

On convergence of successive approximations of some generalized contraction mappings

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Abstract. In this paper we consider the relation $x \in Fx$, where $F: X \rightarrow CN(X)$, X is a metric space and $CN(X)$ denotes the space of all non-empty and closed subsets of X . In Section 2 we formulate some existence theorems for the relation $x \in Fx$ by the method of successive approximations imposing certain general contraction conditions on the operator F . In particular, we generalize some results of [7], [9], [20], [21], [32], [33].

In Section 3 we use the results obtained to the case of the generalized metric space which was introduced in paper [41] (see also [27], [30]). This gives a possibility of an application of our theorems to certain functional and integro-functional equations (Theorems 3.1–3.2).

All our results have been obtained by using the idea of Ważewski's method [27], [28], [41] to general contraction mappings in orbitally complete metric spaces.

1. Notation, definitions and lemmas. Let $R_+ = \{g \in R: g \geq 0\}$, where R is the set of real numbers. Let $S(R_+)$ denote the set of all non-negative, non-increasing and bounded sequences $(g_n)_{n \in N} \in R_+^N$. In the sequel $S_0(R_+) \subset S(R_+)$ will denote the subclass of all sequences convergent to zero and for $(g_n)_{n \in N} \in S(R_+)$ we will write $g_n \searrow g$ if $\lim g_n = g$.

Let the mappings $a^i: R_+^5 \rightarrow R_+$, $i = 1, 2$, satisfy the following conditions:

(A) a^i is a mapping non-decreasing in each coordinate separately and monotonically continuous for $i = 1, 2$, i.e., the conditions $g_{k,n} \searrow g_k$, $k = 1, 2, \dots, 5$, imply $\lim a^i(g_{1,n}, \dots, g_{5,n}) = a^i(g_1, \dots, g_5)$ for any $(g_1, \dots, g_5) \in \Delta^5$, $\Delta = [0, h] \subset R_+$;

(B) equations

$$(1.1)_i \quad g = \hat{a}^i(g),$$

where $\hat{a}^i(g) = (a^i \circ \varphi_i)(g)$, $i = 1, 2$, $\varphi_1(g) = (g, g, 2g, g, 2g)$, $\varphi_2(g) = (g, 2g, g, 2g, g)$, have in Δ the unique solution $g = 0$.

For $c \in \Delta$ and $(q_n)_{n \in N_0} \in S_0(\Delta)$ ($S_0(\Delta)$ denotes the set of sequences (q_n) such that $q_n \searrow 0$ and $q_n \in \Delta$, $n = 0, 1, \dots$) we define the sequence of successive approximations;

$$(A_n^i(c, q_0, \dots, q_n))_{n \in N_0}, \quad N_0 = N \cup \{0\},$$

where

$$A_0^i(c, q_0) = c, \quad A_n^i(c, q_0, \dots, q_n) = q_n + \hat{a}^i(A_{n-1}^i(c, q_0, \dots, q_{n-1})),$$

$$n = 1, 2, \dots, i = 1, 2.$$

Let us consider the equations

$$(1.2_i) \quad g = \hat{a}^i(g) + q$$

on $[0, b]$, where $q, 2b \in \Delta$, $i = 1, 2$. Let b^i be a solution of (1.2_i) on $[0, b]$, $i = 1, 2$. If for any solution g^i of equation (1.2_i) on the interval $[0, b]$ the inequality $g^i \leq b^i$ holds, then b^i is called a *maximal solution* of (1.2_i) on $[0, b]$ and is denoted by $m_i(q, b)$, $i = 1, 2$.

Lemmas 1.1–1.3 formulated below are a simple adaptation of the well-known lemmas included in papers [27], [28] (see also [30], p. 20–22). Thus we may omit their proofs.

LEMMA 1.1. *If assumptions (A) and (B) are satisfied, $(q_n)_{n \in \mathbb{N}_0} \in S_0(\Delta)$ and if there exists $c \in \Delta$ such that $q_0 + \hat{a}^i(c) \leq c$, then the sequence $(A_n^i(c, q_0, \dots, q_n))_{n \in \mathbb{N}_0}$ is convergent to 0 for $i = 1, 2$.*

LEMMA 1.2. *If assumptions (A) and (B) are satisfied and if there exist $q, 2b \in \Delta$ such that $q + \hat{a}^i(b) \leq b$, then there exists a maximal solution $m_i(q, b) \in \mathring{E}[0, b]$ of (1.2_i), $i = 1, 2$. Moreover, if there exists $p \in [0, b]$ such that $p \leq \hat{a}^i(p) + q$, then $p \leq m_i(q, b)$ for $i = 1, 2$.*

LEMMA 1.3. *Let assumptions (A) and (B) be satisfied on $\Delta = R_+$ and let for any $q \in R_+$ there exist a maximal solution $m_i(q)$ of the equation $g = q + \hat{a}^i(g)$, $i = 1, 2$. Then we have:*

1° *if for some $b \in R_+$ the inequality $b \leq \hat{a}^i(b) + q$ holds, then $b \leq m_i(q)$ for $i = 1, 2$,*

2° *if $q' \leq q$, then $m_i(q') \leq m_i(q)$.*

Let (X, d) be a metric space and let $N(X)$ be the family of all non-empty subsets of X . Let $F_n: X \rightarrow N(X)$ for $n = 1, 2, \dots$ and $w: N \times X \rightarrow N$ be given functions.

Any sequence $(x_n)_{n \in \mathbb{N}_0}$, $x_0 \in X$ and $x_n \in F_n^{w(n, x_{n-1})} x_{n-1}$, $n = 1, 2, \dots$, is called an *orbit of the sequence of mappings $(F_n)_{n \in \mathbb{N}}$* with the start-point x_0 , where $F^{n+1}(x) = F(F^n(x)) = \bigcup_{u \in F^n(x)} F(u)$ for each $x \in X$ and $n = 1, 2, \dots$. The

set of all orbits of a sequence $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ starting from x_0 will be denoted by $O(\mathcal{F}, w, x_0)$.

A fundamental sequence $(x_n) \in O(\mathcal{F}, w, x_0)$ is called a *Cauchy orbit*. Obviously, $(x_n)_{n \in \mathbb{N}} \in X^\mathbb{N}$ is fundamental if and only if $d(x_n, x_{n+p}) \leq q_n$ for some $(q_n)_{n \in \mathbb{N}} \in S_0(R_+)$ and $n \geq n_0$, $p \in \mathbb{N}_0$. The set of all Cauchy orbits $(x_n) \in O(\mathcal{F}, w, x_0)$ is denoted by $CO(\mathcal{F}, w, x_0)$.

We say that (X, d) is an (\mathcal{F}, w, x_0) -orbitally complete space $[(\mathcal{F}, w)-$

orbitally complete space] if each $(x_n) \in CO(\mathcal{F}, w, x_0)$ is convergent to an element of X [(X, d) is an (\mathcal{F}, w, x) -orbitally complete space for each $x \in X$ separately].

We say that (X, d) is an (\mathcal{F}, w, x_0) -orbitally precompact space [(\mathcal{F}, w)-orbitally precompact space] if each $(x_n) \in O(\mathcal{F}, w, x_0)$ has a Cauchy subsequence [(X, d) is an (\mathcal{F}, w, x) -orbitally precompact space for each $x \in X$ separately].

We say that (X, d) is an (\mathcal{F}, w, x) -orbitally compact space [(\mathcal{F}, w)-orbitally compact space] if each $(x_n) \in O(\mathcal{F}, w, x)$ has a convergent subsequence [(X, d) is an (\mathcal{F}, w, x) -orbitally compact space for each $x \in X$ separately].

Let $CN(X)$ be the family of all closed and non-empty subsets of X . Assume that a function $H: N(X) \times N(X) \rightarrow R_+$ has the properties:

$$(H_1) \quad H(A, A) = 0 \text{ for each } A \in N(X),$$

$$(H_2) \quad H(A, B) = H(B, A) \text{ for each } A, B \in N(X),$$

$$(H_3) \quad H(A, B) \leq H(A, C) + H(C, B) \text{ for each } A, B, C \in N(X),$$

(H_4) for every pair of sequences $(A_n)_{n \in N}, (B_n)_{n \in N}, A_n, B_n \in CN(X), n = 1, 2, \dots$, every sequence $(x_n)_{n \in N}$ such that $x_n \in A_n, n \in N$, and every $(q_n)_{n \in N} \in S_0(R_+)$ there exists $(y_n)_{n \in N}$ such that $y_n \in B_n$ and $d(x_n, y_n) \leq H(A_n, B_n) + q_n$ for $n \in N$.

Obviously, the sequence $(A_n)_{n \in N}, A_n \in CN(X), n \in N$, is convergent to $A, A \in CN(X)$, if and only if there exists $(q_n)_{n \in N} \in S_0(R_+)$ and a positive integer n_0 that $H(A_n, A) \leq q_n$ for $n \geq n_0$.

The mapping $T: X \rightarrow CN(X)$ will be called (\mathcal{F}, w, x) -orbitally continuous at $\bar{x} \in X$ [(\mathcal{F}, w)-orbitally continuous on X] if for every $(x_n) \in O(\mathcal{F}, w, x)$ and $\lim x_n = \bar{x}$ we have $\lim Tx_n = T\bar{x}$ [T is (\mathcal{F}, w, x) -orbitally continuous at each point $x \in X$ for every start-point $x \in X$ separately].

The above definitions of (\mathcal{F}, w, x) -orbital continuity of $T: X \rightarrow CN(X)$ are slight modifications of the well-known corresponding definitions considered e.g. in the papers of Ćirić.

Let the function $\delta: X \times CN(X) \rightarrow R_+$ be such that

$$(\delta_1) \quad \delta(x, A) = 0 \Leftrightarrow x \in A \text{ for each } A \in CN(X) \text{ and } x \in X,$$

$$(\delta_2) \quad \delta(x, A) \leq d(x, y) + \delta(y, B) + H(A, B) \text{ for each } x, y \in X \text{ and each } A, B \in CN(X).$$

2. Some fixed-point theorems for multivalued mappings. Let (X, d) be a metric space, H and δ be as in Section 1. Assume that the mappings $F_n: X \rightarrow N(X), n = 1, 2, \dots$, have the properties:

(2.1) there exist a function $w: N \times X \rightarrow N$, a point $x_0 \in X$ and $2b \in N$ such that

(a) $V_n x_{n-1} \in CN(B(x_0, b))$ for each $n = 1, 2, \dots$ and $(x_n) \in O(\mathcal{F}, w, x_0)$, where $V_n = F_n^{w(n, x_{n-1})}$,

(b) (X, d) is an (\mathcal{F}, w, x_0) -orbitally complete space,

(2.2) for every orbit $(x_n) \in O(\mathcal{F}, w, x_0)$ and for each $n, m \in N, n \neq m$, the inequality holds:

$$\begin{aligned} & H(V_n x_{n-1}, V_m x_{m-1}) \\ & \leq a^{i(m,n)}(d(x_{n-1}, x_{m-1}), d(x_{n-1}, x_n), d(x_{m-1}, x_m), d(x_{n-1}, x_m), d(x_n, x_{m-1})), \\ & \qquad i(m, n) = \begin{cases} 1 & \text{for } n < m, \\ 2 & \text{for } n > m. \end{cases} \end{aligned}$$

a^i fulfils (A) and (B) on $\Delta = [0, h]$ for $i = 1, 2$ and there exist $q_0, q_1 \in \Delta$ such that

(a) $d(x_0, x_1) \leq q_0$ for some $x_1 \in V_1 x_0$,

(b) $q_0 + q_1 + \hat{a}^i(h) \leq h$ and $m_i(q_0) + q_1, h \in [0, b]$, where $m_i(q_0 + q_1, h)$ is a maximal solution of the equation $q_0 + q_1 + \hat{a}^i(g) = g$ in $\Delta, i = 1, 2$,

(c) if $g \in R_+$ fulfils $g \leq q_0 + q_1 + \hat{a}^i(g)$, then $g \in \Delta$,

(2.3) for every convergent orbit $(x_n) \in O(\mathcal{F}, w, x_0)$ such that $\lim x_n = \bar{x}$ and for each $k, l \in N, k \neq l$, the inequality holds

$$\begin{aligned} & H(V_k x_{k-1}, V_l x) \\ & \leq a^3(d(x_{k-1}, x), \delta(x_{k-1}, V_k x_{k-1}), \delta(\bar{x}, V_l \bar{x}), \delta(x_{k-1}, V_l x_{k-1}), \delta(\bar{x}, V_k x_{k-1})), \end{aligned}$$

where $a^3: \Delta^5 \rightarrow R_+$ is a non-decreasing and monotonically continuous mapping such that $g = 0$ is the unique solution of the equation

$$g = a^3(0, 0, g, g, 0) \quad \text{or} \quad g = a^3(0, g, 0, 0, g) \quad \text{on } \Delta.$$

THEOREM 2.1. *If assumptions (2.1)–(2.3) are fulfilled, then there exists $x \in B(x_0, b)$ such that $x \in V_n x$ for $n = 1, 2, \dots$*

Proof. For $(q_n)_{n=2,3,\dots} \in S_0(\Delta)$ such that $q_2 \leq q_1$ we define a sequence $(x_n)_{n \in N_0} \in O(\mathcal{F}, w, x_0)$ as follows:

(2.4) x_0, x_1 – as in (2.2) (a), $x_n \in V_n x_{n-1}$ and

$$d(x_{n-1}, x_n) \leq H(V_{n-1} x_{n-2}, V_n x_{n-1}) + q_{n-1} - q_n, \quad n = 2, 3, \dots$$

At first we prove that for each $n = 1, 2, \dots$ we have $x_n \in B(x_0, b)$. Obviously, $d(x_0, x_1) \leq q_0 \leq b$. For any $n \in N$ and $d(x_0, x_{n-1}) \leq b$ we have

$$\begin{aligned} d_n &= d(x_0, x_n) \leq d(x_0, x_1) + d(x_1, x_n) \\ &\leq q_0 + q_1 + H(V_1 x_0, V_n x_{n-1}) \\ &\leq q_0 + q_1 + a^1(d_{n-1}, d_{n-1}, d_{n-1} + d_n, d_n, 2d_n). \\ &\leq q_0 + q_1 + a^1(b, b, b + d_n, d_n, b + d_n). \end{aligned}$$

On account of (2.2) (b)–(c) we get $d_n \leq b$. Indeed, for $d_n > b$ and $d_n \leq q_0 + q_1 + \hat{a}^1(d_n)$ we would have by (2.2) (c) that $d_n \in \Delta$ and (2.2) (b) imply that $d_n \leq m_1(q_0 + q_1, h) \leq b$.

Now we prove that $(x_n) \in CO(\mathcal{F}, w, x_0)$. Obviously, for each $n = 1, 2, \dots$ we have

$$d(x_0, x_n) \leq b = A_0^1(b, q_0),$$

where $A_0^1(b, q_0)$ is the first element of the sequence $(A_n^1(b, q_0, \dots, q_n))_{n \in N_0}$ (see Section 1).

Suppose that $d_p = d(x_p, x_{n+p}) \leq A_p^1(b, q_0, \dots, q_p)$ for some $p \in N_0$. Then

$$\begin{aligned} d_{p+1} &= d(x_{p+1}, x_{n+p+1}) \\ &\leq q_{p+1} - q_{n+p+1} + H(V_{p+1}x_p, V_{n+p}) \\ &\leq q_{p+1} + a^1(d(x_p, x_{n+p}), d(x_p, x_{p+1}), d(x_{n+p}, x_{n+p+1}); \\ &\quad d(x_p, x_{n+p+1}), d(x_{p+1}, x_{n+p})) \\ &\leq q_{p+1} + a^1(d(x_p, x_{n+p}), d(x_p, x_{1+p}), d(x_p, x_{n+p}) + d(x_p, x_{n+1+p}), \\ &\quad d(x_p, x_{n+1+p}), d(x_p, x_{1+p}) + d(x_p, x_{n+p})) \\ &\leq q_{p+1} + \hat{a}^1(A_p^1(b, q_0, \dots, q_p)) = A_{p+1}^1(b, q_0, \dots, q_p). \end{aligned}$$

Analogously we get

$$d(x_p, x_{n+p}) \leq A_p^2(b, q_0, \dots, q_{p-1})$$

and consequently $(x_n) \in CO(\mathcal{F}, w, x_0)$.

Now, (2.1) (b) implies that there exists $\bar{x} \in X$ such that $\lim x_n = \bar{x}$. From (2.1) (a) and from (δ_2) we get

$$\delta_n = \delta(\bar{x}, V_n \bar{x}) \leq d(\bar{x}, x_m) + H(V_m x_{m-1}, V_n \bar{x})$$

and from (2.3)

$$\begin{aligned} \delta_n &= \delta(x, V_n x) \\ &\leq d(\bar{x}, x_m) + a^3(d(x_{m-1}, \bar{x}), \delta(x_{m-1}, V_m x_{m-1}), \\ &\quad \delta(x, V_n x), \delta(x_{m-1}, V_n \bar{x}), \delta(\bar{x}, V_m x_{m-1})). \end{aligned}$$

Thus

$$\delta_n \leq a^3(0, 0, \delta_n, \delta_n, 0)$$

and from the properties of a^3 we infer that $\delta_n = 0$. Therefore

$$x \in V_n x \quad \text{for } n = 1, 2, \dots,$$

which completes the proof.

Remark 2.1. We may replace condition (2.1) (c) of Theorem 2.1 by an equivalent condition of the form (2.1) (c'): (X, d) is an (\mathcal{F}, w, x_0) -orbitally compact space.

Remark 2.2. Let $\psi_1(g) = (g, g, g, 2g, g)$ and $\psi_2(g) = (g, g, g, g, 2g)$ for $g \in R_+$ and let all assumptions (2.1)–(2.3) except (2.2) (c) be fulfilled for

the functions $\hat{a}^1 = a \circ \psi_1$ and $\hat{a}^2 = a \circ \psi_2$. If for some orbit $(x_n) \in O(\mathcal{F}, w, x_0)$ of the form (2.4) the inequality $d(x_n, x_{n+1}) \leq b$ holds for $n = 1, 2, \dots$, then there exists $\bar{x} \in B(x_0, b)$ such that $\bar{x} \in V_n \bar{x}$, $n = 1, 2, \dots$

Remark 2.3. It is easy to see that if $\chi_i(g) = (g, g, g, g, g)$ for $i = 1, 2$, $g \in R_+$ and all assumptions (2.1)–(2.3) except (2.2)(c) are fulfilled for the function $\hat{a}^i = a \circ \chi_i$, $i = 1, 2, \dots$, $g \in R_+$ and if $d(x_0, x_n) \leq b$ for each $n = 2, 3, \dots$, then there exists $\bar{x} \in B(x_0, b)$ such that $\bar{x} \in V_n \bar{x}$ for $n = 1, 2, \dots$

Now we give the global version of Theorem 2.1.

Let (X, d) , H and δ be as above and let $F_n: X \rightarrow N(X)$, $n = 1, 2, \dots$. Moreover, suppose that:

(2.5) there exists a function $w: N \times X \rightarrow N$ such that

(a) $V_n x_{n-1} \in CN(X)$ for $n = 1, 2, \dots$, V_n as in Theorem 2.1, $(x_n) \in O(\mathcal{F}, w, x)$, $x \in X$,

(b) (X, d) is an (\mathcal{F}, w) -orbitally complete space,

(2.6) for every orbit $(x_n) \in O(\mathcal{F}, w, u)$, $u \in X$, and for each $n, m \in N$, $n \neq m$, the inequality of (2.2) holds and a^i fulfils (A) and (B) on $\Delta = R_+$ and for each $q \in R_+$ there exists a maximal solution $m_i(q) \in R_+$ of the equation $q + \hat{a}^i(b) = b$ for $i = 1, 2$,

(2.7) for every convergent orbit $(x_n) \in O(\mathcal{F}, w, x)$, $x \in X$, such that $\lim x_n = \bar{x}$ and for each $k, l \in N$, $k \neq l$, inequality (2.3) holds and $a^3: R_+^5 \rightarrow R_+$ has all the properties of (2.3) on $\Delta = R_+$.

From Theorem 2.1 and Lemma 1.3 we get

THEOREM 2.2. *If assumptions (2.5)–(2.7) are fulfilled, then there exists $\bar{x} \in X$ such that $\bar{x} \in V_n \bar{x}$ for $n = 1, 2, \dots$*

EXAMPLE 2.1. Let (X, d) be a complete metric space and let $F_n: X \rightarrow CB(X)$, $n = 1, 2, \dots$, where $CB(X)$ denotes the class of all non-empty, closed and bounded subsets of X . Let $\delta(x, A) = \inf \{d(x, y) : y \in A\}$ and H be a Hausdorff metric. If assumptions (2.5)–(2.7) are fulfilled, then the assertion of Theorem 2.2 is true.

Remark 2.4. If in Example 2.1 we take $a^i(g) = \alpha \cdot g$ for $i = 1, 2$, $g \in R_+$, $0 \leq \alpha < 1$, then we obtain the theorem of Nadler [32].

Remark 2.5. Let us consider the single-valued mapping $F: X \rightarrow X$, where (X, d) is a metric space. If $F_n = F$, $n = 1, 2, \dots$, then condition (2.6) of Theorem 2.2 has the form

$$(2.6) \quad d(x_{n+1}, x_{m+1}) \leq a^{i(m,n)}(d(x_n, x_m), d(x_n, x_{n+1}), d(x_m, x_{m+1}), d(x_n, x_{m+1}), d(x_{n+1}, x_m)),$$

where

$$i(m, n) = \begin{cases} 1 & \text{for } n < m, \\ 2 & \text{for } n > m, \end{cases}$$

and a^i , $i = 1, 2$, is as in Theorem 2.2. If, additionally, (X, d) is an F -orbitally complete space and F is F -orbitally continuous, then there exists a fixed point of F (see also, for example, Massa [30]). If, furthermore, the inequality

$$(W) \quad d(Fx, Fy) \leq a^i(d(x, y), d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx))$$

holds for each $(x, y) \in X^2$, where $i = 1$ or $i = 2$ and a^i is as in Theorem 2.2, then the fixed point is unique.

In the case of $a(g) = a^i(g, 0, 0, 0, 0)$ we obtain a Ważewski-type condition (see [27] and [41]). It is easy to see that if condition (W) holds, then for each $x_0 \in X$ we have $F^n x_0 \in B(x_0, b)$ for $n = 1, 2, \dots$, $b = \max\{b_1, b_2\}$, where b_i is a maximal solution on R_+ of the equation $q + \hat{a}^i(g) = g$.

Husain and Sehgal prove in [18] that if (X, d) is a complete metric space, $a: R_+^5 \rightarrow R_+$ is a continuous, non-decreasing function in each coordinate variable and satisfies the condition $a(t, t, t, t, t) < t$ for any $t > 0$, and if the mapping $F: X \rightarrow X$ satisfies the following conditions:

(a) $F^n x_0 \in B(x_0, r)$, $n = 1, 2, \dots$ for some $x_0 \in X$, $r > 0$,

(b) $d(Fx, Fy) \leq a(d(x, y), d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx))$ for all $x, y \in X$, then F has a unique fixed point in X .

At first, it is easy to see that the above assertion is true if we assume that the function a is monotonically continuous and (X, d) is an (F, x_0) -orbitally complete metric space (see Remark 2.3). On the other hand, in our case we may additionally write the following estimation: $d(\bar{x}, F^n x_0) \leq m_i(q_n)$ for $n \in N$, where $d(F^n x_0, F^{n+1} x_0) \leq q_n$ and $m_i(q_n)$ is the maximal solution of the equation $q_n + \hat{a}^i(g) = g$, $i = 1, 2$ (see [27] and [41]). Obviously, there exist Husain–Sehgal type functions for which condition (2.2)(c) is not fulfilled. Thus, in general, for Husain–Sehgal-type functions we cannot obtain the above estimates.

Remark 2.6. Let $F: X \rightarrow X$ and $d(Fx, Fy) \leq a(d(x, y))$, where $a: P \rightarrow R_+$ is an upper semicontinuous function from the right on the closure of $P = \{d(x, y): (x, y) \in X^2\}$ and satisfies $a(t) < t$ for all $t \in P \setminus \{0\}$, where (X, d) is an F -orbitally complete space. Then:

(a) $F^n x_0 \in B(x_0, r)$, $n = 1, 2, \dots$, $r > 0$,

(b) F has a unique fixed point $\bar{x} \in X$ and $\lim F^n x = \bar{x}$ for each $x \in X$ (see Boyd and Wong [2]).

But, in general (see Remark 2.2), for Boyd–Wong type functions we cannot obtain the estimation: $d(\bar{x}, F^{n+1} x_0) \leq m(q_n)$ for $n \in N$, where $d(F^n x_0, F^{n+1} x_0) \leq q_n$ and $m(q_n)$ is the maximal solution of the equation $q_n + a(g) = g$.

Remark 2.7. Some theorems of Ćirić and Kubiaczyk result from Theorem 2.2:

(a) Ćirić [40] proves the theorem on the existence and uniqueness of the solution of the equation $x = Fx$ in an F -orbitally complete space, where $F: X \rightarrow X$ fulfils condition (b) of Remark 2.5 for $a(t_1, \dots, t_5) = \max \{t_i: 1 \leq i \leq 5\}$, $0 \leq \lambda < 1$.

Let us consider the orbit (x_n) , $x_{n+1} = Fx_n$, $x_0 \in X$, $n = 0, 1, \dots$. Then we have

$$d(x_{n+1}, x_{m+1}) \leq \begin{cases} \lambda(1-\lambda)^{-1} \max \{d(x_n, x_m), d(x_n, x_{n+1}), d(x_n, x_{m+1})\} & \text{for } n < m, \\ \lambda(1-\lambda)^{-1} \max \{d(x_n, x_m), d(x_m, x_{m+1}), d(x_{n+1}, x_m)\} & \text{for } n > m. \end{cases}$$

Therefore we can take

$$\begin{aligned} a^1(g_1, g_2, g_3, g_4, g_5) &= \lambda(1-\lambda)^{-1} \max \{g_1, g_2, g_4\} & \text{for } n < m, \\ a^2(g_1, g_2, g_3, g_4, g_5) &= \lambda(1-\lambda)^{-1} \max \{g_1, g_3, g_5\} & \text{for } n > m. \end{aligned}$$

The functions a^1 and a^2 fulfil all the assumptions of Theorem 2.2 and therefore the result of Ćirić follows from our Theorem 2.2.

(b) Kubiacyk [24] consider mappings $F_1, F_2: X \rightarrow CB(X)$ for which the inequality holds:

(I) $H(F_1x, F_2y) \leq k\psi(\delta(x, F_1x), \delta(y, F_2y), d(x, y))$ for $(x, y) \in X^2$, $k < 1$, where $\psi: P \rightarrow [0, \infty)$, $P = Q^3$, $Q = \{d(x, y): (x, y) \in X^2\}$, is right continuous and such that:

- (α) $\psi(a, b, c) = \psi(b, a, c)$,
- (β) $\psi(a, b, a) \leq \max \{a, b\}$ for all $(a, b, c), (a, b, a), (b, a, c) \in P$,
- (γ) the orbit (x_n) defined by:

$$\begin{aligned} x_0 \in X, \quad x_1 \in F_1x_0, \\ x_{2n} \in F_2x_{2n-1} \quad \text{and} \quad d(x_{2n-1}, x_{2n}) \leq 1/\sqrt{k} H(F_1x_{2n-2}, F_2x_{2n-1}), \\ x_{2n+1} \in F_1x_{2n} \quad \text{and} \quad d(x_{2n}, x_{2n+1}) \leq 1/\sqrt{k} H(F_2x_{2n-1}, F_1x_{2n}) \end{aligned}$$

contains a convergent subsequence.

Then (see [24]) $\lim x_n = \bar{x}$ and $\bar{x} \in F_i \bar{x}$, $i = 1, 2$.

For the orbit (x_n) we have

$$d(x_{n+1}, x_{m+1}) \leq \begin{cases} k(1-k)^{-1} \max \{d(x_n, x_m), d(x_n, x_{n+1})\} & \text{for } n < m, \\ k(1-k)^{-1} \max \{d(x_n, x_m), d(x_m, x_{m+1})\} & \text{for } n > m. \end{cases}$$

Therefore we can take

$$\begin{aligned} a^1(g_1, g_2, g_3, g_4, g_5) &= k \cdot (1-k)^{-1} \max \{g_1, g_2\} & \text{for } n < m, \\ a^2(g_1, g_2, g_3, g_4, g_5) &= k \cdot (1-k)^{-1} \max \{g_1, g_3\} & \text{for } n > m. \end{aligned}$$

Thus, by Remark 2.6, this orbit is a Cauchy orbit. It is easy to see that the convergence of Cauchy orbits is equivalent to the existence of convergent subsequences of such orbits. Therefore the result of Kubiacyk follows from Theorem 2.2.

Remark 2.8. Tasković [40] showed that Banach's result on contractions can be extended to f -contractions. $F: X \rightarrow X$ is an f -contraction mapping of a metric space (X, d) if for every $x, y \in X$ the inequality holds:

$$(T) \quad d(Fx, Fy) \leq f(\alpha_1(x, y)d(x, y), \alpha_2(x, y)d(x, Fx), \alpha_3(x, y)d(y, Fy), \alpha_4(x, y)d(x, Fy), \alpha_5(x, y)d(y, Fx)),$$

where $f: R_+^5 \rightarrow R_+$ is increasing, semihomogeneous and such that the function $u: R_+ \rightarrow R_+$, $u(g) = f(g_1 \cdot g, \dots, g_5 \cdot g)$ is continuous at the point $g = 1$ for each $(g_1, \dots, g_5) \in R_+^5$, $\alpha_i: X^2 \rightarrow R_+$, $i = 1, \dots, 5$. If condition (T) holds, then the Ćirić condition holds, too. Indeed, we have

$$f(\alpha_1(x, y)d(x, y), \alpha_2(x, y)d(x, Fx), \dots, \alpha_5(x, y)d(y, Fx)) \leq \lambda \max \{d(x, y), d(x, Fx), \dots, d(y, Fx)\},$$

where $\lambda = \sup \{f(\alpha_1(x, y), \dots, \alpha_5(x, y)): (x, y) \in X^2\} < 1$.

Now let (X, d) , H and δ be as in Theorem 2.2. Let $F_n: X \rightarrow CN(X)$, $n = 1, 2, \dots$ fulfil (2.5) and let, moreover,

(2.8) for every orbit $(x_n) \in O(\mathcal{F}, w, x)$, $x \in X$, and each $n, m \in N$, $n \neq m$ the inequality holds:

$$H(V_n x_{n-1}, V_m x_{m-1}) \leq \alpha d(x_{n-1}, x_{m-1}) + \beta (d(x_{n-1}, x_n) + d(x_{m-1}, x_m)) + \gamma (d(x_{n-1}, x_m) + d(x_n, x_{m-1})),$$

where $\alpha, \beta, \gamma \in R_+$ and $\alpha + 2\beta + 3\gamma < 1$,

(2.9) for every convergent orbit $(x_n) \in O(\mathcal{F}, w, x)$, $x \in X$, such that $\lim x_n = \bar{x}$ and for each $k, l \in N$, $k \neq l$, the inequality holds:

$$H(V_k x_{k-1}, V_l \bar{x}) \leq \beta' \delta(\bar{x}, V_l \bar{x}) + \gamma' \delta(x_{k-1}, V_l \bar{x}),$$

where $\beta', \gamma' \in R_+$ and $\beta' + \gamma' < 1$.

THEOREM 2.3. *If assumptions (2.5) and (2.8)–(2.9) are fulfilled, then there exists $z \in X$ such that $z \in V_n z$ for $n = 1, 2, \dots$. Moreover, if in (2.8)–(2.9), $w(n, x) = 1$ for each $(n, x) \in N \times X$, then this assertion is also true with $\alpha + 2\beta + 2\gamma < 1$.*

Proof. Let $z_0 \in X$ and let z_1 be an arbitrary element of $V_1 z_0$. Now let $(q_n) \in S_0(R_+)$ and z_2 be such that $z_2 \in V_2 z_1$ and $d(z_1, z_2) \leq H(V_1 z_0, V_2 z_1) + q_1 - q_2$, $(1 - \mu)d(z_1, z_2) \geq vq_2$, where $\mu = (\alpha + \beta + \mu)(1 - \beta - \gamma)^{-1}$ and $v = (1 - \beta - \gamma)^{-1}$.

Inductively, we assume that for $n = 2, 3, \dots$, $z_n \in V_n z_{n-1}$ and $(1 - \mu) \times d(z_{n-1}, z_n) \geq vq_n$, where μ and v are as above.

Then we have for each $n \in N$

$$\begin{aligned} d(z_n, z_{n+1}) &\leq H(V_n z_{n-1}, V_{n+1} z_n) + q_n - q_{n+1} \\ &\leq \alpha d(z_{n-1}, z_n) + \beta (d(z_{n-1}, z_n) + d(z_n, z_{n+1})) + \\ &\quad + \gamma d(z_{n-1}, z_{n+1}) + q_n - q_{n+1}. \end{aligned}$$

Therefore

$$d(z_n, z_{n+1}) \leq \mu d(z_{n-1}, z_n) + \nu (q_n - q_{n+1}).$$

Now, for each $n, p \in N$ we have

$$\begin{aligned} H(V_n z_{n-1}, V_{n+p} z_{n+p-1}) \\ \leq \alpha d(z_{n-1}, z_{n+p-1}) + \beta d(z_{n-1}, z_n) + \beta \mu^p d(z_{n-1}, z_n) + \\ + \gamma d(z_{n-1}, z_{n+p}) + \gamma d(z_n, z_{n+p-1}) + \nu (q_n - q_{n+p}). \end{aligned}$$

Finally we obtain (for sufficiently large n, p)

$$\begin{aligned} H(V_n z_{n-1}, V_{n+p} z_{n+p-1}) \\ \leq a^1 (d(z_{n-1}, z_{n+p-1}), d(z_{n-1}, z_n), 0, d(z_{n-1}, z_{n+p}), d(z_n, z_{n+p-1})) + q_{n,p}, \end{aligned}$$

where

$$a^1(g_1, g_2, g_3, g_4, g_5) = \alpha g_1 + 2\beta g_2 + \gamma g_4 + \gamma g_5$$

for each $(g_1, \dots, g_5) \in R_+^5$ and $q_{n,p} \searrow 0$ as $n, p \rightarrow \infty$.

Obviously, in the same way we show that

$$a^2(g_1, g_2, g_3, g_4, g_5) := \alpha g_1 + (2\beta + \gamma)g_3 + \gamma g_5$$

for each $(g_1, \dots, g_5) \in R_+^5$.

All the assumptions of Theorem 2.2 for any orbit (z_n) as above are satisfied and Theorem 2.3 is true.

Remark 2.9. It is easy to see that conditions (2.8) and

$$\begin{aligned} (2.8') \quad H(V_n x_{n-1}, V_m x_{m-1}) \\ \leq \alpha d(x_{n-1}, x_{m-1}) + \beta_1 d(x_{n-1}, x_n) + \beta_2 d(x_{m-1}, x_m) + \\ + \gamma_1 d(x_{n-1}, x_m) + \gamma_2 d(x_n, x_{m-1}), \end{aligned}$$

where $\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2 \in R_+$ and $\alpha + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 < 1$, are equivalent. This fact is well known for linear conditions (see e.g. Ghosh [14]) and follows from the symmetry of the metric d . Indeed, we have

$$\begin{aligned} d(x_n, x_m) &\leq \alpha d(x_{n-1}, x_{m-1}) + \frac{1}{2}(\beta_1 + \beta_2) (d(x_{n-1}, x_n) + d(x_{m-1}, x_m)) \\ &\quad + \frac{1}{2}(\gamma_1 + \gamma_2) (d(x_{n-1}, x_m) + d(x_n, x_{m-1})) \end{aligned}$$

and we can take

$$\beta = \frac{(\beta_1 + \beta_2)}{2}, \quad \gamma = \frac{(\gamma_1 + \gamma_2)}{2}.$$

EXAMPLE 2.2. Let (X, d) , H and δ be as in Example 2.1. Let $S, T: X \rightarrow CN(X)$ and let there exist $\alpha, \beta, \gamma \in R_+$ such that $\alpha + 2\beta + 2\gamma < 1$ and for each $x, y \in X$

$$H(Sx, Ty) \leq \alpha d(x, y) + \beta(\delta(x, Sx) + \delta(y, Ty)) + \gamma(\delta(x, Ty) + \delta(y, Tx)).$$

Then there exists a common fixed point of S and T in X . Indeed, if we take $w(n, x) = 1$, $F_{2k+1} = S$, $F_{2k+2} = T$ for $k = 0, 1, \dots$, then the assertion follows from Theorem 2.3.

Remark 2.10. The theorem in Example 2.2 for $\gamma = 0$ and $S, T: X \rightarrow CB(X)$ was proved by Reich (see [35]).

Remark 2.11. If in Example 2.2 $S = T$ and $T: X \rightarrow X$, then we obtain the Kannan-type condition:

$$(K) \quad d(Tx, Ty) \leq \alpha d(x, y) + \beta(d(x, Tx) + d(y, Ty)) + \gamma(d(x, Ty) + d(y, Tx)), \\ (x, y) \in X^2, \alpha, \beta, \gamma \geq 0, \alpha + 2\beta + 2\gamma < 1.$$

Obviously, the assertion of the theorem in Example 2.2 is true if condition (K) holds (the fixed point of T is unique).

Remark 2.12. Non-linear contraction type conditions for single-valued mappings, which generalize the Lanach contraction principle, have been considered by Browder [3], Rakotch [35], Ważewski [41], Kwapisz [27], [28], Boyd and Wong [2], Husain and Sehgal [18], Ćirić [7], [8], [9], Tasković [40] and others. Kannan [20] and Chatterja [6] considered condition (K) in particular cases. Bryant [4], Sehgal [18], Chi Song Wong [42], [43] and others impose (K)-type conditions on some iterate F^k rather than on mapping F . Sequences of generalized contractions are considered, for example, in [29] and [15].

Now let (X, d) , H and δ be as in Theorem 2.3. Let $F_n: X \rightarrow N(X)$, $n = 1, 2, \dots$, fulfil (2.5) and let, moreover,

(2.10) for every orbit $(x_n) \in O(\mathcal{F}, w, x)$, $x \in X$, and for each $n, m \in N$, $n \neq m$, the inequality holds:

$$H(V_n x_{n-1}, V_m x_{m-1}) \\ \leq \alpha(x_{n-1}, x_{m-1}) d(x_{n-1}, x_{m-1}) + \beta(x_{n-1}, x_{m-1}) (d(x_{n-1}, x_n) + d(x_{m-1}, x_m)) + \\ + \gamma(x_{n-1}, x_{m-1}) (d(x_{n-1}, x_m) + d(x_n, x_{m-1})),$$

where $\alpha, \beta, \gamma: X^2 \rightarrow R_+$ and

$$\sup \{ \alpha(x, y) + 2\beta(x, y) + 3\gamma(x, y) : (x, y) \in X^2 \} < 1,$$

(2.11) for every convergent orbit (x_n) , $(x_n) \in O(\mathcal{F}, w, x)$, such that $\lim x_n = \bar{x}$ and for $k, l \in N$, $k \neq l$, the inequality holds:

$$H(V_k x_{k-1}, V_l \bar{x}) \leq \beta'(x_{k-1}, \bar{x}) \delta(x, V_l \bar{x}) + \gamma'(x_{k-1}, \bar{x}) \delta(x_{k-1}, V_l \bar{x}),$$

where $\beta', \gamma': X^2 \rightarrow R_+$ and $\sup \{ \beta'(x, y) + \gamma'(x, y) : (x, y) \in X^2 \} < 1$.

THEOREM 2.4. *If all assumptions (2.5) and (2.10)–(2.11) are fulfilled, then there exists $x \in X$ such that $x \in V_n x$ for $n = 1, 2, \dots$*

Proof. At first we notice that

$$\mu = \sup \{z_1(x, y) : (x, y) \in X^2\} < 1,$$

where

$$z_1(x, y) = (\alpha(x, y) + \beta(x, y) + \gamma(x, y)) (1 - \beta(x, y) - \gamma(x, y))^{-1}.$$

Therefore we can take

$$v = \sup \{(1 - \beta(x, y) - \gamma(x, y))^{-1} : (x, y) \in X^2\}$$

and thus the proof reduces to the proof of Theorem 2.3, where μ and v are as above and, obviously, condition (2.11) imply the existence and uniqueness of the solution x .

Remark 2.12. From Theorem 2.4 we can obtain Theorem 1, proved by Kubiacyk in [24]. For this purpose it is sufficient to take $w(n, x) = 1$, $F_{2k+1} = S$, $F_{2k+2} = T$ for each $k = 0, 1, \dots$ in Theorem 2.4, assuming that (X, d) is complete and $S, T: X \rightarrow CB(X)$.

From Remark 2.11 and Theorem 2.4 we get

CONCLUSION 2.1(Ćirić [7]). *If $F: X \rightarrow X$, (X, d) is an F -orbitally complete space and the inequality*

$$d(Fx, Fy) \leq \alpha(x, y)d(x, y) + \beta_1(x, y)(d(x, Fx) + \beta_2(x, y)d(y, Fy)) + \gamma(x, y)(d(x, Fy) + d(y, Fx))$$

holds for all $(x, y) \in X^2$, where

$$\sup \{\alpha(x, y) + \beta_1(x, y) + \beta_2(x, y) + 2\gamma(x, y) : (x, y) \in X^2\} = \lambda < 1,$$

$\alpha, \beta_1, \beta_2, \gamma: X^2 \rightarrow R_+$, then

- (i) there is a unique fixed-point \bar{x} of F in X ,
- (ii) $\lim F^n x = \bar{x}$ for every $x \in X$,
- (iii) $d(\bar{x}, F^n x) \leq \lambda(1 - \lambda)^{-1} d(x, Fx)$.

Let (X, d) , H and δ be as in Theorem 2.1. Suppose that $F_{2k+1} = S$, $F_{2k+2} = T$ for $k = 0, 1, \dots$ and $S, T: X \rightarrow N(X)$ are such that (2.1) is fulfilled for a function w , where $w(2k-1, x) = w_1$ and $w(2k, x) = w_2$ for $k \in N$, $x \in X$. Assume that (2.2) is fulfilled and that

$$(2.12) \quad S^{w_1}, T^{w_2} \text{ are } (\mathcal{F}, w, x_0)\text{-orbitally continuous on } B(x_0, b).$$

THEOREM 2.5. *If assumptions (2.1)–(2.2) and (2.12) are fulfilled, then there exists $\bar{x} \in B(x_0, b)$ such that $\bar{x} \in S^{w_1} \bar{x}$ and $\bar{x} \in T^{w_2} \bar{x}$.*

Proof. Obviously, by Theorem 2.1, there exists $\bar{x} \in B(x_0, b)$ such that $\lim x_n = \bar{x}$. Thus

$$\delta(\bar{x}, S^{w_1} \bar{x}) \leq d(\bar{x}, x_{2k+1}) + H(S^{w_1} x_{2k}, S^{w_1} \bar{x})$$

and from (2.12)

$$\delta(\bar{x}, S^{w_1} \bar{x}) = 0.$$

On account of

$$\delta(\bar{x}, T^{w_2} \bar{x}) \leq d(\bar{x}, x_{2k+1}) + H(S^{w_1} x_{2k+1}, T^{w_2} \bar{x})$$

we also have $\bar{x} \in T^{w_2} \bar{x}$.

Now we are going to formulate two theorems which are simple generalizations of theorems of Kubiacyk [24].

THEOREM 2.6. *Let (X, d) , H and δ be as above and let $S, T: X \rightarrow CN(X)$. Moreover, suppose that:*

(2.14) *for each $(x, A) \in X \times CN(X)$ there exists $\bar{x} \in A$ such that $d(x, \bar{x}) = \delta(x, A)$,*

(2.15) *there exists $\min \varphi_1(X)$ or $\min \varphi_2(X)$, where $\varphi_1(x) = \delta(x, Sx)$ and $\varphi_2(x) = \delta(x, Tx)$, $x \in X$,*

(2.16) *there exists a function $M: [CN(X)]^2 \rightarrow R_+$ such that*

(a) *for each $A, B \in CN(X)$, $H(A, B) \leq M(A, B)$,*

(b) *$d(x, y) \leq M(A, B)$ for each $A, B \in CN(X)$ and $x \in A, y \in B$,*

(c) *for all distinct elements x, y of X the inequality holds:*

$$M(Sx, Ty) < a(d(x, y), \delta(x, Sx), \delta(y, Ty), \delta(x, Ty), \delta(y, Sx)),$$

where $a: R_+^5 \rightarrow R_+$ is a non-decreasing function with $a(g, g, g, 2g, 2g) \leq g$ for each $g \in R_+$.

Then

(1) *there exists a fixed point of S or T in X ,*

(2) *if $u \in Su$ and $v \in Tv$, then $u = v$.*

THEOREM 2.7. *If all assumptions (2.14)–(2.15) and (2.16) (a) (b) are fulfilled and if*

(2.16') (c) *for all distinct elements x, y of X the inequality holds:*

$$M(Sx, Ty) < \alpha(x, y)d(x, y) + \beta(x, y)(\delta(x, Sx) + \delta(y, Ty)) + \gamma(x, y)(\delta(x, Ty) + \delta(y, Sx)),$$

where $\alpha, \beta, \gamma: X^2 \rightarrow R_+$ and

$$\sup \{ \alpha(x, y) + 2\beta(x, y) + 2\gamma(x, y) : (x, y) \in X^2 \} \leq 1,$$

then the assertion of Theorem 2.6 is true.

Indeed, it is easy to see that

$$\sup \{ (\alpha(x, y) + \beta(x, y) + \gamma(x, y))(1 - \beta(x, y) - \gamma(x, y))^{-1} : (x, y) \in X^2 \} \leq 1$$

and the above assertion results from the proof of Theorem 2.4.

CONCLUSION 2.2. Let (X, d) be a compact metric space and $S, T: X \rightarrow CB(X)$. Let either S or T be a continuous mapping. Suppose that assumptions (2.16) (a) (b) and (2.16') (c) of Theorem 2.7 are fulfilled. Then the assertion of Theorem 2.7 is true.

CONCLUSION 2.3. (Kubiacyk [24]). Let (X, d) and S, T be as in Conclusion 2.2. If assumption (2.16') (c) is fulfilled for $M = \Delta$, where $\Delta(A, B) = \sup \{d(x, y): x \in A, y \in B\}$, then the assertion of Theorem 2.7 is true.

3. Some generalizations. Now let G be a non-empty set equipped with a binary operation $(x, y) \rightarrow x + y$ mapping $G \times G$ into G , such that $(G, +)$ is an Abelian group. Let \leq be a partial ordering of G and suppose that the partially ordered set (G, \leq) has the compatibility property:

(1) $g_1 \leq g_2$ implies $g_1 + g \leq g_2 + g$ for each $g_1, g_2, g \in G$.

Let $G_+ = \{g \in G: g \geq 0\}$ and assume that the partially ordered set (G_+, \leq) has the additional property:

(2) $g_1 + g_2 \leq g$ implies $g_1, g_2 \leq g$ for each $g_1, g_2, g \in G_+$.

Let $S(G_+)$ denote the set of all non-increasing sequences $(g_n) \in G_+^{\mathbb{N}}$ with an operator "lim", $S(G_+) \ni (g_n)_{n \in \mathbb{N}} \rightarrow \lim g_n = g \in G_+$ such that for $(g_n), (g'_n) \in S(G_+)$ we have:

(1) $\lim (g_n + g'_n) = \lim g_n + \lim g'_n$;

(2) $g_n \leq g'_n$ for $n \in \mathbb{N}$ implies $\lim g_n \leq \lim g'_n$;

(3) $(g_n) = (g)$ implies $\lim g_n = g$.

In the sequel $S_0(G_+) \subset S(G_+)$ will denote the subclass of all sequences convergent to zero.

We define a partial ordering \leq on G^5 in the usual way: $\bar{g} \leq \bar{g}'$ or $\bar{g}' \geq \bar{g}$ if and only if $\bar{g}' - \bar{g} \in G_+^5$, $\bar{g} = (g_1, \dots, g_5)$, $\bar{g}' = (g'_1, \dots, g'_5)$. The new operator $\lim: S(G_+^5) \rightarrow G_+^5$, where $\lim (g_{1,n}, \dots, g_{5,n}) := (\lim g_{1,n}, \dots, \lim g_{5,n})$, has the properties analogous to (1)–(3) (obviously, $(g_n) = (g_{1,n}, \dots, g_{5,n}) \in S(G_+^5)$ if and only if $(g_{i,n}) \in S(G_+)$, $i = 1, \dots, 5$).

Remark 3.1. It is easy to see that Lemmas 1.1–1.3 are true for the case, where $d^i: G_+^5 \rightarrow G_+$, $i = 1, 2$. In Lemmas 1.1–1.2 we must additionally assume that also $b \in \Delta$.

We say that (X, d) is a *generalized metric space*, if $d: X^2 \rightarrow G_+$ is a function such that

(a) $d(x, y) = 0 \Leftrightarrow x = y$ for each $x, y \in X$,

(b) $d(x, y) \leq d(x, z) + d(y, z)$ for each $x, y, z \in X$.

Let the operator $\lim: X^{\mathbb{N}} \rightarrow X$ fulfil the condition: a sequence $(x_n) \in X^{\mathbb{N}}$ is convergent to $x \in X$ if there exists a sequence $(q_n) \in S_0(G_+)$ and a positive integer n_0 such that $d(x_n, x) \leq q_n$ for $n \geq n_0$; we then write, as usual,

$\lim x_n = x$. By A we denote the sequential closure of the set $A \subset X$, i.e. the set $\{x \in X: \text{there exists } (x_n) \in A^{\mathbb{N}} \text{ such that } x = \lim x_n\}$. In general, the pair (X, \lim) is not an L -space of Fréchet type (see for example Kuratowski [25], Kiszyński [22]).

Remark 3.2. Mutatis mutandis we can write generalized versions of Theorems 2.1–2.7 in a generalized metric space; we will call them in the sequel Theorems 3.1–3.7, respectively. Proofs of Theorems 2.1–2.7 can be easily transformed to work in such a generalized setting. Let us state explicitly three of these theorems:

THEOREM 3.1. *Let (X, d) be a generalized metric space and let the order relation \leq be linear on Δ . If assumptions (2.1)–(2.3) are fulfilled, then there exists $\bar{x} \in B(x_0, b)$ such that $\bar{x} \in V_n \bar{x}$ for $n = 1, 2, \dots$*

THEOREM 3.2. *Let (X, d) be a generalized metric space. If assumptions (2.5)–(2.7) are fulfilled, then there exists $\bar{x} \in X$ such that $\bar{x} \in V_n \bar{x}$ for $n = 1, 2, \dots$*

THEOREM 3.7. *Let (X, d) be a generalized metric space and suppose that assumptions (2.14)–(2.16) (a) (b) and (2.16') are fulfilled. If, in addition, (G, \leq) is a complete structure and the ordering is linear on $\varphi_1(X) \cup \varphi_2(X)$, then the assertion of Theorem 2.7 is true.*

EXAMPLE 3.1. Let (X, d) be a metric space and let $N(\varepsilon, A) = \{x \in X: d(x, c) < \varepsilon \text{ for some } c \in A, A \in CN(X)\}$. Then the function $H: CN(X) \times CN(X) \rightarrow R_+^*$, given by

$$H(A, B) = \begin{cases} \inf \{ \varepsilon > 0: A \subset N(\varepsilon, B) \text{ and } B \subset N(\varepsilon, A) \} & \text{if the infimum exists,} \\ \infty, & \text{otherwise,} \end{cases}$$

for $A, B \in CN(X)$, is a generalized Hausdorff metric, where R^* is the extended set of real numbers (see for example [17]). Let $\delta(x, A) = \inf \{d(x, y): y \in A\}$ for $x \in X, A \in CN(X)$. Obviously, the functions H and δ are as in Theorem 3.2, where $G_+ = R_+^* = \{g \in R^*: g \geq 0\}$. If $F_n: X \rightarrow N(X), n = 1, 2, \dots, 2b \in \Delta$ and assumptions (2.5)–(2.7) are fulfilled, then the assertion of Theorem 3.2 is true.

CONCLUSION 3.1 (Czerwik [10]). Let (X, d) be a complete metric space and let H and δ be as in Example 3.1. Let $F: X \rightarrow CN(X)$ have the property:

$$H(Fx, Fy) \leq a^i (d(x, y), \delta(x, Fx), \delta(y, Fy), \delta(x, Fy), \delta(y, Fx)), \quad i = 1, 2.$$

If a^1, a^2 are as in Theorem 3.2 with $G_+ = R_+^*$, then there exists $\bar{x} \in X$ such that $\bar{x} \in F\bar{x}$. Indeed, it suffices to take $w(n, x) = 1, F_n = F$ for $n = 1, 2, \dots$ and from Theorem 3.2 we obtain the assertion.

Let the function $d: X \times X \rightarrow (R_+^*)^*$ and the space (X, d) be as above, H and δ be as in Theorem 3.2. Let $F: X \rightarrow N(X)$ fulfil the conditions:

- (3.1) there exists $U \subset X$ such that
- (a) $F(U) \in CN(X)$,

(b) $F: F(U) \rightarrow CN(F(U))$,

(c) $\text{diam } F(U) < \infty$;

(3.2) $(F(U), d)$ is an F -orbitally complete subspace of (X, d) ;

(3.3) for every orbit $(x_n) \in O(F, u)$, $u \in F(U)$ and for each $n, m \in N$, $m \neq n$, the inequality holds:

$$H(Fx_{n-1}, Fx_{m-1}) \leq a^{i(m,n)}(d(x_{n-1}, x_{m-1}), d(x_{n-1}, x_n), d(x_{m-1}, x_m), d(x_{n-1}, x_m), d(x_n, x_{m-1})),$$

where

$$i(m, n) = \begin{cases} 1 & \text{for } n < m, \\ 2 & \text{for } n > m, \end{cases}$$

and $a^i: ((R_+^*)^n)^5 \rightarrow (R_+^*)^n$ are such that

(a) the restriction of a^i to $(R_+^n)^5$ fulfils assumptions (A) and (B) of Theorem 3.2, $i = 1, 2$,

(b) for each $q \in R_+^n$ there exists a maximal solution $m_i(q) \in R_+^n$ of the equation $q + a^i(b) = b$ for $i = 1, 2$;

(3.4) for every convergent orbit $(x_n) \in O(F, u)$, $u \in F(U)$, with $\lim x_n = \bar{x}$ and for each $k, l \in N$, $k \neq l$, the inequality holds:

$$H(Fx_{k-1}, Fx_l) \leq a^3(d(x_{k-1}, x_l), \delta(x_{k-1}, Fx_{k-1}), \delta(x_l, Fx_l), \delta(x_{k-1}, Fx_l), \delta(x_l, Fx_{k-1})),$$

where $a^3: ((R_+^*)^n)^5 \rightarrow (R_+^*)^n$ is a non-decreasing and monotonically continuous mapping such that $g = 0$ is the unique solution of the equation $g = a^3(0, 0, g, g, 0)$ or $g = a^3(0, g, 0, 0, g)$ in R_+^n .

THEOREM 3.8. *If assumptions (3.1)–(3.4) are fulfilled, then there exists $x \in F(U)$ and $x \in Fx$.*

This assertion follows from Theorem 3.2.

CONCLUSION 3.2 (a). If (X, d) , H and δ are as above and (3.1)–(3.2) are fulfilled for $F: X \rightarrow N(X)$ and if, moreover, the inequality

$$(3.5) \quad H(Fx, Fy) \leq a^i(d(x, y), \delta(x, Fx), \delta(y, Fy), \delta(x, Fy), \delta(y, Fx))$$

holds for each $x, y \in X$, where a^i , $i = 1, 2$, are as in Theorem 3.1, then the assertion of this theorem is true.

(b) In particular, if

$$a^i(g_1, g_2, g_3, g_4, g_5) = A \cdot g_1 + B_1 \cdot g_2 + B_2 \cdot g_3 + C_1 \cdot g_4 + C_2 \cdot g_5$$

for $(g_1, g_2, g_3, g_4, g_5) \in (R_+^n)^5$, $i = 1, 2$, where A, B_1, B_2, C_1, C_2 are non-negative $n \times n$ -matrices such that

(3.6) there exists either the product

$$L_1 = (I - B_2 - C_1)^{-1}(A + B_1 + C_1)$$

or

$$L_2 = (I - B_1 - C_2)^{-1}(A + B_2 + C_2)$$

satisfying $r(L_1) < 1$ and $r(B_2 + C_1) < 1$ [$r(L_2) < 1$ and $r(B_1 + C_2) < 1$ respectively], where $r(A)$ denotes the spectral radius of A , then the assertion of Theorem 3.1 is true.

Remark 3.4. If in Conclusion 3.1 (b) $F: X \rightarrow X$, (X, d) is a complete metric space, $a^1(g_1, g_2, g_3, 0, 0) = \psi(g_1, g_2, g_3)$, where ψ is continuous and such that $\psi(0, 0, g_3) \geq g_3$ implies $g_3 = 0$, then we obtain the Rus type theorem ([37], Theorem 3).

Now let (X, d) be a complete metric space and H be as above. Let the continuous function $\delta: X \times N(X) \rightarrow (R_+)^n$ be such that the restriction of δ to $X \times B(X)$ ($B(X)$ denotes the class of all non-empty and bounded subsets of X) has the properties:

(δ'_1) $x \in A$ implies $\delta(x, A) = 0$ for each $A \in B(X)$ and $x \in X$,

(δ'_2) $\delta(x, A) \leq d(x, y) + \delta(y, B) + H(A, B)$ for each $x, y \in X$ and each $A, B \in B(X)$,

(δ'_3) if $A \in C(X)$, where $C(X)$ denotes the family of all non-empty and compact subsets of X , then $\delta(x, A) = 0$ implies $x \in A$.

Assume that there exists a function $M: [N(X)]^2 \rightarrow (R_+)^n$ such that

(3.6) (a) for each $A, B \in B(X)$, $H(A, B) \leq M(A, B)$,

(b) $d(x, y) \leq M(A, B)$ for each $A, B \in B(X)$ and $x \in A, y \in B$.

Let $F: X \rightarrow N(X)$ and suppose that there exists $U \in N(X)$ such that

(3.7) $F: F(U) \rightarrow B(X)$, $F(U) \in B(X)$ and $F(F(U))$ is a compact set,

(3.8) for all distinct elements $x, y \in F(U)$ the inequality holds:

$$M(Fx, Fy) < a(d(x, y) \delta(x, Fx), \delta(y, Fy), \delta(x, Fy), \delta(y, Fx)),$$

where $a: (R_+^n)^5 \rightarrow R_+^n$ is a non-decreasing function with $a(g, g, g, 2g, 2g) \leq g$ for each $g \in R_+^n$.

THEOREM 3.9. *If assumptions (3.6)–(3.8) are fulfilled, then there exists a unique fixed point of F in $F(U)$.*

It is easy to see that $\varphi(X)$ has the minimal element, $\varphi(x) = \delta(x, Fx)$, $x \in \overline{F(U)}$, where δ is as in Theorem 3.6. Therefore the above assertion is a conclusion of Theorem 3.6.

Remark 3.5. In particular, we can take

$$a(g_1, g_2, g_3, g_4, g_5) = Ag_1 + B_1g_2 + B_2g_3 + C(g_4 + g_5),$$

where A, B_1, B_2 and C are non-negative matrices such that $r(A + B_1 + B_2 + 2C) < 1$. If $F: X \rightarrow X$, $B_1 = B_2 = 0$, then assumptions (3.6)–(3.8) are fulfilled.

Remark 3.6. To emphasize the importance of the generalizations stated in Section 3, we now show how they work in the context of the results of the recent papers of Kwapisz and Turo (see [28], [29]). We take here $(G_+, \leq) = (C_0(D, R_+), \leq)$, where $C_0(D, R_+)$ is the class of all upper semi-continuous functions from D to R_+ and $z_1 \leq z_2 \Leftrightarrow z_1(x, y) \leq z_2(x, y)$ for all $(x, y) \in D$; $D = [0, a] \times [0, b]$; $a, b > 0$; and let $d: [C(D, E)]^2 \rightarrow G_+$, where $d(z_1, z_2)(x, y) := \|z_1(x, y) - z_2(x, y)\|$, $(E, \|\cdot\|)$ is a Banach space. Suppose that the operator $f: D \times [C(D, E)]^4 \times E \rightarrow E$ satisfies the inequality

$$\begin{aligned} \|Fu^1 - Fu^2\|(x, y) &:= \|Fu^1(x, y) - Fu^2(x, y)\| \\ &\leq a_i(x, y, \|u^1 - u^2\|(\cdot, \cdot), \|u^1 - Fu^1\|(\cdot, \cdot), \|u^2 - Fu^2\|(\cdot, \cdot), \\ &\quad \|u^1 - Fu^2\|(\cdot, \cdot), \|u^2 - Fu^1\|(\cdot, \cdot)) \\ &\quad \text{for } u^1, u^2 \in C(D, E), \end{aligned}$$

where F and a_i are some integro-functional operators defined with the aid of f and Ω_i , respectively. Some additional assumptions imposed on f and Ω_i give a possibility of proving the generalizations of Theorems 1–4 of [29].

Remark 3.7. Let us consider two generalized metric spaces: (X, d) and (X, e) . Assume that

(a) (X, e) is an (\mathcal{F}, w) -orbitally complete space, where \mathcal{F} is the sequence of mappings $F_n: X \rightarrow X$, $n = 1, 2, \dots$,

(b) every Cauchy orbit of \mathcal{F} in (X, d) is a Cauchy orbit in (X, e) ,

(c) assumptions (2.5)–(2.6) of Theorem 3.2 are fulfilled with respect to the metric d except (2.5) (b),

(d) each mapping F_i , $i = 1, 2, \dots$, is (\mathcal{F}, w) -orbitally continuous in the space (X, e) .

Then there exists $\bar{x} = \lim x_n$ such that $\bar{x} \in V_n \bar{x}$, $n = 1, 2, \dots$, where (x_n) is the orbit as in Theorem 3.2 and the limes-operator “lim” is generated by the metric e .

On the other hand, if instead of condition (d) we assume (2.7) of Theorem 3.2 for the metric e , then the common fixed point of F_i , $i = 1, 2, \dots$, is unique. In particular, we can take $e(x, y) \leq kd(x, y)$, $x, y \in X$, $k > 0$, and condition (2.7) for the metric d (see for example Iseki [19] and Singh [38]).

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