A simple boundedness theorem
for a Liénard equation with damping

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As we can see in the notable book [4], in order to see the boundedness of the equation
(1) \[ x'' + f(x)x' + g(x) = e(t) \]
or the equivalent system
(2) \[ x' = y - F(x) + E(t), \quad y' = -g(x) \quad \text{on } \mathbb{R}^2, \]
where \( F(x) := \int_{0}^{x} f(u)du \) and \( E(t) := \int_{0}^{t} e(s)ds \), there are quite various methods in constructing a family of closed curves on \( \mathbb{R}^2 \) which serve to prove the boundedness of the solutions of (2). It is well known that under the conditions
(3) \[ xF(x) \geq 0, \quad xg(x) \geq 0, \quad G(x) \to \infty \quad (|x| \to \infty), \quad e(t) \equiv 0, \]
where \( G(x) := \int_{0}^{x} g(u)du \), the continuous family
(4) \[ V(x, y) := \frac{1}{2}y^2 + G(x) = c \quad \text{with a parameter } c \]
provides such a family. However, in the most of the cases, some of the conditions in (3) are replaced with weaker conditions, and such a closed curve is obtained by connecting several arcs in a skillful way. It is also attempted to get such a family as a continuous family caused by a single Lyapunov function, but an eligible Lyapunov function is usually constructed in a patched work by a finesse, cf. [6]; p. 41.

When \( xg(x) \geq 0 \ (|x| \geq a), \ F(x) \operatorname{sgn} x \to \infty \ (|x| \to \infty) \) and \( |E(t)| \leq M \) for constants \( a \) and \( M \), Mizohata and Yamaguti [2] have presented an idea to construct a Lyapunov function in a rather simple manner, that is,
\[ V(x, y) := \frac{1}{2}(y - d(x))^2 + G(x) \]
is a desirable Lyapunov function if \( d(x) \) is given by

\[
d(x) := \begin{cases} 
L, & x \geq b, \\
\frac{L}{b} x, & |x| \geq b, \\
-L, & x \leq -b
\end{cases}
\]

for an \( L > 0 \) and a \( b \geq a \) such that \( F(x) \text{sgn} x > M + L \ (|x| \geq b) \). It is not difficult to see that \( V(t, x, y) \leq 0 \) outside a bounded set, which guarantees the boundedness of the \( y \)-component of the solution, while that of the \( x \)-component follows from the first equation of (2) when \( y(t) \) is bounded.

In this article, it will be shown that by generalizing their idea we shall state a boundedness theorem whose proof is supplied by a Lyapunov function constructed in a simple manner similar to \([2]\) but which is general enough to make the theorem due to Graef \([1]\) as its corollary.

Consider the system

\[
x' = \varphi(y) - F(t, x, y), \quad y' = -g(x),
\]

where \( \varphi(y), F(t, x, y) \) and \( g(x) \) are continuous in their arguments. Then, we have the following theorem.

**Theorem.** Suppose the following conditions:

(i) \( xg(x) \geq 0 \ (|x| \geq a) \) for an \( a > 0 \);

(ii) there are constants \( \delta > 0 \) and \( c, M \in \mathbb{R} \) such that

\[
\{F(t, x, y) - c\} \text{sgn} x \geq \delta \quad \text{if} \ |x| \geq a,
\]

and

\[
|F(t, x, y)| \leq M \quad \text{if} \ |x| \leq a,
\]

whatever \( t \geq 0 \) and \( y \in \mathbb{R} \) are;

(iii) for any \( \gamma \) and \( \beta > 0 \) there are \( x^- < 0 \) and \( x^+ > 0 \) such that

\[
G(x^\pm) \pm F(t, x^\pm, y) > \gamma \quad \text{for all} \ t \geq 0 \ \text{and} \ y \in [-\beta, \beta];
\]

(iv) \( \varphi(y) \text{sgn} y \to \infty \) as \( |y| \to \infty \).

Then the solutions of (5) are uniformly bounded, and if condition (i) is strengthened to

(i*) \( xg(x) > 0 \ (|x| \geq a) \),

then they are uniformly ultimately bounded.

**Proof.** Set

\[
V(x, y) := G(x) + \Phi(y) - d(x)y,
\]
where $\Phi(y) := \int_0^y \phi(u) du$ and

$$d(x) := c + \begin{cases} 
\varepsilon & \text{if } x \geq a, \\
\frac{\varepsilon}{a} x & \text{if } |x| \leq a, \\
-\varepsilon & \text{if } x \leq -a 
\end{cases}$$

for an $\varepsilon \in (0, \delta)$. Since $|d(x)| \leq |c| + \varepsilon =: d_0$ and since

$$G(x) \geq \int_{a \cdot \text{sgn } x}^x g(u) du - \int_{-a}^0 |g(u)| du,$$

$$\Phi(y) - d(x)y \geq \int_0^y (\phi(u) - d_0 \text{sgn } u) du,$$

the function $V(x, y)$ is bounded from below and we have

$$V(x, y) \to \infty \quad \text{as } |y| \to \infty \quad \text{uniformly in } x \in \mathbb{R}.$$  \hspace{1cm} (6)

On the other hand, by calculating the derivative

$$V'(t, x, y) = \phi(y)(-g(x)) + g(x)(\phi(y) - F(t, x, y))$$

$$-d'(x)y(\phi(y) - F(t, x, y)) - d(x)(-g(x))$$

$$= -g(x)(F(t, x, y) - d(x)) - d'(x)y(\phi(y) - F(t, x, y))$$

along the solution of (5). Therefore, we have

$$V'(t, x, y) \leq -g(x)(F(t, x, y) - c - \varepsilon \cdot \text{sgn } x) \leq - (\delta - \varepsilon)|g(x)|$$

if $|x| \geq a$, while when $|x| \leq a$,

$$V'(t, x, y) \leq -\frac{\varepsilon}{a} y(\phi(y) - M \cdot \text{sgn } y) + (M + \varepsilon) \max_{|x| \leq a} |g(x)|,$$

which implies

$$V'(t, x, y) \leq -\eta \quad (|x| \leq a, \ |y| \geq b)$$

for a $b > 0$ and an $\eta > 0$ by condition (iv). Hence, we have

$$V'(t, x, y) \leq -\min \{(\delta - \varepsilon)|g(x)|, \ \eta\} \leq 0$$  \hspace{1cm} (7)

if $V(x, y) > \alpha_0$, where $\alpha_0 := \max \{V(x, y) : |x| \leq a, \ |y| \leq b\}$. This implies that for any $\alpha > \alpha_0$ the set

$$V_\alpha := \{(x, y) : V(x, y) \leq \alpha\}$$

is positively invariant, that is, for any $(\xi, \zeta) \in V_\alpha$ a solution of (5) starting at $(\xi, \zeta)$ at $t = \tau$ belongs to $V_\alpha$ for all $t \geq \tau$.

By (6) there is a $\beta = \beta(\alpha) > 0$ such that

$$V(x, y) \leq \alpha \quad \text{implies} \quad |y| \leq \beta(\alpha),$$
while condition (iii) guarantees the existence of a $q = q(x, r) > r$ for any $r \geq a$ such that

$$G(q^+) > \gamma_0(\alpha) \quad \text{or} \quad \inf_{|y| \leq \beta(\alpha), t} F(t, q^+, y) > \gamma_1(\alpha)$$

for a $q^+ \in (r, q(\alpha, r)]$ and

$$G(q^-) > \gamma_0(\alpha) \quad \text{or} \quad \sup_{|y| \leq \beta(\alpha), t} F(t, q^-, y) < -\gamma_1(\alpha)$$

for a $q^- \in [-q(\alpha, r), -r)$, where

$$\gamma_0(\alpha) := \alpha + d_0 \beta(\alpha) - \inf_{|y| \leq \beta(\alpha)} \Phi(y) \quad \text{and} \quad \gamma_1(\alpha) := \sup\{|\varphi(y)|: |y| \leq \beta(\alpha)\}.$$

Thus, we can show that for any given $r \geq a$ we can choose $\alpha = \alpha(r) > \alpha_0$ so that $D_r := \{(x, y): x^2 + y^2 \leq r^2\} \subseteq \mathcal{V}_{a_0}$, and hence any solution of (5) starting from $D_r$ at $t = \tau$ must stay in $\{(x, y): |x| \leq q(\alpha(r), r), |y| \leq \beta(\alpha(r))\}$ for all $t \geq \tau$. This means that the solutions of (5) are uniformly bounded, and also we can find an $A(r) > \alpha_0$ such that any solution of (5) starting from $D_r$ stays in $\mathcal{V}_{A(r)}$ for ever.

If condition (i*) is satisfied, then we have

$$V'(t, x, y) \leq -\eta^* \quad \text{on the complement of } \mathcal{V}_{a_0}$$

for an $\eta^* > 0$. Therefore, any solution of (5) starting from $D_r$ at $t = \tau$ can not keep off from $\mathcal{V}_{a_0}$ in the whole interval $[\tau, \tau + T_r]$, where $T_r := (A(r) - \alpha_0)/\eta^*$, and hence the solution must stay in $\mathcal{V}_{a_0}$ for all $t \geq \tau + T_r$, while we can see that the $x$-component will remain in $\{x: |x| \leq q(\alpha_0, a)\}$ if $t \geq \tau + T_r + T_2(r)$, where $T_2(r) := 2\beta(\alpha(r))/v(r)$ and $v(r) := \inf\{|g(x)|: a \leq |x| \leq q(\alpha(r)), r\}$.

This proves the uniform ultimate boundedness of the solutions of (5).

**Example 1.** For the system

$$x' = y - F(x) + E(t), \quad y' = -g(x),$$

Graef [1] has shown that the solutions are uniformly ultimately bounded if

(a) $xg(x) > 0$ ($|x| \geq a$);

(b) $|E(t)| \leq \gamma$;

(c) $xF(x) \geq \gamma |x|$ ($|x| \geq a$);

(d) $xF(x) \geq (\gamma + \epsilon)|x|$ for $x \geq a$ or for $x \leq -a$, where $\epsilon > 0$ is any given number;

(e) $G(x) + F(x)\text{sgn } x \to \infty$ ($|x| \to \infty$).

Since conditions (c) and (d) imply that

$$\left(F(x) - E(t) - \sigma \frac{\epsilon}{2}\right)\text{sgn } x \geq \frac{\epsilon}{2}$$

for $|x| \geq a$, where $\sigma = 1$ if the relation in (d) holds for $x \geq a$ and $\sigma = -1$ otherwise, all the conditions in the theorem are satisfied.
Example 2. Wu [5] has considered the existence of a nontrivial periodic solution of the system

\[ x' = \varphi(y) - F(x), \quad y' = -g(x) \]

under the conditions:

(a) \( xg(x) > 0 \) (\( x \neq 0 \)), \( G(x) \to \infty \) (\( |x| \to \infty \));
(b) \( xF(x) < 0 \) for small \( |x| > 0 \), \( F(x) \geq k \) (\( x \geq a \)), \( F(x) \leq k' < k \) (\( x \leq -a \));
(c) \( y\varphi(y) > 0 \) (\( y \neq 0 \)), \( \varphi(y) \sgn y \to \infty \) (\( |y| \to \infty \)).

Obviously, all the conditions in the theorem are fulfilled, because

\[ (F(x) - c)\sgn x \geq \delta > 0 \quad (|x| \geq a) \]

with \( c = (k + k')/2 \) and \( \delta = (k - k')/2 \). Therefore, the solutions of (9) are uniformly ultimately bounded, and hence the conclusion follows from the Theorem of Poincaré–Bendixson, where we should note that the origin is a unique critical point and unstable.

Remark. It should be stated that there are many examples which are not covered by our theorem. For example, Opial [3] has shown that under the conditions:

(a) \( |e(t)| \leq m \);
(b) \( \liminf_{x \to \infty} g(x) > m, \limsup_{x \to -\infty} g(x) < -m \);
(c) \( \lim_{|x| \to \infty} F(x)\sgn x = \infty \);
(d) \( \liminf_{|x| \to \infty} \{|F(x)|[|F(x)| - 2m] - 4m|x|\} > -\infty \) for a \( 0 < p \leq m/2 \)

the solutions of the system

\[ x' = y - F(x), \quad y' = -g(x) + e(t), \]

which is equivalent to (1), are uniformly bounded. Our theorem does not cover this example. However, when \( e(t) \) is \( \omega \)-periodic, there is a \( \mu := \frac{1}{\omega} \int_0^\omega e(s)ds \) and \( E(t) - \mu t \) is bounded, and hence by applying the theorem to the system

\[ x' - y - F(x) + E(t) - \mu t, \quad y' = -g(x) + \mu \]

instead of the system (10) we can see the ultimately boundedness of the solutions of (1), and we can present weaker conditions for it.

References


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