ANNALES
POLONICI MATHEMATICI
XLV (1985)

A functional equation arising from the Joukowski transformation

by Hiroshi Haruki (Waterloo, Canada)

Dedicated to Professor Dr. J. D. Aczél on his 60th Birthday

Abstract. The purpose of this paper is to solve a functional equation which arises from the Joukowski transformation in aerodynamics and to present a geometric interpretation of the equation.

1. Introduction and statement of theorem. The transformation

$$w = f(z) = \tfrac{1}{2}(z+1/z) \quad (z \neq 0) \tag{1}$$

satisfies the functional equation

$$f(z)^2 + f(iz)^2 = 1 \quad \text{for } 0 < |z| < +\infty. \tag{2}$$

The transformation (1) is said to be the *Joukowski transformation* (see [6], p. 271) which will be denoted by $j(z)$ throughout this paper.

The purpose of this paper is to solve (2), i.e., to prove (see Section 3) the following theorem and to present a geometric interpretation of (2) (see Section 4).

Theorem 1. *Suppose that a complex-valued function f of a complex variable z is analytic for $0 < |z| < +\infty$ and is either analytic or has a pole at $z = 0$. The only solutions of* (2) *are the following* (a) *and* (b):

$$f(z) = j(\varphi(z)), \tag{a}$$

where φ is a meromorphic function of z which has the form $\varphi(z) = \sum_{n=m}^{+\infty} b_{4n+3} z^{4n+3}$ and never vanishes for $0 < |z| < +\infty$. Here m is an arbitrarily fixed integer and each of b_{4n+3} $(n = m, m+1, m+2, \ldots)$ is a complex constant;

$$f(z) = j(\exp(\psi(z))) = \cosh(\psi(z)), \tag{b}$$

where ψ *is an entire function of* z *which has the form*

$$\psi(z) = \frac{\pi i}{4} + m\pi i + \sum_{n=0}^{+\infty} b_{4n+2} z^{4n+2} \quad \text{for } |z| < +\infty .$$

Here m *is an arbitrarily fixed integer and each of* b_{4n+2} $(n = 0, 1, 2, \ldots)$ *is a complex constant.*

Remark. If we put $m = -1$, $b_{-1} = 1$ and $b_{4n+3} = 0$ $(n = 0, 1, 2, \ldots)$, then we obtain $\varphi(z) = 1/z$. Hence the solution (a) in the above theorem becomes $f(z) = j(1/z) = \frac{1}{2}(z + 1/z)$.

2. A lemma. In order to prove Theorem 1 we shall apply the following lemma:

LEMMA 1. *If* f *is any entire function of* z *which is never zero, then there exists an entire function* g *of* z *such that* $f(z) = \exp(g(z))$ *holds for* $|z| < +\infty$.

Proof. See [2], p. 192 and [4], p. 127.

3. Proof of Theorem 1. If we put

$$\varphi(z) = f(z) + if(iz) \tag{3}$$

for $0 < |z| < +\infty$, then, by (2), we have for $0 < |z| < +\infty$

$$\varphi(z)\bigl(f(z) - if(iz)\bigr) = 1 . \tag{4}$$

Hence we obtain

$$\varphi(z) \neq 0 \quad \text{for } 0 < |z| < +\infty . \tag{5}$$

By (4), (5) we have for $0 < |z| < +\infty$

$$1/\varphi(z) = f(z) - if(iz). \tag{6}$$

Adding (3) and (6) side by side yields for $0 < |z| < +\infty$

$$f(z) = \tfrac{1}{2}\bigl(\varphi(z) + 1/\varphi(z)\bigr) = j\bigl(\varphi(z)\bigr). \tag{7}$$

If we replace z $(\neq 0)$ by iz in (2), then we obtain for $0 < |z| < +\infty$

$$f(iz)^2 + f(-z)^2 = 1 . \tag{8}$$

By (2), (8) we obtain for $0 < |z| < +\infty$

$$f(-z)^2 = f(z)^2 . \tag{9}$$

Since the field of all meromorphic functions for $|z| < +\infty$ has no divisors of zero, by (9) we see that f is either an odd function of z for $0 < |z| < +\infty$ or an even function of z for $0 < |z| < +\infty$. We discuss two cases.

Case 1. The case where f is an odd function of z for $0 < |z| < +\infty$.

Replacing z ($\neq 0$) by iz in (3) and using the fact that f is odd for $0 < |z| < +\infty$ yields

$$\varphi(iz) = f(iz) + if(-z) = -i\big(f(z) + if(iz)\big) = -i\varphi(z),$$

and so

(10) $$\varphi(iz) = -i\varphi(z) \quad \text{for } 0 < |z| < +\infty.$$

Since, by hypothesis, f is analytic for $0 < |z| < +\infty$ and is either analytic or has a pole at $z = 0$, so is φ. Therefore, $\varphi(z)$ is expanded into a power series of the following form for $0 < |z| < +\infty$:

(11) $$\varphi(z) = \sum_{n=l}^{+\infty} b_n z^n,$$

where l is an integer and each of b_n ($n = l, l+1, l+2, \ldots$) is a complex constant with $b_l \neq 0$. Substituting (11) back into (10) and equating the coefficients of z^n ($n = l, l+1, l+2, \ldots$) yields

(12) $$b_n i^n = -ib_n \quad (n = l, l+1, l+2, \ldots).$$

If we put $n = l$ in (12), then, by $b_l \neq 0$ we have $i^l = -i$. Hence we obtain

(13) $$l = 4m+3,$$

where m is an integer.

By (12), (13) we obtain

(14) $$b_n = 0 \quad \text{for all } n \equiv 0, 1, 2 \pmod 4.$$

By (5), (7), (11), (13), (14) we obtain (a) $f(z) = j\big(\varphi(z)\big)$ in the theorem.

Conversely, direct substitution shows, by using (10), that $f(z) = j\big(\varphi(z)\big)$ satisfies our original functional equation (2).

Case 2. The case where f is an even function of z for $0 < |z| < +\infty$.

Replacing z ($\neq 0$) by iz in (3) and using the fact that f is even for $0 < |z| < +\infty$ yields

$$\varphi(iz) = f(iz) + if(-z) = f(iz) + if(z) = i\big(f(z) - if(iz)\big),$$

and so

(15) $$\varphi(iz) = i\big(f(z) - if(iz)\big) \quad \text{for } 0 < |z| < +\infty.$$

By (4), (15) we obtain for $0 < |z| < +\infty$

(16) $$\varphi(z)\,\varphi(iz) = i.$$

Next we shall prove that f is analytic at $z = 0$ and so f is an entire function of z.

The proof is by contradiction. Assume contrary. Then f is not analytic at $z = 0$. So, by hypothesis f has a pole at $z = 0$. Let its order be l ($\in N$). By Laurent's Theorem, f is expanded into a Laurent series of the following form for $0 < |z| < +\infty$:

$$f(z) = \frac{c_{-l}}{z^l} + \frac{c_{-l+1}}{z^{l-1}} + \frac{c_{-l+2}}{z^{l-2}} + \dots, \tag{17}$$

where each of c_n ($n = -l, -l+1, -l+2, \dots$) is a complex constant with $c_{-l} \neq 0$.

Since f is even for $0 < |z| < +\infty$, by (17) l is an even positive integer. We put

$$l = -2m, \tag{18}$$

where m is a negative integer.

Substituting (17) into (2), equating the coefficients of $1/z^{2l}$ on both sides and taking (18) into account yields

$$c_{2m}^2 + \frac{1}{i^{-4m}} c_{2m}^2 = 0, \quad \text{or} \quad 2c_{2m}^2 = 0,$$

and so

$$c_{-l} = 0,$$

which contradicts $c_{-l} \neq 0$.

Consequently, f is analytic at $z = 0$ and so f is an entire function of z. Hence, by (3), so is φ. Since φ is analytic at $z = 0$, φ is continuous at $z = 0$. So (16) holds for $|z| < +\infty$. Therefore, φ is never zero. Hence, by Lemma 1 in Section 2 there exists an entire function ψ of z such that

$$\varphi(z) = \exp(\psi(z)) \tag{19}$$

holds for $|z| < +\infty$.

By (16), (19) we have for $|z| < +\infty$

$$\exp(\psi(z) + \psi(iz)) = i. \tag{20}$$

Differentiating both sides of (20) yields for $|z| < +\infty$

$$\exp(\psi(z) + \psi(iz)) \frac{d}{dz}(\psi(z) + \psi(iz)) = 0,$$

or

$$\frac{d}{dz}(\psi(z) + \psi(iz)) = 0.$$

So $\psi(z) + \psi(iz)$ is a complex constant, say K. By (20) we have $\exp(K) = i$ and so $K = \frac{1}{2}\pi i + 2m\pi i$, where m is an integer. Therefore, we obtain for $|z| < +\infty$

$$\psi(z) + \psi(iz) = \tfrac{1}{2}\pi i + 2m\pi i, \tag{21}$$

where m is an integer.

Since ψ is an entire function of z, by Taylor's Theorem ψ is expanded into a power series of the following form for $|z| < +\infty$:

$$\psi(z) = \sum_{n=0}^{+\infty} b_n z^n, \tag{22}$$

where each of b_n ($n = 0, 1, 2, \ldots$) is a complex constant.

Substituting (22) back into (21) and equating the coefficients of z^n ($n = 0, 1, 2, \ldots$) on both sides yields

$$b_0 = \tfrac{1}{4}\pi i + m\pi i \tag{23}$$

and

$$b_n(1+i^n) = 0 \quad (n = 1, 2, 3, \ldots). \tag{24}$$

By (24) we obtain

$$b_n = 0 \quad \text{for all } n \equiv 0, 1, 3 \pmod 4. \tag{25}$$

By (22), (23), (25) we have for $|z| < +\infty$

$$\psi(z) = \tfrac{1}{4}\pi i + m\pi i + \sum_{n=0}^{+\infty} b_{4n+2} z^{4n+2}, \tag{26}$$

where m is an integer and each of b_{4n+2} ($n = 0, 1, 2, \ldots$) is a complex constant.

By (7), (19), (26) we obtain (b) $f(z) = j(\exp(\psi(z))) = \cosh(\psi(z))$ in the theorem.

Conversely, direct substitution shows, by using (21), that (b) $f(z) = j(\exp(\psi(z))) = \cosh(\psi(z))$ satisfies our original functional equation (2). ∎

4. A geometric interpretation of the functional equation (2). We consider an ellipse E in the w-plane with foci at 1 and -1 and with centre at 0. Then there exists a positive real number r ($\neq 1$) such that $w = j(z) = \frac{1}{2}(z + 1/z)$ maps the circle $|z| = r$ in the z-plane in a one-to-one manner onto the ellipse E (see [6], p. 270). Let z_0 be an arbitrarily fixed point on the circle $|z| = r$ in the z-plane. Then we shall prove that the two points $j(z_0)$ and $j(iz_0)$ represent end points of two semiconjugate diameters of E.

The angle between the directed tangent to the circle $|z| = r$ at $z = iz_0$ and the positive real axis in the z-plane is given by $\arg(z_0)+\pi$. Hence, by a basic theorem in conformal mapping the angle β between the directed tangent to E at $w = j(iz_0)$ and the positive real axis in the w-plane is given by $(\arg(z_0)+\pi)+\arg(j'(iz_0))$. Since $j'(iz_0) = \frac{1}{2}(1+1/z_0^2)$, we obtain

$$\begin{aligned} \beta &\equiv (\arg(z_0)+\pi)+\arg(j'(iz_0)) \equiv \arg(z_0)+\pi+\arg(\tfrac{1}{2}(1+1/z_0^2)) \\ &\equiv \pi+\arg(z_0\cdot\tfrac{1}{2}(1+1/z_0^2)) \equiv \pi+\arg(\tfrac{1}{2}(z_0+1/z_0)) \\ &\equiv \pi+\arg(j(z_0)) \equiv \pi+\alpha \pmod{2\pi}. \end{aligned} \tag{27}$$

Here α denotes one argument of $j(z_0)$.

By (27) the directed tangent to E at $j(iz_0)$ is parallel to the straight line joining the point 0 and the point $j(z_0)$. Therefore, we obtain the following lemma:

LEMMA 2. *With the same notation as above the two points $j(z_0)$ and $j(iz_0)$ represent end points of two semiconjugate diameters of the ellipse E.*

We may now present a geometric interpretation of the functional equation (2).

THEOREM 2. *Let OP and OQ be two semiconjugate diameters of an ellipse E in the w-plane with foci at F and F' and with centre at 0. We denote by v_1, v_2, v_3, v the four vectors $\overrightarrow{FP}$, $\overrightarrow{F'P}$, $\overrightarrow{OQ}$, $\overrightarrow{F'F}$, respectively. Then*

(i) $\overline{PF}\cdot\overline{PF'} = \overline{OQ}^2$ (see [7], p. 178);

(ii) $\widehat{vv_3} \equiv \frac{1}{2}(\widehat{vv_1}+\widehat{vv_2}+\pi) \pmod{\pi}$.

Remark. Let V_1 and V_2 be two nonzero vectors. Then we denote by $V_1 V_2$ a directed angle of rotation to coincide the positive direction of V_1 with that of V_2.

Proof. We may assume that F and F' are 1 and -1, respectively. We consider the mapping $w = j(z) = \frac{1}{2}(z+1/z)$ from the z-plane into the w-plane. Then there exists a positive real number r ($\neq 1$) such that $w = j(z)$ maps the circle $|z| = r$ in the z-plane in a one-to-one manner onto the ellipse E. Hence there exists a complex number z_0 on the circle $|z| = r$ such that $j(z_0) = P$ and $j(iz_0) = Q$. Since $f(z) = j(z)$ satisfies (2), we obtain

$$j(z_0)^2+j(iz_0)^2 = 1. \tag{28}$$

Proof of (i). By (28), we have

$$\begin{aligned}\overline{PF}\cdot\overline{PF'} &= |j(z_0)-1|\,|j(z_0)-(-1)|\\ &= |j(z_0)^2-1| = |-j(iz_0)^2| = |j(iz_0)|^2 = \overline{OQ}^2. \quad \blacksquare\end{aligned}$$

Proof of (ii). By (28) we have

$$\begin{aligned}2\widehat{vv_3} &\equiv 2\arg\big(j(iz_0)\big) \equiv \arg\big(j(iz_0)^2\big)\\ &\equiv \arg\big(1-j(z_0)^2\big) \equiv \arg\big(-(j(z_0)-1)(j(z_0)-(-1))\big)\\ &\equiv \arg(-1)+\arg\big(j(z_0)-1\big)+\arg\big(j(z_0)-(-1)\big)\\ &\equiv \pi+\widehat{vv_1}+\widehat{vv_2} \pmod{2\pi}. \quad \blacksquare\end{aligned}$$

5. A remark on the functional equation (2). The functional equation (2) is of the Ganapathy Iyer and Montel type (see [3] and [5], p. 65). Ganapathy Iyer and Montel solved the following functional equation:

$$f(z)^2+g(z)^2 = 1, \tag{29}$$

where f and g are unknown entire functions of z.

They obtained the following theorem.

THEOREM A. *The only system of solutions of* (29) *is*

$$f(z) = \cos(\varphi(z)), \quad g(z) = \sin(\varphi(z)),$$

where φ *is an arbitrary entire function of* z.

Since, by hypothesis, f in (2) is a meromorphic function for $0 < |z| < +\infty$ and is either analytic or has a pole at $z = 0$ while f and g in (29) are entire functions of z, we cannot apply Theorem A to (2).

The mappings $w = \cos z$ and $w = \sin z$ are closely related to the mapping $w = j(z)$ since $\cos z = j(\exp(-iz))$ and $\sin z = j(i\exp(-iz))$ for all complex z. (For applications of the mappings $w = \cos z$ and $w = \sin z$ to geometry, see [1].) These two equalities show that $j(z)$ is a solution of (2) for $0 < |z| < +\infty$, since $\cos^2 z + \sin^2 z = 1$ holds and since $\exp(-iz)$ takes every complex value except 0.

This work was supported by the Natural Sciences and Engineering Research Council of Canada Grant Nr. A-4012.

References

[1] J. Aczél and H. Haruki, *Commentary to Einar Hille's collected works*, The MIT Press, Cambridge, Mass., and London, England (edited by R. R. Kallman) (1975), 651–658.

[2] Lars V. Ahlfors, *Complex analysis*, second edition, McGraw-Hill Book Company, 1966.

[3] V. Ganapathy Iyer, *On certain functional equations*, J. Indian Math. Soc. 3 (1939), 312–315.

[4] M. Heins, *Complex function theory*, Academic Press, New York and London 1968.

[5] A. I. Markushevich, *Entire functions*, American Elsevier Publishing Company, 1966.

[6] Z. Nehari, *Conformal mapping*, first edition, McGraw-Hill Book Company, 1952.

[7] G. Salmon, *A treatise on conic sections*, sixth edition, Chelsea Publishing Company, New York 1954.

UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO, CANADA

Reçu par la Rédaction le 5. 10. 1983