)(x, y) = f(yx^{-1})$. Then $P[(f \otimes g) Mh] = hP(f \otimes g)$ (see [2], p. 276) and $A_p(G) = P(L^p \otimes_\gamma L^{p'})$.

Let U and V be open subsets of G with compact closure such that $\lambda(U)^{-1} P(\chi_U \otimes \chi_V) = 1$ on E where λ denotes Haar measure. For any $h \in A_p^E$,

$$\lambda(U)^{-1} P\big((Mh)\, \chi_U \otimes \chi_V\big) = h.$$

We have thus the following isomorphism of A_p^E into $L^p \otimes L^{p'}(G \times G)_{U \times V}$:

$$h \to [Mh(\chi_U \otimes \chi_V)] \in L^p \otimes_\gamma L^{p'}$$

(see [12], p. 98) and

$$A\|h\|_{A_p} \leqslant \|Mh(\chi_U \otimes \chi_V)\|_{L^p \otimes_\gamma L^{p'}} \leqslant B\|h\|_{A_p} \qquad \text{for some } A, B > 0.$$

This is the case since P is a contraction by [12], p. 98.

However $L^p \otimes L^{p'}$ is weakly sequentially complete, by Theorem 1 of [17]. Therefore, so is every closed subspace. This implies that A_p^E is weakly sequentially complete.

Routine arguments reduce the general case to the second countable case (see [12], p. 106).

REFERENCES

[1] M. Cowling, *An application of Littlewood–Paley theory in harmonic analysis*, Mathematische Annalen 241 (1979), p. 83–96.

[2] – *Some applications of Grothendieck's theory of topological tensor products in harmonic analysis*, ibidem 232 (1978), p. 273–285.

[3] Ching Chou, *Topological invariant means on the von Neumann algebra $VN(G)$*, Transactions of the American Mathematical Society 273 (1982), p. 207–229.

[4] C. F. Dunkl and D. E. Ramirez, *C^*-algebras generated by Fourier–Stieltjes transforms*, ibidem 164 (1972), p. 435–441.

[5] – – *Existence and nonuniqueness of invariant means*, Proceedings of the American Mathematical Society 32 (1972), p. 525–530.

[6] – – *Weakly almost periodic functionals on the Fourier algebra*, Transactions of the American Mathematical Society 185 (1973), p. 501–514.

[7] N. Dunford and J. T. Schwartz, *Linear operators, I*, Interscience, N. Y. 1958.

[8] E. E. Granirer, *Exposed points of convex sets and weak sequential convergence*, Memoirs of the American Mathematical Society 123 (1972).

[9] – *Weakly almost periodic and uniformly continuous functionals on the Fourier algebra of any locally compact group*, Transactions of the American Mathematical Society 189 (1974), p. 371–382.

[10] – *Density theorems for some linear subspaces and some C^*-subalgebras of $VN(G)$*, Symposia of Mathematics, Istituto Nazionale di Alta Matematica, vol. XXII, 1977, p. 61–70.

[11] – *Properties of the set of topological invariant means on P. Eymard's W^* algebra $VN(G)$*, Indagationes Mathematicae 36 (1974), p. 116–121.

[12] C. Herz, *Harmonic synthesis for subgroups*, Annales de l'Institut Fourier (Grenoble) 23 (1973), p. 91–123.

[13] – *Une généralisation de la notion de transformé de Fourier–Stieltjes*, ibidem 24 (1974), p. 145–157.

[14] L. Hörmander, *Estimates for translation invariant operators...*, Annals of Mathematics 104 (1960), p. 93–140.

[15] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Vols. I and II, Springer-Verlag, Berlin 1963 and 1970.

[16] R. Larsen, *The multiplier problem*, Lecture Notes in Mathematics 105, Springer-Verlag.

[17] F. Lust, *Produits tensoriels projectifs d'espaces de Banach faiblement séquentiellement complets*, Colloquium Mathematicum 36 (1976), p. 255–267.

[18] P. F. Renaud, *Invariant means on a class of von Neumann algebras*, Transactions of the American Mathematical Society 170 (1972), p. 285–291.

[19] W. Rudin, *Functional analysis*, Mc Graw-Hill, 1973.

[20] M. Zafran, *The spectra of multiplier transformations on the L^p spaces*, Annals of Mathematics (to appear).

[21] E. E. Granirer, *Geometric and topological properties of certain w^* compact convex subsets of double duals of Banach spaces which arise from the study of invariant means*, Illinois Journal of Mathematics 30 (1986), p. 148–174.

UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, CANADA

Reçu par la Rédaction le 05. 11. 1981