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THE UNIT BALL OF EVERY INFINITE-DIMENSIONAL NORMED LINEAR SPACE CONTAINS A $(1+\varepsilon)$-SEPARATED SEQUENCE

BY

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In this paper* we prove the following result:

THEOREM 1. *If X is an infinite-dimensional normed linear space, then there are an $\varepsilon > 0$ and a sequence $(x_n) \subseteq X$ with $\|x_n\| = 1$ and $\|x_n - x_m\| > 1+\varepsilon$ if $n \neq m$.*

This verifies a conjecture of Kottman [4] who proved Theorem 1 in the case $\varepsilon = 0$. For an infinite-dimensional space X, let

$$\lambda(X) = \sup\{1+\varepsilon:\ \exists (x_n) \subseteq X,\ \|x_n\| = 1 \text{ and } \|x_n - x_m\| > 1+\varepsilon \text{ if } n \neq m\}.$$

Kottman also proved that if X is isomorphic to l_ϱ $(1 \leqslant \varrho < \infty)$, then $\lambda(X) \geqslant 2^{1/\varrho}$, while if X is isomorphic to c_0, then $\lambda(X) = 2$. Since Tsirelson [1] has shown that there exist infinite-dimensional Banach spaces that contain no isomorph of c_0 or l_ϱ $(1 \leqslant \varrho < \infty)$, one possible method for proving Theorem 1 is eliminated. Our approach shall be to focus on the question of whether or not X contains c_0.

We shall always assume (as we clearly may) that X is an infinite-dimensional Banach space.

We begin with the following lemma due to W. B. Johnson. We wish to thank Professor Johnson for allowing us to reproduce here his result.

LEMMA 1. *Let (x_n) be a normalized basic sequence in X such that, for all infinite $M \subseteq N$, there is a subsequence $L = (l_i)$ of M such that*

$$\sup_k \Big\| \sum_{i=1}^{k} (-1)^i x_{l_i} \Big\| < \infty. \tag{1}$$

Then X contains an isomorph of c_0.

Proof. By a standard application of the combinatorial result that Borel sets are Ramsey [3] we infer that there is a subsequence (m_i) of N, so that if (l_i) is a subsequence of M, then (1) holds. Thus if $y_i = x_{m_{2i}} - x_{m_{2i+1}}$,

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then for all subsequences (j_i) of N we have

$$\sup_k \Big\| \sum_{i=1}^{k} y_{j_i} \Big\| < \infty .$$

It is well known that this implies that (y_i) is equivalent to the unit vector basis of c_0.

Proof of Theorem 1. Let (x_n) be a normalized basic sequence in X which is *asymptotically monotone*. By this we mean for all n and scalars $(a_i)_{i=1}^{\infty}$

$$\Big\| \sum_{i=1}^{n} a_i x_i \Big\| \leqslant (1+20^{-n}) \Big\| \sum_{i=1}^{\infty} a_i x_i \Big\| .$$

If X contains c_0, we are done, so assume that X does not contain c_0. Then by Lemma 1 we may assume (by passing to a subsequence if necessary) that

$$\sup_k \Big\| \sum_{i=1}^{k} (-1)^i x_{m_i} \Big\| = \infty \quad \text{for all } (m_i) \subseteq N. \tag{2}$$

Notation. Let a be a limit point of the real sequence

$$(1/\|x_n - x_{n+1} + x_{n+2}\|)_{n=1}^{\infty} .$$

Of course, $1/3 \leqslant a \leqslant 1$.

For $\delta > 0$ we call an element $b \in X$ a *δ-block* of (x_n) if

$$b = \beta \sum_{i=1}^{l} (-1)^{i+1} x_{m_i} \quad \text{with } \|b\| = 1,$$

$m_1 < m_2 < \ldots < m_l$, $|a/\beta - 1| < \delta$ and $l \geqslant 3$ is odd.

Also we shall write expressions like $n < b_1 < b_2 < \ldots < b_k$ if

$$b_i = \sum_{j=p_i+1}^{p_{i+1}} a_j x_j \quad \text{with } n \leqslant p_1 < p_2 < \ldots < p_{k+1} .$$

Note that, by the definition of a, for all n and all $\delta > 0$ there is a δ-block b with $n < b$.

The method of proof will be to consider the technical condition (*) below and show that if (*) holds, then (1) is contradicted, while if (*) is false, then the conclusion of the theorem is true.

(*) For all $\delta > 0$ and all $n \in N$, there exist δ-blocks $(b_i)_{i=1}^{k}$ with $n < b_1 < b_2 < \ldots < b_k$ such that if b is a δ-block with $b_k < b$, then there exists an i, $1 \leqslant i \leqslant k$, such that $\|b_i - b\| \leqslant 1 + \delta$.

The negation of (*) is

(not*) There exist $\delta > 0$ and $n \in N$ such that for all δ-blocks $(b_i)_{i=1}^k$ with $n < b_1 < b_2 < \ldots < b_k$ there is a δ-block $b > b_k$ such that, for all $1 \leqslant i \leqslant k$, $\|b_i - b\| > 1+\delta$.

If (not*) holds, then an easy induction argument yields δ-blocks $b_1 < b_2 < \ldots$ such that $\|b_i - b_j\| \geqslant 1+\delta$ for $i \neq j$.

Thus we assume that (*) holds. Let $\delta_j = 20^{-j}$ and inductively choose δ_j-blocks $(b_i^j)_{i=1}^{k_j}$ such that $b_1^1 < b_2^1 < \ldots < b_{k_1}^1 < b_1^2 < \ldots$ and if b is a δ_j-block with $b > b_{k_j}^j$, then there exists an i, $1 \leqslant i \leqslant k_j$, such that $\|b_i^j - b\| < 1+\delta_j$.

Let

$$d_i^j = \frac{a}{\beta_i^j} b_i^j, \quad \text{where } b_i^j = \beta_i^j \sum_k (-1)^{k+1} x_{m_k}.$$

CLAIM. *There is a sequence* $(d_{m_j}^j)_{j=1}^\infty$ *such that*

$$\sup_k \Big\| \sum_{j=1}^k (-1)^j d_{m_j}^j \Big\| < \infty. \tag{3}$$

The proof of Theorem 1 will be complete once we prove the Claim since this clearly contradicts (2).

Notation. For $d \in \operatorname{span}(x_n)$ and $j \in N$ let $(d)_j$ be the element of $\operatorname{span}(x_n)$ obtained from d by deleting the last j non-zero terms of its expansion. Thus $(x_1 + x_7 - x_9 + x_{12})_2 = x_1 + x_7$.

We now prove the Claim. Fix an even positive integer l and let i_l, $1 \leqslant i_l \leqslant k_l$, be arbitrary. Then there exists i_{l-1}, $1 \leqslant i_{l-1} \leqslant k_{l-1}$, such that

$$\|b_{i_{l-1}}^{l-1} - b_{i_l}^l\| \leqslant 1 + \delta_{l-1}.$$

Now,

$$\|(d_{i_{l-1}}^{l-1} - d_{i_l}^l) - (b_{i_{l-1}}^{l-1} - b_{i_l}^l)\| \leqslant \left| \frac{a}{\beta_{i_{l-1}}^{l-1}} - 1 \right| + \left| \frac{a}{\beta_{i_l}^l} - 1 \right| < \delta_{l-1} + \delta_l < 2\delta_{l-1}.$$

Thus $\|d_{i_{l-1}}^{l-1} - d_{i_l}^l\| < 1 + 3\delta_{l-1}$, and so

$$\|(d_{i_{l-1}}^{l-1} - d_{i_l}^l)_1\| < (1+3\delta_{l-1})(1+20^{-(l-1)}) < 1+\delta_{l-2}.$$

Also

$$\|(d_{i_{l-1}}^{l-1} - d_{i_l}^l)_1\| \geqslant \frac{\|d_{i_{l-1}}^{l-1}\|}{1+20^{-(l-1)}} > 1 + \delta_{l-2}.$$

We proceed by induction. Assume that $1 \leqslant j' \leqslant l-2$ and that we have found $1 \leqslant i_j \leqslant k_j$ for $j = l-j', l-j'+1, \ldots, l$ such that if

$$z_{j'} = \Big(\sum_{j=l-j'}^{l} (-1)^{l-j'+j} d_{i_j}^j \Big)_{j'},$$

then $1 - \delta_{l-(j'+1)} < \|z_{j'}\| < 1 + \delta_{l-(j'+1)}$. This implies that $z_{j'}/\|z_{j'}\|$ is a

$\delta_{l-(j'+1)}$-block (note that $z_{j'}$ has odd support size). Thus there exists an $i_{l-(j'+1)}$, $1 \leqslant i_{l-(j'+1)} \leqslant k_{l-(j'+1)}$, such that

$$\left\| b^{l-(j'+1)}_{i_{l-(j'+1)}} - \frac{z_{j'}}{\|z_{j'}\|} \right\| \leqslant 1 + \delta_{l-(j'+1)}.$$

Also,

$$\left\| \Big(\sum_{j=l-(j'+1)}^{l} (-1)^{l-(j'+1)+j} d^j_{i_j} \Big)_{j'} - \Big(b^{l-(j'+1)}_{i_{l-(j'+1)}} - \frac{z_{j'}}{\|z_{j'}\|} \Big) \right\|$$

$$= \left\| (d^{l-(j'+1)}_{i_{l-(j'+1)}} - b^{l-(j'+1)}_{i_{l-(j'+1)}}) - \Big(\Big(\sum_{j=l-j'}^{l} (-1)^{l-j'+j} d^j_{i_j} \Big)_{j'} - \frac{z_{j'}}{\|z_{j'}\|} \Big) \right\|$$

$$\leqslant \left| \frac{a}{\beta^{l-(j'+1)}_{i_{l-(j'+1)}}} - 1 \right| + \left\| z_{j'} - \frac{z_{j'}}{\|z_{j'}\|} \right\| < 2\delta_{l-(j'+1)}.$$

So

$$\left\| \Big(\sum_{j=l-(j'+1)}^{l} (-1)^{l-(j'+1)+j} d^j_{i_j} \Big)_{j'+1} \right\|$$
$$< (1+3\delta_{l-(j'+1)})(1+20^{-(l-(j'+1))}) < 1+\delta_{l-(j'+2)}.$$

It follows that if

$$z_{j'+1} = \Big(\sum_{j=l-(j'+1)}^{l} (-1)^{l-(j'+1)+j} d^j_{i_j} \Big)_{j'+1},$$

then

$$1 - \delta_{l-(j'+2)} < \|z_{j'+1}\| < 1 + \delta_{l-(j'+2)}.$$

If we set $j'+1 = l-1$, we get

$$\left\| \Big(\sum_{j=1}^{l} (-1)^j d^j_{i_j} \Big)_{l-1} \right\| < 2,$$

and since each $d^j_{i_j}$ has support size at least 3 and we are deleting only $l-1$ terms, we get

$$\left\| \sum_{j=1}^{l/2} (-1)^{j+1} d^j_{i_j} \right\| < 2(1+20^{-l}) < 3. \tag{4}$$

Now the $d^j_{i_j}$'s in (4) depend upon the fixed even l with which we began the argument above. We will now write $i_{l,j}$ instead of i_j to note this dependence. The set $\{i_{l,1} : l \in N, l \text{ even}\}$ has cardinality less than or equal to k_1, and so there is an infinite set L_1 of positive even integers and $1 \leqslant i_1 \leqslant k_1$ such that $i_{l,1} = i_1$ for all $l \in L_1$. Continuing in this way, we get a sequence of infinite sets $L_1 \supset L_2 \supset \ldots$ and $1 \leqslant i_j \leqslant k_j$ such that if $k \in N$ and $j' \leqslant k$, then $i_{l,j'} = i_{j'}$ for all $l \in L_k$.

Let l_k be the k-th element of L_k. Then $k \leqslant l_k/2$ and, for all $k \in N$,

$$\Big\| \sum_{j=1}^{k} (-1)^{j+1} d_{i_j}^{j} \Big\| \leqslant \Big\| \sum_{j=1}^{l_{k/2}} (-1)^{j+1} d_{i_{l_k},j}^{j} \Big\| (1+20^{-1}) < 3(1+20^{-1}).$$

This proves the Claim, and hence Theorem 1.

Remarks. (1) The proof of Theorem 1 shows that if (x_n) is any weakly null normalized sequence in a space X that does not contain c_0, then there are an $\varepsilon > 0$ and a normalized block (b_n) of (x_n) with $\|b_n - b_m\| > 1+\varepsilon$ for $n \neq m$.

(2) The non-separable analogue of Theorem 1 is false. For example, if $X = c_0(\Gamma)$, where Γ is uncountable and $(x_a)_{a\in A}$ is a set of norm 1 elements in X with $\|x_a - x_\beta\| > 1+\varepsilon$ for $a \neq \beta$, then A must be countable. For suppose A is uncountable. If $x \in c_0(\Gamma)$ and $\delta \geqslant 0$, then let

$$S_\delta(x) = \{\gamma \in \Gamma: |x(\gamma)| > \delta\}.$$

For $a \in A$ let $y_a(\gamma) = x_a(\gamma)$ if $\gamma \in S_\varepsilon(x_a)$ and $y_a(\gamma) = 0$ otherwise. Then $(y_a)_{a\in A}$ is a set of norm 1 elements in X with $\|y_a - y_\beta\| > 1+\varepsilon$ for $a \neq \beta$ and $S_0(y_a)$ is finite for each a. Thus there are an uncountable subset $A_1 \subset A$ and a finite $F \subset \Gamma$ such that if $a \neq \beta \in A_1$, then $S_0(y_a) \cap S_0(y_\beta) = F$ (see [2]). Since A_1 is infinite, there exists $a \neq \beta \in A_1$ so that $\|y_a - y_\beta\| \leqslant 1$ which is a contradiction.

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