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Value distribution of regular functions of two complex variables*

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1. Introduction. The study of the value distribution of a regular function of two complex variables in a given domain in the space C^2 of two complex variables is of a different character to the analogous problem in one variable. Riemann's mapping theorem shows that any simply connected plane domain is conformally equivalent to either the open plane or the unit disk. The absence of any comparable theorem in C^2 produces the problem of finding suitable domains in C^2 with which to work and in which techniques from the classical one variable study can be used. Such a type of domain (analytic polyhedron) was introduced by Bergman, and on the three dimensional boundary of an analytic polyhedron, $\mathfrak{M}$, lies the two dimensional distinguished boundary, $\mathfrak{F}^2$(1) (Bergman-Šilov), on which any function regular in $\overline{\mathfrak{M}}$ must assume the maximum of its absolute value. A function regular in $\overline{\mathfrak{M}}$ can be represented in terms of its values on $\mathfrak{F}^2$ by (i) generalizing the Cauchy formula (Bergman, Weil(2)) or (ii) generalizing Szegö's orthonormal boundary functions (Bergman [1]). Consequently upper bounds for the absolute value of a function regular in $\overline{\mathfrak{M}}$ can be given in terms of its values on $\mathfrak{F}^2$. We consider in this paper the relations between the value distribution of a regular function in $\mathfrak{M}$ and on the part of $\partial\mathfrak{M}$(3) complementary to $\mathfrak{F}^2$.

The three dimensional boundary, $\mathfrak{m}^3$, of $\mathfrak{M}$ consists of a finite number of segments of analytic hypersurfaces. An analytic hypersurface is a one parameter family of analytic surfaces called *laminas*. If we suppose that $f(z_1, z_2)$ considered as a function of one complex variable in each lamina belongs to a given normal family, then bounds for $|f(z_1, z_2)|$ in a general analytic surface, $\mathfrak{A}_0^2$, meeting $\mathfrak{M}$, can be given in terms of the values of $f(z_1, z_2)$ on $\mathfrak{m}^3-\mathfrak{F}^2$, under suitable extra hypotheses (Berg-

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(1) German letters denote manifolds, and the superscripts 1, 2, 3 indicate the dimension.

(2) See [1], for literature.

(3) $\partial\mathfrak{M}$ denotes the boundary of $\mathfrak{M}$.

man [1], [2], Charzyński [3], Śladkowska [5]). It is the nature of these extra hypotheses that concerns us here. Charzyński assumes that $\mathfrak{A}_0^2$ meets only one of the analytic hypersurfaces which constitute $\mathfrak{m}^3$ (see also [1], p. 186). In general, the analytic surface $\mathfrak{A}_0^2$ meets $\mathfrak{m}^3$ in a closed curve $\mathfrak{a}_0^1$ and each point of $\mathfrak{a}_0^1$ belongs to one or more of the analytic hypersurfaces which constitute $\mathfrak{m}^3$. Points lying on both $\mathfrak{a}_0^1$ and $\mathfrak{F}^2$ lead to difficulties when the Poisson formula is applied, and Śladkowska introduces geometrical hypotheses on the behavior of $\mathfrak{A}_0^2 \cap \mathfrak{M}$ near such points to overcome this (see also [1], p. 189). We shall show firstly how to make extra hypotheses on the function class, instead of the domain, to get over this problem. Secondly the bounds we give for $|f(z_1, z_2)|$ will be in terms of its values on a one dimensional boundary set, which need not be connected. The ideas will be illustrated by assuming that $f(z_1, z_2)$ omits two distinct values in each lamina. We note here that Bergman [2] has given bounds for $|f(z_1, z_2)|$ in terms of its value at a single boundary point for a more special type of domain.

2. Definitions. (Analytic polyhedra and analytic surfaces.)

HYPOTHESIS 1. *The boundary $\mathfrak{m}^3$ of the analytic polyhedron $\mathfrak{M}$ consists of finitely many closed segments $\bar{\mathfrak{e}}_k^3$ $(k = 1, \ldots, l)$ of analytic hypersurfaces.*

Further

$$\bar{\mathfrak{e}}_k^3 = \bigcup_{\lambda_k=0}^{1} \bar{\mathfrak{J}}_k^2(\lambda_k) \qquad (k = 1, \ldots, l)$$

and $\mathfrak{J}_k^2(\lambda_k)$ is, for fixed λ_k, a domain lying in the analytic surface

$$\Phi_k(z_1, z_2, \lambda_k) = 0,$$

where Φ_k is a regular function of (z_1, z_2) in $\overline{\mathfrak{M}}$ and is continuously differentiable in λ_k.

HYPOTHESIS 2. *We assume that each lamina $\bar{\mathfrak{J}}_k^2(\lambda_k)$ can be uniformized so that to each $\Phi_k(z_1, z_2, \lambda_k)$ we associate two continuously differentiable functions*

$$z_1 = h_{k1}(Z_k, \lambda_k), \qquad z_2 = h_{k2}(Z_k, \lambda_k) \qquad (|Z_k| \leqslant 1)$$

mapping the closed unit Z_k-disk in a (1-1) manner onto $\bar{\mathfrak{J}}_k^2(\lambda_k)$ such that

$$\Phi_k\big(h_{k1}(Z_k, \lambda_k), h_{k2}(Z_k, \lambda_k), \lambda_k\big) = 0 \qquad (|Z_k| \leqslant 1).$$

For each fixed λ_k, $h_{kj}(Z_k, \lambda_k)$ $(j = 1, 2)$ are assumed regular in $|Z_k| < 1$. Thus

(1) $$\bar{\mathfrak{e}}_k^3 = \big\{(z_1, z_2):\ z_1 = h_{k1}(Z_k, \lambda_k), z_2 = h_{k2}(Z_k, \lambda_k);\ |Z_k| \leqslant 1, 0 \leqslant \lambda_k \leqslant 1\big\}.$$

HYPOTHESIS 3. *Suppose that* $\overline{\mathfrak{I}}_k^2(\lambda_k') \cap \overline{\mathfrak{I}}_k^2(\lambda_k'') = \emptyset$, *if* $\lambda_k' \neq \lambda_k''$. *Also, for every* λ_{k0}, Z_{k0} $(0 < \lambda_{k0} < 1\colon |Z_{k0}| < 1;\ k = 1, \ldots, l)$, *and for sufficiently small* $\varepsilon > 0$, *the set of points* (1) *for which* $|Z - Z_{k0}| < \varepsilon$, $|\lambda_k - \lambda_{k0}| < \varepsilon$ *shall include all points of* $\mathfrak{m}^3$ *near enough to* $(h_{k1}(Z_{k0}, \lambda_{k0}), h_{k2}(Z_{k0}, \lambda_{k0}))$.

Let $\mathfrak{A}_0^2$ be a segment of an analytic surface in C^2, that is

$$\mathfrak{A}_0^2 = \{(z_1, z_2)\colon z_1 = g_1(\zeta),\ z_2 = g_2(\zeta),\ \zeta \in D\},$$

where D is a simply connected domain in the ζ-plane which we assume contains the closure of the unit ζ disk in its interior, and $g_j(\zeta)$ $(j = 1, 2)$ are regular in D. We assume that $\partial\mathfrak{A}_0^2$ lies in $C^2 - \mathfrak{M}$, and make the following hypotheses on $\mathfrak{A}^2 = \mathfrak{A}_0^2 \cap \mathfrak{M}$.

HYPOTHESIS 4. $\mathfrak{A}^2 = \{(z_1, z_2)\colon z_1 = g_1(\zeta),\ z_2 = g_2(\zeta),\ |\zeta| < 1\}$. *The boundary curve* $\mathfrak{a}^1$ *of* $\mathfrak{A}^2$ *is* $\mathfrak{A}_0^2 \cap \mathfrak{m}^3$, *and we suppose that*

$$\mathfrak{a}^1 = \{(z_1, z_2)\colon z_1 = g_1(e^{i\psi}), z_2 = g_2(e^{i\psi}), 0 \leqslant \psi \leqslant 2\pi\}$$

can be divided into P *parts*, $\mathfrak{a}_j^1$ $(j = 1, \ldots, P)$ *given by a* ψ-*range*, $\psi_j \leqslant \psi \leqslant \psi_{j+1}$ $(j = 1, \ldots, P\colon \psi_1 \equiv \psi_{P+1})$ *so that* $\mathfrak{a}_j^1 \subset \bar{e}_{k(j)}^3$, *for some* $k(j)$. *The end points of* $\mathfrak{a}_j^1$, *corresponding to* $\psi = \psi_j, \psi_{j+1}$, *lie on the distinguished boundary and we assume that no other points of* $\mathfrak{a}_j^1$ *belong to* $\mathfrak{F}^2$.

The last assumption is only for the sake of simplicity.

HYPOTHESIS 5. *We know that each point of* $\mathfrak{a}_j^1$ *lies in a certain lamina*

$$\mathfrak{I}_{k(j)}^2(\lambda_{k(j)}) = \{(z_1, z_2)\colon z_1 = h_{k(j)1}(Z_{k(j)}, \lambda_{k(j)}), z_2 = h_{k(j)2}(Z_{k(j)}, \lambda_{k(j)})\}$$

and so we can find functions $\lambda_{k(j)}(\psi), Z_{k(j)}(\psi)$ $(\psi_j \leqslant \psi \leqslant \psi_{j+1})$ *such that*

$$\mathfrak{a}_j^1 = \{(z_1, z_2)\colon z_1 = h_{k(j)1}(Z_{k(j)}(\psi), \lambda_{k(j)}(\psi)),$$
$$z_2 = h_{k(j)2}(Z_{k(j)}(\psi), \lambda_{k(j)}(\psi));\ \psi_1 \leqslant \psi \leqslant \psi_2\}.$$

3. Definition of the function class $\mathfrak{S}(\mathfrak{M}, \mathfrak{A}_0^2)$.

HYPOTHESIS 6. *Suppose* $f(z_1, z_2)$ *is regular in* $\mathfrak{M}$ *and on almost every lamina* $\mathfrak{I}_k^2(\lambda_k)$ $(0 \leqslant \lambda_k \leqslant 1;\ k = 1, \ldots, l)$,

$$f(z_1, z_2) = f(h_{k1}(Z_k, \lambda_k), h_{k2}(Z_k, \lambda_k)),$$

considered as a function in the unit Z_k-*disk, omits the two distinct values* $u(k, \lambda_k)$ *and* $v(k, \lambda_k)$, *where we suppose* $|u(k, \lambda_k)| \leqslant |v(k, \lambda_k)|$.

We have seen in Hypothesis 5 that $\mathfrak{a}^1 = \bigcup_{j=1}^{P} \mathfrak{a}_j^1$, where $\mathfrak{a}_j^1 \subset e_{k(j)}^3$ except for its endpoints. The final hypothesis demands that $f(z_1, z_2)$ satisfy a slightly stronger condition in the laminas of $\mathfrak{m}^3$ which are "close to" the laminas in which $\mathfrak{a}^1$ meets $\mathfrak{F}^2$.

HYPOTHESIS 7. *There are positive numbers* ε_0, δ_j $(j = 1, \ldots, P\colon \delta_1 \equiv \delta_{p+1})$ *such that*

(i) *for fixed* $\lambda_{k(j)}(\psi)$ $(\psi_j - \delta_j \leqslant \psi \leqslant \psi_j + \delta_j)$, *the function*

$$f\left(h_{k(j)1}\left(Z_{k(j)}(\psi), \lambda_{k(j)}(\psi)\right), h_{k(j)2}\left(Z_{k(j)}(\psi), \lambda_{k(j)}(\psi)\right)\right) \tag{2}$$

is regular in the disk $|Z_{k(j)}(\psi)| \leqslant 1 + \varepsilon_0$;

(ii) *function* (2) *omits the values* $u\left(k(j), \lambda_{k(j)}(\psi)\right)$ *and* $v\left(k(j), \lambda_{k(j)}(\psi)\right)$ *in the disk* $|Z_{k(j)}(\psi)| \leqslant 1 + \varepsilon_0$.

DEFINITION. If the above hypotheses are all satisfied, we say that $f \in \mathfrak{S}(\mathfrak{M}, \mathfrak{A}_0^2)$.

4. Bounds for $f(z_1, z_2) \in \mathfrak{S}(\mathfrak{M}, \mathfrak{A}_0^2)$. We now give bounds for $|f(z_{10}, z_{20})|$, when $(z_{10}, z_{20}) \in \mathfrak{A}^2$, in terms of the values of $f(z_1, z_2)$ on a one dimensional boundary set $\mathfrak{b}^1$. For the laminas of $\bar{e}_k^3$ $(k = 1, \ldots, l)$, we suppose, for a fixed λ_k $(0 \leqslant \lambda_k \leqslant 1)$, that

$$f\left(h_{k1}\left(Z_k^0(\lambda_k), \lambda_k\right), h_{k2}\left(Z_k^0(\lambda_k), \lambda_k\right)\right) = \omega(k, \lambda_k), \quad \text{where } |Z_k^0(\lambda_k)| < 1.$$

We assume further that if the lamina λ_k corresponds to a value ψ, of the parameter describing $\mathfrak{a}^1$, for which $\psi \in \bigcup_{j=1}^{P} [\psi_j + \delta_j, \psi_{j+1} - \delta_{j+1}]$, then $|Z_k^0(\lambda_k)| \leqslant r_0 < 1$. The set of points

$$\mathfrak{b}^1 = \left\{(z_1, z_2)\colon z_1 = h_{k1}\left(Z_k^0(\lambda_k), \lambda_k\right),\right.$$
$$\left. z_2 = h_{k2}\left(Z_k^0(\lambda_k), \lambda_k\right);\ 0 \leqslant \lambda_k \leqslant 1,\ k = 1, \ldots, l\right\}.$$

We fix a point $G = \left(g_1(e^{i\psi}), g_2(e^{i\psi})\right)$ $(\psi \neq \psi_j, \psi_{j+1})$ of $\mathfrak{a}_j^1$ and give an upper bound for $\left|f\left(g_1(e^{i\psi}), g_2(e^{i\psi})\right)\right|$. Set

$$u(\psi) = u\left(k(j), \lambda_{k(j)}(\psi)\right), \quad v(\psi) = v\left(k(j), \lambda_{k(j)}(\psi)\right).$$

The point G lies in the lamina $\bar{\mathfrak{J}}_{k(j)}^2\left(\lambda_{k(j)}(\psi)\right)$ in which $f(z_1, z_2)$ (when considered as a function in the unit $Z_{k(j)}(\psi)$ disk) omits $u(\psi), v(\psi)$ and assumes the value

$$\omega(\psi) \equiv \omega\left(k(j), \lambda_{k(j)}(\psi)\right) \quad \text{when } Z_{k(j)}(\psi) = Z_{k(j)}^0\left(\lambda_{k(j)}(\psi)\right) \equiv Z_{k(j)}^0(\psi).$$

We put

$$\tilde{Z}\left(Z_{k(j)}(\psi), \psi\right) = \frac{Z_{k(j)}(\psi) + Z_{k(j)}^0(\psi)}{1 + Z_{k(j)}(\psi)\overline{Z_{k(j)}^0(\psi)}},$$

and consider

$$F\left(Z_{k(j)}(\psi), \psi\right) = \frac{f\left(\tilde{Z}\left(Z_{k(j)}(\psi), \psi\right)\right) - u(\psi)}{v(\psi) - u(\psi)}.$$

Then $F(Z_{k(j)}(\psi), \psi)$ is regular in $|Z_{k(j)}(\psi)| < 1$ and omits the values 0, 1 there.

Hence, by Schottky's theorem [4],

$$\log|f(\tilde{Z}(Z_{k(j)}(\psi), \psi)) - u(\psi)| \leqslant \log|v(\psi) - u(\psi)| + \frac{1 + |\tilde{Z}(Z_{k(j)}(\psi), \psi)|}{1 - |\tilde{Z}(Z_{k(j)}(\psi), \psi)|}\{\pi + \log\mu(\psi)\},$$

for $|Z_{k(j)}(\psi)| < 1$, and where

$$\mu(\psi) = \max\left(1, \left|\frac{\omega(\psi) - u(\psi)}{v(\psi) - u(\psi)}\right|\right).$$

Thus

$$\log|f(g_1(e^{i\psi}), g_2(e^{i\psi}))| \leqslant \log^+|v(\psi) - u(\psi)| + \log^+|u(\psi)| + \log 2 + \frac{1 + |\tilde{Z}(\psi)|}{1 - |\tilde{Z}(\psi)|}\{\pi + \log\mu(\psi)\}, \tag{3}$$

where $\tilde{Z}(\psi) = \tilde{Z}(Z^0_{k(j)}(\psi), \psi)$ $(0 \leqslant \psi \leqslant 2\pi;\ \psi \neq \psi_j,\ j = 1, \ldots, P)$.

Estimate (3) may not be adequate to give an upper bound for $|f(z_{10}, z_{20})|$, $(z_{10}, z_{20}) \in \mathfrak{U}^2$, since the second term may diverge rapidly as $\psi \to \psi_j$ so that (3) when integrated over ψ will become infinite. To overcome this we use Hypothesis 7 instead of Hypothesis 6 to give an alternative estimate for $\log|f(g_1(e^{i\psi}), g_2(e^{i\psi}))|$ when ψ lies in an interval of the type $[\psi_j - \delta_j, \psi_j + \delta_j]$.

For $\psi \in \bigcup_{j=1}^{P} [\psi_j - \delta_j, \psi_j + \delta_j]$, we now put

$$\tilde{\tilde{Z}}(Z_{k(j)}(\psi), \psi) = \frac{(1 + \varepsilon_0)(Z_{k(j)}(\psi) + Z^0_{k(j)}(\psi))}{(1 + \varepsilon_0)^2 + Z_{k(j)}(\psi)\overline{Z^0_{k(j)}}(\psi)},$$

$$\tilde{\tilde{Z}}(\psi) = \tilde{\tilde{Z}}(Z^0_{k(j)}(\psi), \psi),$$

and obtain

$$\log|f(g_1(e^{i\psi}), g_2(e^{i\psi}))| \leqslant \log^+|v(\psi) - u(\psi)| + \log^+|u(\psi)| + \log 2 + \frac{1 + |\tilde{\tilde{Z}}(\psi)|}{1 - |\tilde{\tilde{Z}}(\psi)|}\{\pi + \log\mu(\psi)\}. \tag{4}$$

The function $\log|f(g_1(\zeta), g_2(\zeta))|$ is subharmonic in $|\zeta| < 1$ and if we denote by $P(\psi, \zeta_0)$ the Poisson kernel $\mathrm{Re}\left\{\frac{e^{i\psi} + \zeta_0}{e^{i\psi} - \zeta_0}\right\}$, we obtain, using (3), (4),

(5) $$\log|f(z_{10}, z_{20})| = \log\left|f\big(g_1(\zeta_0), g_2(\zeta_0)\big)\right|$$

$$\leqslant \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\big(g_1(e^{i\psi}), g_2(e^{i\psi})\big)\right| P(\psi, \zeta_0)\,d\psi$$

$$\leqslant \frac{1}{2\pi}\int_0^{2\pi} \left\{\log^+|v(\psi)-u(\psi)|+\log^+|u(\psi)|+\log 2\right\} P(\psi, \zeta_0)\,d\psi+$$

$$+\sum_{j=1}^{P}\frac{1}{2\pi}\int_{\psi_j+\delta_j}^{\psi_{j+1}-\delta_{j+1}} \frac{1+|\tilde{Z}(\psi)|}{1-|\tilde{Z}(\psi)|}\big(\pi+\log\mu(\psi)\big)P(\psi, \zeta_0)\,d\psi+$$

$$+\sum_{j=1}^{P}\frac{1}{2\pi}\int_{\psi_j-\delta_j}^{\psi_j+\delta_j} \min\left\{\frac{1+|\tilde{Z}(\psi)|}{1-|\tilde{Z}(\psi)|}, \frac{1+|\tilde{\tilde{Z}}(\psi)|}{1-|\tilde{\tilde{Z}}(\psi)|}\right\}\big(\pi+\log\mu(\psi)\big)P(\psi, \zeta_0)\,d\psi.$$

But, if $\psi\in\bigcup_{j=1}^{P}[\psi_j-\delta_j, \psi_j+\delta_j]$, then

$$|\tilde{\tilde{Z}}(\psi)|\leqslant\frac{(1+\varepsilon_0)\big(1+|Z^0_{k(j)}(\psi)|\big)}{(1+\varepsilon_0)^2+|Z^0_{k(j)}(\psi)|}, \quad 1-|\tilde{\tilde{Z}}(\psi)|\geqslant\frac{\varepsilon_0\big(1+\varepsilon_0-|Z^0_{k(j)}(\psi)|\big)}{(1+\varepsilon_0)^2+|Z^0_{k(j)}(\psi)|}\geqslant\frac{\varepsilon_0^2}{3}$$

(ε_0 can be taken sufficiently small).

Next, the points of $\mathfrak{a}^1$ corresponding to $\psi\in\bigcup_{j=1}^{P}[\psi_j+\delta_j, \psi_{j+1}-\delta_{j+1}]$ will lie in the disk $|Z_k(\psi)|\leqslant R_0<1$, where R_0 depends on $\mathfrak{m}^3, \delta_1, \ldots, \delta_P$ but can be taken independent of ψ on the set $\bigcup_{j=1}^{P}[\psi_j+\delta_j, \psi_{j+1}-\delta_{j+1}]$.

Thus

$$|\tilde{Z}(\psi)|\leqslant\frac{R_0+r_0}{1+r_0R_0}, \quad 1-|\tilde{Z}(\psi)|\geqslant\frac{(1-r_0)(1-R_0)}{2}.$$

Thus we have established, with the above notation,

THEOREM. *If* $f(z_1, z_2)\in\mathfrak{S}(\mathfrak{M}, \mathfrak{A}_0^2)$, *and if* $(z_{10}, z_{20}) = \big(g_1(\zeta_0), g_2(\zeta_0)\big)\in\mathfrak{M}\cap\mathfrak{A}_0^2$, *then*

$$\log|f(z_{10}, z_{20})|\leqslant\frac{1}{2\pi}\int_0^{2\pi}\{\log^+|v(\psi)-u(\psi)|+\log^+|v(\psi)|+\log 2\}P(\psi, \zeta_0)\,d\psi+$$

$$+\sum_{j=1}^{P}\frac{1}{\pi}\left\{\frac{2}{(1-r_0)(1-R_0)}\int_{\psi_j+\delta_j}^{\psi_{j+1}-\delta_{j+1}}\big(\pi+\log\mu(\psi)\big)P(\psi, \zeta_0)\,d\psi+\right.$$

$$\left.+\frac{3}{\varepsilon_0^2}\int_{\psi_j-\delta_j}^{\psi_j+\delta_j}\big(\pi+\log\mu(\psi)\big)P(\psi, \zeta_0)\,d\psi\right\}.$$

5. Notes on the theorem.

A. If we suppose that $f(z_1, z_2) \neq 0$ in $\overline{\mathfrak{M}}$, then we may set

$$u(k, \lambda_k) = 0 \quad (0 \leqslant \lambda_k \leqslant 1, k = 1, \ldots, l).$$

The function $[f(z_1, z_2)]^{-1} \in \mathfrak{S}(\mathfrak{M}, \mathfrak{A}_0^2)$ and hence we deduce

COROLLARY. *If the hypotheses of the theorem are satisfied and if, in addition, $f(z_1, z_2) \neq 0$ in $\overline{\mathfrak{M}}$, then*

$$-\sum_{j=1}^{P} \frac{1}{\pi} \left\{ \frac{2}{(1-r_0)(1-R_0)} \int_{\psi_j+\delta_j}^{\psi_{j+1}-\delta_{j+1}} \{\pi + \log \nu(\psi)\} P(\psi, \zeta_0)\, d\psi + \right.$$

$$\left. + \frac{3}{\varepsilon_0^2} \int_{\psi_j-\delta_j}^{\psi_j+\delta_j} \{\pi + \log \nu(\psi)\} P(\psi, \zeta_0)\, d\psi \right\} + \frac{1}{2\pi} \int_0^{2\pi} \log |v(\psi)| P(\psi, \zeta_0)\, d\psi$$

$$\leqslant \log |f(z_{10}, z_{20})| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \log |v(\psi)| P(\psi, \zeta_0)\, d\psi +$$

$$+ \sum_{j=1}^{P} \frac{1}{\pi} \left\{ \frac{2}{(1-r_0)(1-R_0)} \int_{\psi_j+\delta_j}^{\psi_{j+1}-\delta_{j+1}} \{\pi + \log \mu(\psi)\} P(\psi, \zeta_0)\, d\psi + \right.$$

$$\left. + \frac{3}{\varepsilon_0^2} \int_{\psi_j-\delta_j}^{\psi_j+\delta_j} \{\pi + \log \mu(\psi)\} P(\psi, \zeta_0)\, d\psi \right\},$$

where

$$\mu(\psi) = \max \left\{ 1, \left| \frac{\omega(k, \lambda_k(\psi))}{v(\psi)} \right| \right\}, \qquad \nu(\psi) = \max \left\{ 1, \left| \frac{v(\psi)}{\omega(k, \lambda_k(\psi))} \right| \right\}.$$

We note that the $\log 2$ term in the theorem is not required for the corollary.

B. Suppose $f(z_1, z_2)$ satisfies similar hypotheses to those of the theorem in each of a sequence of analytic polyhedra $\{\mathfrak{M}_n\}_1^\infty$ for which $\partial \mathfrak{M}_n \subset \mathfrak{M}_{n+1}$ $(n \geqslant 1)$, and $\mathfrak{M}_n \to \mathfrak{M}_0$ as $n \to \infty$. Then in any analytic surface $\mathfrak{A}_0^2$ intersecting $\mathfrak{M}_0$ we can give bounds for $|f(z_{10}, z_{20})|$, since any point $(z_{10}, z_{20}) \in \mathfrak{M}_0 \cap \mathfrak{A}_0^2$ also lies in $\mathfrak{M}_n \cap \mathfrak{A}_0^2$ if $n > n_0 = n_0(z_{10}, z_{20})$. By the theorem we can find a bound for $|f(z_{10}, z_{20})|$ if $n > n_0$ and the infimum of these bounds yields an upper bound for $|f(z_{10}, z_{20})|$ independent of n. Thus in special circumstances it is possible to give bounds for functions which are defined in domains in C^2 which are not necessarily analytic polyhedra.

C. If, in B, we suppose $f(z_1, z_2) \neq 0$ in $\mathfrak{M}_0$ and that γ^1 joins a point of $\mathfrak{M}_0$ to Q on $\partial\mathfrak{M}_0$, then the lower bounds on $|f|$ exhibited in the corollary can be used to give estimates on the rate of growth (or of decrease) of $|f(q)|$ as $q \to Q$ on γ^1.

D. If we assume in the theorem that the curve $\mathfrak{a}^1$ lies completely in e_k^3, then the upper bound for $|f(z_{10}, z_{20})|$ is greatly simplified since Hypothesis 7 is no longer required, and the estimate on $|\tilde{Z}(\psi)|$ in terms of r_0, R_0 holds for all ψ, $0 \leqslant \psi \leqslant 2\pi$. In this case, we find, for $(z_{10}, z_{20}) \in \mathfrak{A}^2$,

$$\log|f(z_{10}, z_{20})| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\Big\{\log^+|v(\psi)-u(\psi)|+\log^+|u(\psi)|+\log 2+ \\ + \frac{4}{(1-r_0)(1-R_0)}\big(\pi+\log\mu(\psi)\big)\Big\}P(\psi, \zeta_0)\,d\psi$$

(cf. [1], p. 186, [3]).

E. The bounds in the theorem are in terms of the values of f on a fairly arbitrary one dimensional boundary set $\mathfrak{b}^1$. If we had normalized $\mathfrak{b}^1$ and assumed $f\big(h_{k1}(0, \lambda_k), h_{k2}(0, \lambda_k)\big) = \omega(k, \lambda_k)$ for $0 \leqslant \lambda_k \leqslant 1$, $k = 1, \ldots, l$, then $\tilde{Z}(\psi)$, $\tilde{\tilde{Z}}(\psi)$ in (5) could be replaced by $Z_k(\psi)$, $Z_k(\psi)/(1+\varepsilon_0)$, respectively.

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