ANNALES
POLONICI MATHEMATICI
LI (1990)

A simple boundedness theorem for a Liénard equation with damping

by JUNJI KATO (Sendai)

Zdzisław Opial in memoriam

As we can see in the notable book [4], in order to see the boundedness of the equation

$$x''+f(x)x'+g(x) = e(t) \tag{1}$$

or the equivalent system

$$x' = y-F(x)+E(t), \quad y' = -g(x) \quad \text{on } \mathbf{R}^2, \tag{2}$$

where $F(x):=\int_0^x f(u)du$ and $E(t):=\int_0^t e(s)ds$, there are quite various methods in constructing a family of closed curves on $\mathbf{R}^2$ which serve to prove the boundedness of the solutions of (2). It is well known that under the conditions

$$xF(x)\geqslant 0, \quad xg(x)\geqslant 0, \quad G(x)\to\infty \quad (|x|\to\infty), \quad e(t)\equiv 0, \tag{3}$$

where $G(x):=\int_0^x g(u)du$, the continuous family

$$V(x, y):=\tfrac{1}{2}y^2+G(x) = c \quad \text{with a parameter } c \tag{4}$$

provides such a family. However, in the most of the cases, some of the conditions in (3) are replaced with weaker conditions, and such a closed curve is obtained by connecting several arcs in a skillful way. It is also attempted to get such a family as a continuous family caused by a single Lyapunov function, but an eligible Lyapunov function is usually constructed in a patched work by a finesse, cf. [6]; p. 41.

When $xg(x)\geqslant 0$ $(|x|\geqslant a)$, $F(x)\operatorname{sgn}x\to\infty$ $(|x|\to\infty)$ and $|E(t)|\leqslant M$ for constants a and M, Mizohata and Yamaguti [2] have presented an idea to construct a Lyapunov function in a rather simple manner, that is,

$$V(x, y):=\tfrac{1}{2}(y-d(x))^2+G(x)$$

is a desirable Lyapunov function if $d(x)$ is given by

$$d(x) := \begin{cases} L, & x \geqslant b, \\ \dfrac{L}{b}x, & |x| \geqslant b, \\ -L, & x \leqslant -b \end{cases}$$

for an $L > 0$ and a $b \geqslant a$ such that $F(x)\operatorname{sgn} x > M + L$ $(|x| \geqslant b)$. It is not difficult to see that $V'_{(2)}(t, x, y) \leqslant 0$ outside a bounded set, which guarantees the boundedness of the y-component of the solution, while that of the x-component follows from the first equation of (2) when $y(t)$ is bounded.

In this article, it will be shown that by generalizing their idea we shall state a boundedness theorem whose proof is supplied by a Lyapunov function constructed in a simple manner similar to [2] but which is general enough to make the theorem due to Graef [1] as its corollary.

Consider the system

$$(5) \qquad x' = \varphi(y) - F(t, x, y), \qquad y' = -g(x),$$

where $\varphi(y)$, $F(t, x, y)$ and $g(x)$ are continuous in their arguments. Then, we have the following theorem.

THEOREM. *Suppose the following conditions:*

(i) $xg(x) \geqslant 0$ $(|x| \geqslant a)$ *for an* $a > 0$;

(ii) *there are constants* $\delta > 0$ *and* c, $M \in \boldsymbol{R}$ *such that*

$$\{F(t, x, y) - c\}\operatorname{sgn} x \geqslant \delta \qquad \text{if } |x| \geqslant a,$$

and

$$|F(t, x, y)| \leqslant M \qquad \text{if } |x| \leqslant a,$$

whatever $t \geqslant 0$ *and* $y \in \boldsymbol{R}$ *are;*

(iii) *for any* γ *and* $\beta > 0$ *there are* $x^- < 0$ *and* $x^+ > 0$ *such that*

$$G(x^\pm) \pm F(t, x^\pm, y) > \gamma \qquad \text{for all } t \geqslant 0 \text{ and } y \in [-\beta, \beta];$$

(iv) $\varphi(y)\operatorname{sgn} y \to \infty$ *as* $|y| \to \infty$.

Then the solutions of (5) *are uniformly bounded, and if condition* (i) *is strengthened to*

(i*) $xg(x) > 0$ $(|x| \geqslant a)$,

then they are uniformly ultimately bounded.

Proof. Set

$$V(x, y) := G(x) + \Phi(y) - d(x)y,$$

where $\Phi(y) := \int_0^y \varphi(u)du$ and

$$d(x) := c + \begin{cases} \varepsilon & x \geqslant a, \\ \frac{\varepsilon}{a}x & |x| \leqslant a, \\ -\varepsilon & x \leqslant -a \end{cases}$$

for an $\varepsilon \in (0, \delta)$. Since $|d(x)| \leqslant |c| + \varepsilon =: d_0$ and since

$$G(x) \geqslant \int_{a\cdot\operatorname{sgn} x}^{x} g(u)du - \int_{-a}^{a} |g(u)|du,$$

$$\Phi(y) - d(x)y \geqslant \int_0^y (\varphi(u) - d_0 \operatorname{sgn} u)du,$$

the function $V(x, y)$ is bounded from below and we have

(6) $$V(x, y) \to \infty \quad \text{as } |y| \to \infty \text{ uniformly in } x \in \boldsymbol{R}.$$

On the other hand, by calculating the derivative

$$\begin{aligned} V'(t, x, y) &= \varphi(y)(-g(x)) + g(x)(\varphi(y) - F(t, x, y)) \\ &\quad - d'(x)y(\varphi(y) - F(t, x, y)) - d(x)(-g(x)) \\ &= -g(x)(F(t, x, y) - d(x)) - d'(x)y(\varphi(y) - F(t, x, y)) \end{aligned}$$

along the solution of (5). Therefore, we have

$$V'(t, x, y) \leqslant -g(x)(F(t, x, y) - c - \varepsilon\cdot\operatorname{sgn} x) \leqslant -(\delta - \varepsilon)|g(x)|$$

if $|x| \geqslant a$, while when $|x| \leqslant a$,

$$V'(t, x, y) \leqslant -\frac{\varepsilon}{a}y(\varphi(y) - M\cdot\operatorname{sgn} y) + (M + \varepsilon)\max_{|x| \leqslant a}|g(x)|,$$

which implies

$$V'(t, x, y) \leqslant -\eta \qquad (|x| \leqslant a, \ |y| \geqslant b)$$

for a $b > 0$ and an $\eta > 0$ by condition (iv). Hence, we have

(7) $$V'(t, x, y) \leqslant -\min\{(\delta - \varepsilon)|g(x)|, \eta\} \leqslant 0$$

if $V(x, y) > \alpha_0$, where $\alpha_0 := \max\{V(x, y)\colon |x| \leqslant a, \ |y| \leqslant b\}$. This implies that for any $\alpha > \alpha_0$ the set

$$V_\alpha := \{(x, y)\colon V(x, y) \leqslant \alpha\}$$

is positively invariant, that is, for any $(\xi, \zeta) \in V_\alpha$ a solution of (5) starting at (ξ, ζ) at $t = \tau$ belongs to V_α for all $t \geqslant \tau$.

By (6) there is a $\beta = \beta(\alpha) > 0$ such that

$$V(x, y) \leqslant \alpha \quad \text{implies} \quad |y| \leqslant \beta(\alpha),$$

while condition (iii) guarantees the existence of a $\varrho = \varrho(\alpha, r) > r$ for any $r \geqslant a$ such that

$$G(\varrho^+) > \gamma_0(\alpha) \quad \text{or} \quad \inf_{|y| \leqslant \beta(\alpha), t} F(t, \varrho^+, y) > \gamma_1(\alpha)$$

for a $\varrho^+ \in (r, \varrho(\alpha, r)]$ and

$$G(\varrho^-) > \gamma_0(\alpha) \quad \text{or} \quad \sup_{|y| \leqslant \beta(\alpha), t} F(t, \varrho^-, y) < -\gamma_1(\alpha)$$

for a $\varrho^- \in [-\varrho(\alpha, r), -r)$, where

$$\gamma_0(\alpha) := \alpha + d_0\beta(\alpha) - \inf_{|y| \leqslant \beta(\alpha)} \Phi(y) \quad \text{and} \quad \gamma_1(\alpha) := \sup\{|\varphi(y)|: \ |y| \leqslant \beta(\alpha)\}.$$

Thus, we can show that for any given $r > a$ we can choose $\alpha = \alpha(r) > \alpha_0$ so that $D_r := \{(x, y): \ x^2 + y^2 \leqslant r^2\} \subset V_\alpha$, and hence any solution of (5) starting from D_r at $t = \tau$ must stay in $\{(x, y): \ |x| \leqslant \varrho(\alpha(r), r), \ |y| \leqslant \beta(\alpha(r))\}$ for all $t \geqslant \tau$. This means that the solutions of (5) are uniformly bounded, and also we can find an $A(r) > \alpha_0$ such that any solution of (5) starting from D_r stays in $V_{A(r)}$ for ever.

If condition (i*) is satisfied, then we have

$$V'(t, x, y) \leqslant -\eta^* \quad \text{on the complement of } V_{\alpha_0}$$

for an $\eta^* > 0$. Therefore, any solution of (5) starting from D_r at $t = \tau$ can not keep off from V_{α_0} in the whole interval $[\tau, \tau + T_1(r)]$, where $T_1(r) := (A(r) - \alpha_0)/\eta^*$, and hence the solution must stay in V_{α_0} for all $t \geqslant \tau + T_1(r)$, while we can see that the x-component will remain in $\{x: \ |x| \leqslant \varrho(\alpha_0, a)\}$ if $t \geqslant \tau + T_1(r) + T_2(r)$, where $T_2(r) := 2\beta(\alpha(r))/v(r)$ and $v(r) := \inf\{|g(x)|: \ a \leqslant |x| \leqslant \varrho(\alpha(r)), \ r\}$.

This proves the uniform ultimate boundedness of the solutions of (5).

EXAMPLE 1. For the system

$$x' = y - F(x) + E(t), \qquad y' = -g(x), \tag{8}$$

Graef [1] has shown that the solutions are uniformly ultimately bounded if

(a) $xg(x) > 0$ $(|x| \geqslant a)$;

(b) $|E(t)| \leqslant \gamma$;

(c) $xF(x) \geqslant \gamma|x|$ $(|x| \geqslant a)$;

(d) $xF(x) \geqslant (\gamma + \varepsilon)|x|$ for $x \geqslant a$ or for $x \leqslant -a$, where $\varepsilon > 0$ is any given number;

(e) $G(x) + F(x)\operatorname{sgn} x \to \infty$ $(|x| \to \infty)$.

Since conditions (c) and (d) imply that

$$\left(F(x) - E(t) - \sigma\frac{\varepsilon}{2}\right)\operatorname{sgn} x \geqslant \frac{\varepsilon}{2} \qquad (|x| \geqslant a),$$

where $\sigma = 1$ if the relation in (d) holds for $x \geqslant a$ and $\sigma = -1$ otherwise, all the conditions in the theorem are satisfied.

EXAMPLE 2. Wu [5] has considered the existence of a nontrivial periodic solution of the system

$$x' = \varphi(y) - F(x), \qquad y' = -g(x) \tag{9}$$

under the conditions:

(a) $xg(x) > 0$ $(x \neq 0)$, $G(x) \to \infty$ $(|x| \to \infty)$;

(b) $xF(x) < 0$ for small $|x| > 0$, $F(x) \geqslant k$ $(x \geqslant a)$, $F(x) \leqslant k' < k$ $(x \leqslant -a)$;

(c) $y\varphi(y) > 0$ $(y \neq 0)$, $\varphi(y)\operatorname{sgn} y \to \infty$ $(|y| \to \infty)$.

Obviously, all the conditions in the theorem are fulfilled, because

$$(F(x) - c)\operatorname{sgn} x \geqslant \delta > 0 \qquad (|x| \geqslant a)$$

with $c = (k+k')/2$ and $\delta = (k-k')/2$. Therefore, the solutions of (9) are uniformly ultimately bounded, and hence the conclusion follows from the Theorem of Poincaré–Bendixson, where we should note that the origin is a unique critical point and unstable.

Remark. It should be stated that there are many examples which are not covered by our theorem. For example, Opial [3] has shown that under the conditions:

(a) $|e(t)| \leqslant m$;

(b) $\liminf\limits_{x \to \infty} g(x) > m$, $\limsup\limits_{x \to -\infty} g(x) < -m$;

(c) $\lim\limits_{|x| \to \infty} F(x)\operatorname{sgn} x = \infty$;

(d) $\liminf\limits_{|x| \to \infty} \{|F(x)|[|F(x)| - 2p] - 4m|x|\} > -\infty$ for a $0 < p \leqslant m/2$

the solutions of the system

$$x' = y - F(x), \qquad y' = -g(x) + e(t), \tag{10}$$

which is equivalent to (1), are uniformly bounded. Our theorem does not cover this example. However, when $e(t)$ is ω-periodic, there is a $\mu := \frac{1}{\omega}\int_0^\omega e(s)ds$ and $E(t) - \mu t$ is bounded, and hence by applying the theorem to the system

$$x' - y - F(x) + E(t) - \mu t, \qquad y' = -g(x) + \mu$$

instead of the system (10) we can see the ultimately boundedness of the solutions of (1), and we can present weaker conditions for it.

References

[1] J. R. Graef, *On the generalized Liénard equation with negative damping*, J. Differential Equations 12 (1972), 34–62.

[2] S. Mizohata and M. Yamaguti, *On the existence of periodic solutions of the non-linear differential equation* $x'' + a(x)x' + \varphi(x) = p(t)$, Mem. Coll. Sci. Univ. Kyoto, Ser. A, Math. 27 (1951), 109–113.

[3] Z. Opial, *Sur les solutions de l'équation différentielle* $x'' + h(x)x' + f(x) = e(t)$, Ann. Polon. Math. 8 (1960), 71–74.

[4] R. Reissig, G. Sansone and R. Conti, *Qualitative Theorie Nichtlinearer Differentialgleichungen*, Edizioni Cremonese, Roma, 1963.

[5] K. Wu, *The existence of limit cycles for a nonlinear system*, Acta Math. Sinica 25 (1982), 456–463 (Chinese)

[6] T. Yoshizawa, *Stability Theory by Lapunov's Second Method*, Math. Soc. Japan, Tokyo 1966.

Reçu par la Rédaction le 17.09.1987