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CONFORMALLY SYMMETRIC SPACES ADMITTING SPECIAL QUADRATIC FIRST INTEGRALS

BY

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1. Introduction. A non-flat Riemannian space is said to be *of recurrent curvature* (briefly, a *recurrent space*) if its curvature tensor satisfies, for some non-zero vector d_j, the condition (see [8] and [11])

$$R_{hijk,l} = d_l R_{hijk}, \tag{1}$$

where the comma indicates covariant differentiation with respect to the metric of the space.

As a generalization of the concept of a recurrent space, Patterson [5] initiated investigations of Riemannian spaces whose Ricci tensors satisfy relations of the form

$$R_{ij,l} = d_l R_{ij} \tag{2}$$

for some vector d_j. Spaces of such a type, i.e. satisfying (2) for $R_{ij} \neq 0 \neq d_j$, are called *Ricci-recurrent*.

According to Chaki and Gupta [2], an n-dimensional ($n > 3$) Riemannian space is called *conformally symmetric* if its Weyl's conformal tensor

$$C^h{}_{ijk} = R^h{}_{ijk} - \frac{1}{n-2}(g_{ij}R^h{}_k - g_{ik}R^h{}_j + \delta^h_k R_{ij} - \delta^h_j R_{ik}) + \\ + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}) \tag{3}$$

satisfies

$$C^h{}_{ijk,l} = 0. \tag{4}$$

It follows easily from (3) and (4) that every conformally flat n-space ($n > 3$) as well as every symmetric (in the sense of E. Cartan) Riemannian n-space ($n > 3$) is necessarily conformally symmetric. The converse of this is, in general, not true [7].

A Riemannian space is said to admit a *special quadratic first integral* (SQFI) defined by the tensor a_{ij} if a_{ij} is symmetric and satisfies the condition (see [3] and [9])

$$a_{ij,k} = 0. \tag{5}$$

(The metric tensor g_{ij} is also considered as an SQFI).

After Thomas ([10], p. 413; see also [3], p. 318) we make the following definitions:

(A) A set of symmetric covariant-constant tensors $B_1, \ldots, B_t$ of type (0,2) is said to be *linearly independent* if $c_1B_1 + \ldots + c_tB_t = 0$ implies constants $c_1 = \ldots = c_t = 0$.

(B) A set of symmetric covariant-constant linearly independent tensors of type (0, 2) is *complete* if, for any symmetric covariant-constant tensor B of type (0, 2), we have $B = c_1B_1 + \ldots + c_tB_t$, where c_j are constants.

(C) The *index* of a Riemannian space is defined to be the number t ($t \geqslant 1$) of tensors B_j in a complete set as defined in (B).

Thus, the index of a Riemannian space is the greatest number of linearly independent SQFI's which it admits.

The problem of determining all conformally flat spaces ($n \geqslant 3$) admitting special quadratic first integrals was treated in some details by Levine and Katzin in [3] and [4].

This paper deals with analytic conformally symmetric spaces of indefinite metric forms which admit SQFI's.

It will be proved that if a conformally symmetric space (not Cartan--symmetric) is of index $t > 1$, then its scalar curvature vanishes.

If a conformally symmetric space (not Cartan-symmetric) admits more than one linearly independent analytic SQFI, then it admits exactly two such.

Necessary and sufficient conditions will be also obtained for a conformally symmetric space of an index $t > 1$ to be local Ricci-recurrent.

The remainder of the paper is concerned with conformally flat spaces admitting SQFI's as well as with affine collineations in conformally symmetric spaces.

For brevity, we denote a non-Cartan-symmetric conformally symmetric space by a CS_n-space.

2. Preliminary results. We start with a canonical form for the covariant derivative of the Ricci tensor of a CS_n-space whose index $t > 1$. To that end we need several lemmas.

LEMMA 1 ([6], Lemma 1). *If e_j and T_{ij} are numbers satisfying*

$$e_iT_{jm} + e_jT_{im} = 0 \quad or \quad e_iT_{mj} + e_jT_{mi} = 0,$$

then either all the e_j are zero or all the T_{ij} are zero.

LEMMA 2. *If $(T_{ij})_P$ and $(T_{i'j'})_{P'}$ are components of a covariant-constant tensor T_{ij} at any two points P and P', respectively, then there exists a non-singular matrix $(a^i{}_{i'})$ such that*

$$(T_{i'j'})_{P'} = (T_{ij})_P a^i{}_{i'} a^j{}_{j'}.$$

In particular, if T_{ij} is zero at some point, then it is zero at every point.

This lemma can be, for instance, proved in a similar way as in Walker ([8], p. 154). It is sufficient, in his proof, to replace R_{hijk} by T_{ij}.

LEMMA 3. *If a conformally symmetric space admits a symmetric covariant-constant tensor $a_{ij} \neq cg_{ij}$ ($c =$ const), then the Ricci tensor satisfies the equation*

$$a_{rj} R^r{}_{l,p} = \frac{1}{n} Q R_{jl,p}, \quad \textit{where } Q = g^{rs} a_{rs} = \text{const}. \tag{6}$$

Proof. As an immediate consequence of (3) and (4), we have

$$R^h{}_{ijk,l} = \frac{1}{n-2} (g_{ij} R^h{}_{k,l} - g_{ik} R^h{}_{j,l} + \delta^h_k R_{ij,l} - \delta^h_j R_{ik,l}) - \\ - \frac{1}{(n-1)(n-2)} R_{,l} (\delta^h_k g_{ij} - \delta^h_j g_{ik}). \tag{7}$$

Summing in (7) over h and l and taking into account the well-known formulas

$$R^r{}_{ijk,r} = R_{ij,k} - R_{ik,j}, \quad R^r{}_{j,r} = \tfrac{1}{2} R_{,j},$$

we find

$$R_{ij,k} - R_{ik,j} = \frac{1}{2(n-1)} (R_{,k} g_{ij} - R_{,j} g_{ik}). \tag{8}$$

On the other hand, it follows easily from (5) that

$$a_{ij,kl} - a_{ij,lk} = 0. \tag{9}$$

Applying the Ricci identity to (9), we obtain

$$a_{rj} R^r{}_{ikl} + a_{ri} R^r{}_{jkl} = 0,$$

whence, in view of (5),

$$a_{rj} R^r{}_{ikl,p} + a_{ri} R^r{}_{jkl,p} = 0. \tag{10}$$

Contracting now in (10) with g^{pk} and using (8), we get

$$R_{,i} a_{lj} - g_{il} R_{,r} a^r{}_j + R_{,j} a_{li} - g_{lj} R_{,r} a^r{}_i = 0, \tag{11}$$

which, by further contraction, implies

$$R_{,r} a^r{}_j = \frac{1}{n} Q R_{,j}.$$

But the last equation, together with (11), gives

$$\left(a_{kj}-\frac{1}{n}Qg_{kj}\right)R_{,i}+\left(a_{ki}-\frac{1}{n}Qg_{ki}\right)R_{,j}=0. \tag{12}$$

Putting

$$T_{ij}=a_{ij}-\frac{1}{n}Qg_{ij},$$

we see that (12) can be written as

$$R_{,i}T_{kj}+R_{,j}T_{ki}=0.$$

It follows easily from the definition of a_{ij} that $T_{ij}\neq 0$. For otherwise, in view of (5), Lemma 2 would imply

$$a_{ij}=\frac{1}{n}Qg_{ij}\quad \text{with } Q=\text{const},$$

a contradiction. Hence, in view of Lemma 1, we have $R=\text{const}$ which, together with (7) and (10), implies

$$\begin{aligned}g_{ik}a_{rj}R^{r}{}_{l,p}-g_{il}a_{rj}R^{r}{}_{k,p}+a_{lj}R_{ik,p}-a_{kj}R_{il,p}+\\ +g_{jk}a_{ri}R^{r}{}_{l,p}-g_{jl}a_{ri}R^{r}{}_{k,p}+a_{li}R_{jk,p}-a_{ki}R_{jl,p}=0.\end{aligned} \tag{13}$$

Contracting now (13) with g^{ik}, we find

$$(n-1)a_{rj}R^{r}{}_{l,p}-g_{jl}a_{rs}R^{rs}{}_{,p}+a_{rl}R^{r}{}_{j,p}-QR_{jl,p}=0, \tag{14}$$

whence

$$-(n-1)a_{rl}R^{r}{}_{j,p}+g_{lj}a_{rs}R^{rs}{}_{,p}-a_{rj}R^{r}{}_{l,p}+QR_{lj,p}=0. \tag{15}$$

Adding (14) and (15), we obtain

$$a_{rj}R^{r}{}_{l,p}=a_{rl}R^{r}{}_{j,p}.$$

The last formula reduces (14) to the form

$$na_{rj}R^{r}{}_{l,p}-g_{jl}a_{rs}R^{rs}{}_{,p}-QR_{jl,p}=0, \tag{16}$$

whence, by contraction with g^{lp}, we get

$$a_{rs}R^{rs}{}_{,j}=0. \tag{17}$$

But (17), in view of (16), leads immediately to (6). Our lemma is thus proved.

LEMMA 4. *If a CS_n-space admits a symmetric covariant-constant tensor* $a_{ij}\neq cg_{ij}$ ($c=$ const), *then each point has a neighbourhood in which*

$$R_{jk,p}=CA_jA_kA_p \tag{18}$$

and

$$a_{ij} = \frac{1}{n} Qg_{ij} + eA_i A_j, \tag{19}$$

where C is a scalar, $e = \pm 1$ and A_j is a local null parallel vector field.

Proof. Substituting (6) into (13), we get

$$\frac{1}{n} Qg_{ik} R_{jl,p} - \frac{1}{n} Qg_{il} R_{jk,p} + \frac{1}{n} Qg_{jk} R_{il,p} - \frac{1}{n} Qg_{jl} R_{ik,p} + \\ + a_{lj} R_{ik,p} - a_{kj} R_{il,p} + a_{il} R_{jk,p} - a_{ki} R_{jl,p} = 0. \tag{20}$$

A cyclic permutation of i, k and j gives

$$\frac{1}{n} Qg_{kj} R_{il,p} - \frac{1}{n} Qg_{kl} R_{ij,p} + \frac{1}{n} Qg_{ij} R_{kl,p} - \frac{1}{n} Qg_{il} R_{kj,p} + \\ + a_{li} R_{kj,p} - a_{ji} R_{kl,p} + a_{kl} R_{ij,p} - a_{jk} R_{il,p} = 0 \tag{21}$$

and, furthermore,

$$-\frac{1}{n} Qg_{ji} R_{kl,p} + \frac{1}{n} Qg_{jl} R_{ki,p} - \frac{1}{n} Qg_{ki} R_{jl,p} + \frac{1}{n} Qg_{kl} R_{ji,p} - \\ - a_{lk} R_{ji,p} + a_{ik} R_{jl,p} - a_{jl} R_{ki,p} + a_{ij} R_{kl,p} = 0. \tag{22}$$

Adding (20), (21) and (22), we obtain

$$\left(a_{il} - \frac{1}{n} Qg_{il}\right) R_{jk,p} = \left(a_{jk} - \frac{1}{n} Qg_{jk}\right) R_{il,p}. \tag{23}$$

Putting (as in the proof of Lemma 3)

$$T_{jk} = a_{jk} - \frac{1}{n} Qg_{jk},$$

we see that (23) can be written in the form

$$T_{il} R_{jk,p} = T_{jk} R_{il,p}.$$

But the last relation, in view of $T_{ij} = T_{ji}$ and

$$R_{ij,k} = R_{ik,j}, \tag{24}$$

which follows easily from (8) and $R = \text{const}$, yields

$$T_{il} R_{jk,p} = T_{jk} R_{il,p} = T_{jk} R_{pi,l} = T_{pi} R_{jk,l} = T_{ip} R_{jk,l}.$$

Hence

$$T_{il} R_{jk,p} = T_{ip} R_{jk,l}. \tag{25}$$

Since $T_{ij} \neq 0$ everywhere, for each point there exist a neighbourhood W and a real vector field v^j such that in W the condition $v^r v^s T_{rs} = e$ ($e = \pm 1$) holds. Therefore, transvecting formula (25) with $v^i v^l$ and putting $A_j = v^r T_{rj}$, $S_{ij} = v^r R_{ij,r}$, we find

$$R_{jk,p} = eA_p S_{jk}. \tag{26}$$

But it follows easily from (24) that $v^r R_{rk,p} = S_{kp}$.

Transvecting now (26) with v^j, we get

$$S_{kp} = eA_p B_k, \quad \text{where } B_k = v^r S_{rk}. \tag{27}$$

Moreover, as a consequence of (27), we obtain $B_p = eBA_p$ ($B = v^r B_r$). But the last result, together with (27) and (26), leads immediately to (18). Equation (18) is thus proved.

Substituting now (18) into (25), we easily obtain

$$C(A_p T_{il} - A_l T_{ip}) = 0.$$

Since the space is not Cartan-symmetric by assumption, the last formula gives $T_{ij} = eA_i A_j$ which, obviously, is equivalent to (19). Differentiating now (19) covariantly and using $Q = \text{const}$, we find

$$A_i A_{j,k} + A_j A_{i,k} = 0. \tag{28}$$

In view of $A_j \neq 0$, (28), together with Lemma 1, gives $A_{j,k} = 0$, which is the condition for A_j to be a parallel vector field.

Contracting now (19) with g^{ij}, we easily obtain $A^r A_r = 0$. This shows that A_j is a null vector field, which completes the proof.

3. Main results. Now we may proceed to the main results of this paper.

THEOREM 1. *If a CS_n-space admits more than one linearly independent analytic SQFI, then it admits exactly two such.*

Proof. It follows from the assumption that there exists a symmetric tensor $a_{ij} \neq cg_{ij}$ ($c = \text{const}$) such that condition (5) is satisfied. Suppose that $\bar{a}_{ij} \neq cg_{ij}$ ($c = \text{const}$) is another symmetric covariant-constant tensor. Then, in view of Lemma 4, we have

$$R_{jk,p} = \bar{C}\bar{A}_j \bar{A}_k \bar{A}_p, \tag{29}$$

$$\bar{a}_{ij} = \frac{1}{n}\bar{Q}g_{ij} + \bar{e}\bar{A}_i \bar{A}_j, \tag{30}$$

where $\bar{Q} = \text{const}$, $\bar{e} = \pm 1$ and $\bar{A}_j$ is a local null parallel vector field.

Comparing (18) and (29), we easily obtain

$$g\bar{A}_j = CA_j, \quad \text{where } g = \bar{C}h^2 \text{ and } h = v^r \bar{A}_r. \tag{31}$$

It follows from (31) that $g \neq 0$ in some neighbourhood. For otherwise, the assumption $g = 0$ would imply $C = 0$ which, in view of (18) and (7), gives $R^h{}_{ijk,l} = 0$ — a contradiction with the definition of a CS_n-space. Therefore, as a consequence of (31), in this neighbourhood we have

$$\bar{A}_j = \frac{C}{g} A_j.$$

Since A_j and $\bar{A}_j$ are parallel vector fields, the last equation yields $C/g = \text{const}$.

Hence there exists a constant C^* such that

$$\bar{A}_j = C^* A_j. \tag{32}$$

Substituting now (32) into (30) and using (19), we easily obtain

$$\begin{aligned}\bar{a}_{ij} &= \frac{1}{n}\bar{Q}g_{ij} + \bar{e}C^{*2}A_iA_j = \frac{1}{n}\bar{Q}g_{ij} + C_2(eA_iA_j) \\ &= \frac{1}{n}\bar{Q}g_{ij} + C_2\left(a_{ij} - \frac{1}{n}Qg_{ij}\right) = C_1g_{ij} + C_2a_{ij},\end{aligned}$$

which, since $\bar{a}_{ij}$ and $C_1g_{ij} + C_2a_{ij}$ are analytic, completes the proof.

THEOREM 2. *If a CS_n-space admits a symmetric covariant-constant tensor $a_{ij} \neq cg_{ij}$ ($c = \text{const}$), then its scalar curvature vanishes.*

Proof. Differentiating (18) covariantly and applying the Ricci identity, we find

$$R_{rk}R^r{}_{jpl} + R_{rj}R^r{}_{kpl} = (C_{,p}A_l - C_{,l}A_p)A_jA_k. \tag{33}$$

On the other hand, it is known that, for a null parallel vector field, the relations

$$A_rR^r{}_{jpl} = 0, \quad A_rR^r{}_j = 0, \quad A^rA_r = 0 \tag{34}$$

hold.

Therefore, differentiating (33) covariantly and using (18), (7), (34) and $R = \text{const}$ (see the proof of Lemma 3), we get

$$\begin{aligned}CA_m(A_jA_pR_{kl} - A_jA_lR_{pk} + A_kA_pR_{jl} - A_kA_lR_{pj})& \\ = (n-2)(C_{,pm}A_l - C_{,lm}A_p)A_jA_k&.\end{aligned} \tag{35}$$

Transvecting now (35) with A^l and substituting (34), we easily obtain $A^rC_{,rm} = 0$ which, by contraction with g^{kl}, reduces (35) to the form

$$CRA_jA_lA_m = 0.$$

Since C cannot be zero, our theorem is proved.

Now we shall obtain necessary and sufficient conditions for a conformally symmetric space to be Ricci-recurrent.

THEOREM 3. *Let a CS_n-space admit a symmetric covariant-constant tensor $a_{ij} \neq cg_{ij}$ (c = const). Then it is a local Ricci-recurrent space if and only if the condition*

$$R_{rk}R^r{}_{jpl}+R_{rj}R^r{}_{kpl} = 0 \tag{36}$$

is satisfied.

If (36) *holds and $R_{ij} \neq 0$ everywhere, then the recurrence condition* (2) *is satisfied at every point.*

Proof. Adati and Miyazawa [1] proved that the recurrence vector of a conformally symmetric Ricci-recurrent space is a local gradient. Therefore, differentiating (2) covariantly and using the Ricci identity, we easily obtain (36). The sufficiency of the first part of our theorem is thus proved.

Suppose now that (36) is satisfied. Thus, in view of (33) and (35), we find

$$A_jA_pR_{kl}-A_jA_lR_{pk}+A_kA_pR_{jl}-A_kA_lR_{pj} = 0. \tag{37}$$

Transvecting (37) with v^jv^p, we get

$$R_{kl} = eA_lU_k-eA_kU_l+UA_kA_l, \tag{38}$$

where $U_k = v^rR_{rk}$ and $U = v^rv^sR_{rs}$.

But the last equation yields $U_j = eUA_j$, which reduces (38) to the form

$$R_{kl} = UA_kA_l. \tag{39}$$

Since R_{ij} cannot be zero (otherwise the space would be Cartan-symmetric), from (39) and (18) in some neighbourhood we have

$$R_{jk,p} = \frac{C}{U}A_p(UA_jA_k) = d_pR_{jk}. \tag{40}$$

The sufficiency of the first part of our theorem is thus proved.

If now $R_{ij} \neq 0$ everywhere, then each point has a neighbourhood in which (40) is satisfied. But $(d_j-\bar{d}_j)R_{ij} = 0$ implies $\bar{d}_j = d_j$. Therefore there exists a vector field d_j such that (2) is satisfied at every point of the space. The last remark completes the proof.

In what follows we need the following lemma:

LEMMA 5. *The Ricci tensor of a CS_n-space of an index $t > 1$ is of the form*

$$R_{ij} = A_iE_j+A_jE_i, \tag{41}$$

where E_j is some covariant vector.

Proof. Transvecting (35) with $v^m v^j v^p$, we find

$$C(R_{kl}-eA_l U_k+eA_k U_l-UA_k A_l) = (n-2)(DA_l-eD_l)A_k, \tag{42}$$

where $D = v^r v^s C_{,rs}$ and $D_j = v^r C_{,jr}$.

But (42), by transvection with v^k, gives

$$2C(U_l-eUA_l) = e(n-2)(DA_l-eD_l)$$

which, together with (42), implies

$$R_{kl} = eA_k U_l+eA_l U_k-UA_k A_l.$$

The last equation is equivalent to

$$R_{kl} = A_k(eU_l-\tfrac{1}{2}UA_l)+A_l(eU_k-\tfrac{1}{2}UA_k),$$

which leads immediately to (41).

4. Conformally flat spaces admitting SQFI. Every conformally flat space ($n > 3$) is obviously conformally symmetric. Thus all theorems proved for conformally symmetric spaces remain true for conformally flat spaces. Using the previously obtained results we prove now the following theorem (Roman indices take values $1, 2, \ldots, n$ and Greek indices take values $2, 3, \ldots, n-1$):

THEOREM 4. *If a non-Cartan-symmetric conformally flat space ($n > 3$) admits a symmetric covariant-constant tensor $a_{ij} \neq cg_{ij}$ (c = const), then there exists a local coordinate system such that the metric takes the form*

$$ds^2 = \varphi(dx^1)^2+k_{\lambda\mu}dx^\lambda dx^\mu+2dx^1 dx^n, \qquad \varphi = Gk_{\lambda\mu}x^\lambda x^\mu, \tag{43}$$

where $(k_{\lambda\mu})$ is a symmetric and non-singular matrix consisting of constants, and G is a non-constant function of x^1 only. Every space ($n > 3$) with a metric of this form is recurrent and non-special.

Proof. It follows easily from (41) and (34) that $A^r E_r = 0$. This, in view of (3), (34), (41) and $R = 0$, yields

$$R_{ri}R^r{}_{jlm}+R_{rj}R^r{}_{ilm} = \frac{1}{n-2}S(g_{jl}A_iA_m-g_{jm}A_iA_l+g_{il}A_jA_m-g_{im}A_jA_l), \tag{44}$$

where $S = E^r E_r$.

Transvecting (44) with A^l and using $A^r R^h{}_{jrm} = 0$, we find $SA_iA_jA_m = 0$. Hence $S = 0$ and, in view of (44), condition (36) is now satisfied. Thus the space is local Ricci-recurrent by means of Theorem 3.

But in a conformally symmetric Ricci-recurrent space, coordinates can be chosen so that the metric takes the form ([7], Theorem 3)

$$
\begin{aligned}
ds^2 &= \varphi (dx^1)^2 + k_{\lambda\mu} dx^\lambda dx^\mu + 2dx^1 dx^n, \\
\varphi &= \frac{1}{2(n-2)} C \exp\left(\int A\, dx^1\right) k_{\lambda\mu} x^\lambda x^\mu + c_{\lambda\mu} x^\lambda x^\mu,
\end{aligned} \tag{45}
$$

where $k_{\lambda\mu} = k_{\mu\lambda}$ are constants such that $|k_{\lambda\mu}| \neq 0$, $(c_{\lambda\mu})$ is a symmetric matrix of constants satisfying $k^{\alpha\beta} c_{\alpha\beta} = 0$ with $(k^{\alpha\beta}) = (k_{\alpha\beta})^{-1}$, $A(x^1) \neq 0$, and $C \neq 0$ is a constant.

Every n-space ($n > 3$) with a metric of this form is conformally symmetric and satisfies (2) with $R_{ij} \neq 0 \neq d_j$.

Hence, in view of (45), we may write

$$
\varphi = G k_{\lambda\mu} x^\lambda x^\mu + c_{\lambda\mu} x^\lambda x^\mu, \tag{46}
$$

where G is a non-constant function of x^1 only.

But, as follows easily from (2) and (3), every conformally flat ($n > 3$) Ricci-recurrent space is of recurrent curvature. Therefore, all $c_{\lambda\mu}$ in (46) must be zero ([7], Lemma 6), and metric (45) takes form (43).

The second part of our theorem follows from the fact that every space with metric (43) is recurrent and non-special ([8], p. 176).

Remark. Theorem 4 can also be deduced from some considerations of Walker ([11], p. 59-60) and those of Levine and Katzin ([4], p. 257).

5. Affine collineations in CS_n-spaces. A Riemannian space is said to admit a *conformal motion* if there exists a vector field p^i such that

$$
Lg_{ij} = 2sg_{ij},
$$

where L denotes the Lie derivative with respect to this field, and s is a scalar expressible in the form

$$
s = \frac{1}{n} p^r{}_{,r}.
$$

If $s = \text{const}$, then the conformal motion is called *homothetic.*

A Riemannian space is said to admit an *affine collineation* if there exists a vector field q^i such that

$$
L\begin{Bmatrix} h \\ ij \end{Bmatrix} = 0.
$$

It is known that

$$
L\begin{Bmatrix} h \\ ij \end{Bmatrix} = \frac{1}{2} g^{hr} [(Lg_{rj})_{,i} + (Lg_{ri})_{,j} - (Lg_{ij})_{,r}], \tag{47}
$$

which shows that every homothetic motion is an affine collineation. The converse of this is in general not true.

THEOREM 5. *If a conformally symmetric space with non-constant scalar curvature admits an affine collineation, then this collineation is a homothetic motion.*

Proof. It follows easily from (47) that conditions

$$L\begin{Bmatrix} h \\ ij \end{Bmatrix} = 0 \quad \text{and} \quad (Lg_{ij})_{,k} = 0$$

are equivalent. Therefore, in view of (12) and Lemma 2, we obtain

$$a_{ij} = \frac{1}{n} Q g_{ij}, \quad \text{where } a_{ij} = Lg_{ij}.$$

Since $Q = \text{const}$, our theorem is proved.

THEOREM 6. *If a CS_n-space admits an affine collineation, which is not a homothetic motion, then its scalar curvature vanishes.*

This result follows immediately from Theorem 2.

As a consequence of Theorem 6, we have

THEOREM 7. *If a CS_n-space with non-vanishing scalar curvature admits an affine collineation, then this collineation is necessarily a homothetic motion.*

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