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On continuous solutions of the Schröder equation

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Abstract. The author studies continuous solutions in locally convex vector spaces of the Schröder equation.

Let E and F be locally convex topological vector spaces over either the field C of complex numbers or the field R of real numbers. We always assume that E is a Hausdorff space, and $f\colon E \to E$. We prove theorems on the existence and uniqueness of continuous solutions of the Schröder functional equation

(1) $$\varphi[f(x)] = s\varphi(x),$$

where $s \in C$, or $s \in R$ and $0 < |s| < 1$. M. Kuczma in paper [1] studied solutions of class C^p of equation (1), when F is a Banach space and $E = R^n$. By $\Gamma(E)$ and $\Gamma(F)$ we denote any sets of seminorms determining the topology of E and F, respectively.

For a seminorm q defined in E we put

(2) $$h_q(r) := \frac{1}{r} \sup_{q(t) \leqslant r} q[f(t)], \qquad r > 0.$$

Further, we shall deal with the following hypotheses:

(I) *f is homeomorphism E onto $f(E)$.*

(II) *$0 < h_q(r) < 1$ and h_q is the continuous function for $r > 0$, $q \in \Gamma(E)$.*

We denote by $f^k(x)$, $k \in Z$(¹), the k-th iterate of the function $f(x)$:

$$f^0(x) = x, \quad f^{k+1}(x) = f[f^k(x)], \quad f^{k-1}(x) = f^{-1}[f^k(x)].$$

We shall prove some lemmas.

Lemma 1. *$\lim_{k\to\infty} f^k(x) = 0$ for $x \in E$. Moreover, the sequence $\{q[f^k(x)]\}$ is decreasing for $q \in \Gamma(E)$.*

(¹) Z denotes the set of all integers.

Proof. It follows from hypothesis (II) that if $r:=q[f^k(x)]>0$, then

$$q[f^{k+1}(x)]\leqslant \sup_{q(t)\leqslant r} q[f(t)] = rh_q(r)<r=q[f^k(x)],\qquad k=0,1,2,\ldots$$

If $q[f^k(x)]=0$, then according to hypothesis (II) for an arbitrary $r>0$ we have $q[f^{k+1}(x)]<r$; therefore $q[f^{k+1}(x)]=0$. Let $q[f^k(x)]\to r_0>0$, and let $h:=\sup_{[r_0,q(x)]} h_q(r)$. There exists a number $r_1\in[r_0,q(x)]$ such that $h=h_q(r_1)$, because $h_q(r)$ is continuous, thus $h<1$. Since $q[f^{k+1}(x)]\leqslant hq[f^k(x)]$, $k=0,1,\ldots$, we have $r_0\leqslant hr_0$, which is imposible. Thus $r_0=0$. For every $q\in\Gamma(E)$ we have $\lim_{k\to\infty} q[f^k(x)]=0$; so $\lim_{k\to\infty} f^k(x)=0$.

The implication

$$q(x)=0\Rightarrow q[f(x)]=0$$

and hypothesis (I) give the following

COROLLARY 1. $f(0)=0$.

It follows from hypothesis (I) and Corollary 1 that $f(E)$ is an open set and $0\in f(E)$. There exist seminorms $q_1,\ldots,q_m\in\Gamma(E)$ and $\eta>0$ such that

$$\left\{x\in E: \frac{1}{\eta}\max_{i=1,\ldots,m} q_i(x)\leqslant 1\right\}\subset f(E).$$

We put

$$p(x):=\frac{1}{\eta}\max_{i=1,\ldots,m} q_i(x),\qquad x\in E. \tag{3}$$

The function p thus defined is a continuous seminorm on E.

LEMMA 2. *$h_p(r)<1$ for $r>0$.*

Proof. For a fixed $r>0$, and for every x such that $p(x)\leqslant r$, we have

$$p[f(x)]=\frac{1}{\eta}\max_{i=1,\ldots,m} q_i[f(x)]\leqslant \frac{1}{\eta}\max_{i=1,\ldots,m}\Big(\sup_{q_i(t)\leqslant\eta r} q_i[f(t)]\Big) = r\max_{i=1,\ldots,m} h_{q_i}(\eta r).$$

Hence

$$\frac{1}{r}\sup_{p(t)\leqslant r} p[f(t)]\leqslant \max_{i=1,\ldots,m} h_{q_i}(\eta r)<1.$$

This completes the proof.

We define the sets

$$V_0:=\{x\in E:\ p(x)\leqslant 1\}, \tag{4}$$

$$C_0:=\{x\in E:\ p(x)=1\}. \tag{5}$$

LEMMA 3. *$C_0 = \partial V_0$ and $\{x \in E: p(x) < 1\} = \operatorname{Int} V_0$, where ∂V_0 denote the topological boundary of V_0.*

Proof. Let $x_0 \in C_0$ and let U_{x_0} be an arbitrary neighbourhood of x_0. There exist real numbers λ_1 and λ_2 such that

$$0 < \lambda_1 < 1, \quad \lambda_1 x_0 \in U_{x_0}; \quad \lambda_2 > 1, \quad \lambda_2 x_0 \in U_{x_0},$$

because the function $\lambda \to \lambda x_0$ is continuous in $\lambda = 1$. Hence $p(\lambda_1 x_0) = \lambda_1 p(x_0) = \lambda_1 < 1$ and $p(\lambda_2 x_0) = \lambda_2 p(x_0) = \lambda_2 > 1$. Thus $\lambda_1 x_0 \in V_0$ and $\lambda_2 x_0 \notin V_0$, so $x_0 \in \partial V_0$.

Now, let $x_0 \in \partial V_0$ and $\varepsilon > 0$. There exist a neighbourhood U_{x_0} and points $x, y \in U_{x_0}$, such that $x \in V_0$, $y \notin V_0$ and

$$p(x_0) - \varepsilon < p(x), \quad p(y) < p(x_0) + \varepsilon.$$

Thus

$$p(x_0) - \varepsilon < p(x) \leqslant 1 < p(y) < p(x_0) + \varepsilon.$$

Hence

$$-\varepsilon < 1 - p(x_0) < \varepsilon$$

and $p(x_0) = 1$; thus by (5) we get $x_0 \in C_0$.

It follows from the continuity of the seminorm p that the inclusion $\{x \in E: p(x) < 1\} \subset \operatorname{Int} V_0$ holds. Now, let $x_0 \in \operatorname{Int} V_0$. Since $\operatorname{Int} V_0 \subset V_0$, so $p(x_0) \leqslant 1$. But, if $p(x_0) = 1$, then $x_0 \in \partial V_0$. Hence $p(x_0) < 1$.

This completes the proof of the lemma.

By Lemma 2 and by the principle of induction we get the following

LEMMA 4. *$f(V_0) \subset \operatorname{Int} V_0$, $f^{k+1}(V_0) \subset f^k(V_0)$, $k \in Z$.*

We put

$$A_0 := V_0 \setminus f(V_0).$$

LEMMA 5. *$\overline{A}_0 = A_0 \cup f(C_0)$.*

Proof. Let $y_0 \in f(C_0)$. There exists an $x_0 \in C_0$ such that $f(x_0) = y_0$. Let U_{y_0} be an arbitrary neighbourhood of point y_0. We may assume without loss of generality that $U_{y_0} \subset V_0$ because $f(C_0) \subset \operatorname{Int} V_0$. There exists a neighbourhood V_{x_0} of x_0 such that $f(V_{x_0}) \subset U_{y_0} \subset V_0$. It follows from Lemma 3 that there exists a points $x \in V_{x_0} \setminus V_0$. Thus $f(x) \notin f(V_0)$ and $f(x) \in U_{y_0} \subset V_0$. Hence $f(x) \in A_0$. This shows that $A_0 \cup f(C_0) \subset \overline{A}_0$. Now, let $x_0 \in \overline{A}_0$ and $x_0 \notin A_0$. Since V_0 is a closed set, by the definition A_0 we have $x_0 \in f(V_0)$. There exist $y_0 \in V_0$ such that $f(y_0) = x_0$. If $p(y_0) = 1$, then $y_0 \in C_0$ and $x_0 \in f(C_0)$. We suppose that $p(y_0) < 1$; then $y_0 \in \operatorname{Int} V_0$, and thus $x_0 \in \operatorname{Int} f(V_0)$. Hence there exists a neighbourhood U_{x_0} of x_0 such that $U_{x_0} \subset f(V_0)$, which means that $U_{x_0} \cap A_0 = \emptyset$. This is imposible because $x_0 \in \overline{A}_0$. Thus we have proved the inclusion $\overline{A}_0 \subset A_0 \cup f(C_0)$.

LEMMA 6. *$A_0 = C_0 \cup \operatorname{Int} A_0$.*

Proof. It follows from Lemma 2 that $C_0 \subset A_0$. Let $x_0 \in A_0 \setminus C_0$. Thus $x_0 \in V_0 \setminus (C_0 \cup f(V_0))$. Put $B: = \{x \in E: p(x) \geqslant 1\}$. B and $f(V_0)$ are closed sets and $x_0 \notin B \cup f(V_0)$. There exists a neighbourhood U_{x_0} of x_0 such that

$$U_{x_0} \cap B = U_{x_0} \cap f(V_0) = \emptyset.$$

Hence $U_{x_0} \subset V_0$ and $U_{x_0} \cap f(V_0) = \emptyset$. Consequently, $U_{x_0} \subset A_0$ and $x_0 \in \operatorname{Int} A_0$. This completes the proof.

We put

$$A_k: = f^k(A_0), \quad C_k: = f^k(C_0) \quad \text{for } k \in Z.$$

It follows from above definitions that $A_0 \subset f(E)$. For $k < 0$ we may have $A_k = \emptyset$.

LEMMA 7.

$$A_k \cap A_l = \emptyset \quad \textit{for } k \neq l, k, l \in Z, \tag{6}$$

$$\overline{A_k} = A_k \cup C_{k+1}, \quad k \in Z, \tag{7}$$

$$A_k = C_k \cup \operatorname{Int} A_k, \quad k \in Z. \tag{8}$$

Proof. By definitions A_k and by hypothesis (I) we have $A_k = f^k(A_0) = f^k(V_0) \setminus f^{k+1}(V_0)$, $A_l = f^l(V_0) \setminus f^{l+1}(V_0)$. Let $l > k$; then $l \geqslant k+1$ and by Lemma 4 we get $A_l \subset f^l(V_0) \subset f^{k+1}(V_0)$. Therefore $A_l \cap A_k = \emptyset$. This completes the proof of (6). Equalities (7) and (8) follow from Lemmas 5 and 6, respectively, and by induction.

Let

$$P: = \{x \in E: \bigwedge_{k \in Z} p[f^k(x)] = 0\}.$$

It follows from the implication

$$p(x) = 0 \Rightarrow p[f(x)] = 0$$

that

$$P = \bigcap_{k=0}^{\infty} \{x \in E: p[f^{-k}(x)] = 0\}.$$

LEMMA 8. *The set A_0 contains exactly one element of every orbit*([3]) *contained in $E \setminus P$.*

Proof. Suppose that x_0 and $y_0 \in A_0$ and x_0, y_0 belong to the same orbit contained in $E \setminus P$. Then there exists an integer $k \geqslant 1$ such that $y_0 = f^k(x_0)$ In view of Lemma 4, $y_0 \in f(V_0)$, which is imposible. Thus A_0 cannot contain more than one element of every orbit contained in $E \setminus P$. We shall prove that none of the orbits contained in $E \setminus P$ can be separated

([3]) For $x \in E$ the set of all points $f^k(x)$ (for all $k \in Z$ for which $f^k(x)$ is defined) is called the *orbit* determined by x.

from A_0. Let us assume that there exists an orbit separated from A_0. We denote by C the intersection of that orbit with V_0. In view of Lemma 1, we have $C \neq \emptyset$. The inclusion $C \subset f(V_0)$ results from $C \cap A_0 = \emptyset$. Let $x \in C$. There exists a $y \in V_0$ such that $f(y) = x$. If we have $y \notin f(V_0)$, then $y \in A_0$, which is imposible since A_0 is separated from the orbit determined by x. Consequently $y \in f(V_0)$ and by the principle of induction, $f^{-k}(x) \in V_0$, $k = 0, 1, 2, \ldots$ The sequence $p[f^{-k}(x)]$ converges to a positive limit r_0, because by Lemma 1 it is increasing and upper bounded, and, moreover, $x \notin P$. We put

$$r_k := p[f^{-k}(x)], \quad k = 1, 2, \ldots$$

By the definition of h_p we have

$$r_k \leqslant \sup_{p(t) \leqslant r_{k+1}} p[f(t)] = r_{k+1} h_p(r_{k+1}).$$

By the equality

$$rh_p(r) = \sup_{p(t) \leqslant r} p[f(t)]$$

it follows that the function $rh_p(r)$ is increasing. Hence and by above inequality we get

$$r_k \leqslant r_0 h_p(r_0),$$

whence $r_0 \leqslant r_0 h_p(r_0) < r_0$, which is imposible. Thus $C \cap A_0 \neq \emptyset$.

COROLLARY 2. $E \setminus P = \bigcup_{k=-\infty}^{\infty} A_k$.

LEMMA 9. $V_0 \setminus P = \bigcup_{k=0}^{\infty} A_k$.

Proof. First we shall prove the implication

$$x_0 \in A_{-1} \mapsto p(x_0) > 1. \tag{9}$$

If $x_0 \in A_{-1}$, then there exists a $y_0 \in A_0$ such that $y_0 = f(x_0)$. Since $A_0 = V_0 \setminus f(V_0)$, we have $y_0 \notin f(V_0)$. Thus $x_0 \notin V_0$, and $p(x_0) > 1$.

Now, let $x \in V_0 \setminus fP$. By Corollary 2 there exists a $k \in Z$ such that $x \in A_{-k}$. Suppose that $k \geqslant 1$. Then $x \in f^{-k+1}(A_{-1})$ and there exists a $y \in A_{-1}$ such that $y = f^{k-1}(x)$. Hence, by Lemmas 1 and 2 and by (9) we get

$$1 < p(y) \leqslant p(x) \leqslant 1,$$

which is imposible. Consequently we have proved the inclusion

$$V_0 \setminus P \subset \bigcup_{k=0}^{\infty} A_k.$$

The inclusion $\bigcup_{k=0}^{\infty} A_k \subset V_0$ follows by Lemmas 1 and 2. Let $x \in \bigcup_{k=0}^{\infty} A_k$ and let us suppose that $x \in P$. Then by Lemmas 1 and 2 for any k, $k = 1, 2, \ldots$, we have $p[f^k(x)] = 0$. Thus $f^k(x) \in V_0$, $k = 1, 2, \ldots$, which is impossible, since $x \in \bigcup_{k=0}^{\infty} A_k$. Consequently, we get $\bigcup_{k=0}^{\infty} A_k \subset V_0 \setminus P$.

LEMMA 10. $\operatorname{Int} P = \emptyset$.

Proof. Let $x_0 \in P$, $\varepsilon > 0$ and $q \in \Gamma(E)$, i.e., let q be any seminorm determining the topology of E. There exist an $x_1 \in E$ and a $\lambda > 0$ such that $p(x_1) > 0$ and $\lambda q(x_1 - x_0) < \varepsilon$. For

$$x := (1-\lambda)x_0 + \lambda x_1$$

we have

$$q(x - x_0) = q[\lambda(x_1 - x_0)] = \lambda q(x_1 - x_0) < \varepsilon.$$

Moreover,

$$\begin{aligned} p(x) &= p[(1-\lambda)x_0 + \lambda x_1] \geqslant |p[(1-\lambda)x_0] - p(\lambda x_1)| \\ &= |(1-\lambda)p(x_0) - \lambda p(x_1)| = \lambda p(x_1) > 0. \end{aligned}$$

Consequently, in every neighbourhood of point x_0 there is a point x such that $x \notin P$.

THEOREM 1. *Let E and F be locally convex topological vector spaces, let E be a Hausdorff space, and let a function $f: E \to F$ fulfil hypotheses* (I) *and* (II). *Let a function $\psi: A_0 \cup C_1 \to F$ be continuous and fulfil the condition*

$$\psi(x) = s\psi[f^{-1}(x)], \quad x \in C_1,$$

where $A_0 = V_0 \setminus f(V_0)$ and $C_1 = f(C_0)$ (V_0 and C_0 are defined by (4) *and* (5), *respectively). Then there exists exactly one solution φ of the equation*

$$\varphi[f(x)] = s\varphi(x) \tag{10}$$

continuous in $E \setminus P$ and such that

$$\varphi(x) = \psi(x) \quad \text{for } x \in A_0.$$

Proof. According to Theorem 1.1 from paper [1] and Lemma 8 there exists exactly one function φ which is an extension of ψ onto E and satisfies equation (10) in E. This function is defined by the formula

$$\varphi(x) = s^k \psi[f^{-k}(x)], \quad x \in A_k,\ k \in Z. \tag{11}$$

The function φ is continuous in the set $\operatorname{Int} A_k$, $k \in Z$, as a superposition of continuous functions. It follows by Lemma 7 that $\operatorname{Int} A_k \cap \operatorname{Int} A_l = \emptyset$ for $k \neq l$. Thus $\varphi(x)$ defined by (11) is continuous in the set $\bigcup_{k=-\infty}^{\infty} \operatorname{Int} A_k$.

By (11)

(12) $$\varphi(x) = s\psi[f^{-1}(x)] \quad \text{for } x \in A_1$$

holds. Let $x_0 \in C_1 \subset A_1$. Hence, and from the continuity of function ψ on $A_0 \cup C_1$ it follows that for an arbitrary neighbourhood U of the point $\varphi(x_0)$ there exists a neighbourhood $V^1_{x_0}$ of x_0 such that

(13) $$\varphi[V^1_{x_0} \cap (A_0 \cup C_1)] \subset U.$$

There exists a point $y_0 \in C_0$ such that $f(y_0) = x_0$. By (12), $\varphi(x_0) = s\psi(y_0)$ holds; hence there exists a neighbourhood U_1 of the point $\psi(y_0)$ such that $sU_1 \subset U$. Further, it follows from the continuity of the function ψ at $y_0 \in C_0 \subset A_0$ that there exists a neighbourhood V_{y_0} of y_0 such that

$$\psi[V_{y_0} \cap (A_0 \cup C_1)] \subset U_1.$$

Hence

$$s\psi[V_{y_0} \cap (A_0 \cup C_1)] \subset sU_1 \subset U.$$

We get from (12)

(14) $$\varphi[f(x)] = s\psi(x)$$

for $x \in A_0$. Now, we shall prove that equality (14) holds also for $x \in C_1$. Let $x \in C_1$. There exists an $x' \in C_0 \subset A_0$ such that $x = f(x')$ since $C_1 \subset A_1$. Hence

(15) $$s\psi(x) = s\psi[f(x')] = s^2\psi(x'),$$

since ψ fulfils equation (10) in A_0. On the other hand, we get by (11)

(16) $$\varphi[f(x)] = \varphi[f^2(x')] = s^2\psi(x').$$

Equalities (15) and (16) give (14) for $x \in C_1$. By (14) we have

$$s\psi[V_{y_0} \cap (A_0 \cup C_1)] = \varphi\big[f\big(V_{y_0} \cap (A_0 \cup C_1)\big)\big].$$

The function f is a homeomorphism and therefore $V^2_{x_0} := f(V_{y_0})$ is a neighbourhood of x_0 and

(17) $$\varphi[V^2_{x_0} \cap (A_1 \cup C_2)] \subset U.$$

The sets $f^2(V_0)$ and $B := \{x \in E: p(x) \geqslant 1\}$ are closed and they do not contain x_0, and so we can find a neighbourhood $V_{x_0} \subset V^1_{x_0} \cap V^2_{x_0}$ such that

$$V_{x_0} \cap f^2(V_0) = V_{x_0} \cap B = \emptyset.$$

Thus

$$V_{x_0} \subset V_0 \setminus f^2(V_0) = A_0 \cup A_1.$$

In view of (13) and (17) we get

$$\varphi(V_{x_0}) = \varphi\big(V_{x_0} \cap (A_0 \cup A_1)\big) \subset \varphi(V^1_{x_0} \cap A_0) \cup \varphi(V^2_{x_0} \cap A_1) \subset U.$$

We have proved the continuity of φ at points of the set C_1. Hence the continuity of φ may be obtained on C_k, $k \in Z$, by the principle of induction. By Corollary 2 and Lemma 7 we have the continuity of φ on $E \setminus P$.

THEOREM 2. *Every solution of equation* (10), *continuous in* $E \setminus P$ *and bounded on* A_0, *may be extended to a continuous solution of that equation in* E.

Proof. Let $x_0 \in P$, $q \in \Gamma(F)$ and let φ be a continuous solution of equation (10) in $E \setminus P$. It follows by the boundedness of the function φ that there exists a constant $M > 0$ fulfilling the inequality

$$q[\varphi(x)] \leqslant M \tag{18}$$

for $x \in A_0$. Let $x \in A_k$, where k is a positive integer. Then $x = f^k(x_0)$ for some $x_0 \in A_0$. Since $|s| < 1$, we get

$$q[\varphi(x)] = q\big(\varphi[f^k(x_0)]\big) = q\big(s^k \varphi(x_0)\big) = |s|^k q[\varphi(x_0)] \leqslant M.$$

Thus, inequality (18) holds for every $x \in A_k$, $k \in Z$, $k \geqslant 0$. By Lemma 9 $\bigcup_{k=0}^{\infty} A_k = V_0 \setminus P$, and so inequality (18) holds for $x \in V_0 \setminus P$. Let

$$G_k = \{x \in f^k(E) : p[f^{-k}(x)] < 1\}, \quad k = 0, 1, 2, \ldots .$$

The sets G_k are open and $P \subset \bigcup_{k=0}^{\infty} G_k$.

Let $\varepsilon > 0$ and let us fix k such that $|s|^k M < \varepsilon$ and we put $V_{x_0} := G_k$. It is a neighbourhood of x_0. The set $V_{x_0} \cap (E \setminus P)$ is non-empty by Lemma 10. For $x \in V_{x_0} \cap (E \setminus P)$ we have $q[\varphi(x)] = |s|^k q\big(\varphi[f^{-k}(x)]\big) \leqslant |s|^k M < \varepsilon$. Thus $\lim_{x \to x_0} \varphi(x) = 0$ for $x_0 \in P$. Hence and by the definition of P the function

$$\bar{\varphi}(x) := \begin{cases} \varphi(x), & x \in E \setminus P, \\ 0, & x \in P, \end{cases}$$

is a continuous solution of equation (10) in E.

The following corollary results from Theorem 2 and from its proof.

COROLLARY 7. *Every solution of equation* (10) *on* $E \setminus P$ *which is bounded on* A_0 *may be extended to solution of that equation on* E *continuous on* P.

References

[1] M. Kuczma, *On the Schröder equation*, Rozprawy Mat. 34 (1963).
[2] — *Functional equations in a single variable*, Warszawa 1968.

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