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PROCEEDINGS
OF THE FOURTH CONFERENCE
ON ANALYTIC FUNCTIONS

Łódź, September 1-7, 1966

The Conference was organized by the Institute of Mathematics of the Polish Academy of Sciences. The Organizing Committee consisted of: Z. Charzyński (Vice-Chairman), J. Górski, J. Janikowski (General Secretary), W. Janowski, F. Leja (Chairman) and J. Ławrynowicz (Secretary).

There were 69 members of the Conference from Czechoslovakia, Finland, France, the German Democratic Republic, the German Federal Republic, Great Britain, Poland and the U. S. A. The members presented 14 papers and 22 communications. Besides, three seminars were held: (i) on extremal problems in analytic functions, presided by W. Janowski (Łódź, Poland), (ii) on quasiconformal mappings, presided by F. W. Gehring (Ann Arbor, Michigan, U. S. A.), and (iii) on functions of several complex variables, presided by F. Norguet (Strassbourg, France). During the seminars new problems were posed and discussed.

The present Proceedings consist of two parts: abstracts of the papers and communications, and problems posed by the members of the three seminars.

The abstracts are divided into four sections: I. Extremal problems for analytic functions, II. Quasiconformal mappings, III. Functions of several complex variables, IV. Other topics in complex functions.

The problems are divided into three sections corresponding to the three seminars.

J. Ławrynowicz
Secretary

PART ONE. ABSTRACTS

I. EXTREMAL PROBLEMS FOR ANALYTIC FUNCTIONS

Quasi-starlike functions

by I. Dziubiński (Łódź)

Let $\mathfrak{G}$ represent the class of functions $g(z)$ satisfying the equation

$$F\big(g(z)\big) = (1/M)F(z) \quad \text{in } |z|<1\,, \text{ for } M>1\,, \tag{1}$$

where $F(\zeta)$ is a starlike function in relation to the point $\zeta = 0$ having an expansion of the form

$$F(\zeta) = \zeta + b_2\zeta^2 + \ldots \quad \text{in } |\zeta| < 1\,.$$

Further, let $\mathfrak{G}_m$ be the subclass of functions of $\mathfrak{G}$ satisfying equation

$$G\big(g(z)\big) = (1/M)G(z) \quad \text{in } |z|<1\,, \text{ for } M>1\,, \tag{2}$$

where

$$G(\zeta) = \zeta/\prod_{k=1}^{m}(1-\sigma_k\zeta)^{\bar{\beta}_k} \quad \text{in } |\zeta|<1$$

at

$$\sigma_k = e^{i\varphi_k}, \quad \operatorname{im}\varphi_k = 0 \quad (k = 1, 2, \ldots, m)\,,$$

$$\sum_{k=1}^{m}\beta_k = 2\,, \quad \beta_k > 0 \quad (k = 1, 2, \ldots, m)\,.$$

Every function

$$g(z) = a_1z + a_2z^2 + \ldots \quad \text{in } |z|<1$$

satisfying equation (1) or (2) we shall call a *quasi-starlike function.*

Let $H = H(x_2, x_3, \ldots, x_N, y_2, y_3, \ldots, y_N)$ be a real function of class C_1 dependent on $2(N-1)$ real variables, defined in an open set V for which at each point of the set V we have

$$\sum_{k=2}^{N}\left\{\left(\frac{\partial H}{\partial x_k}\right)^2 + \left(\frac{\partial H}{\partial y_k}\right)^2\right\} \neq 0\,,$$

and let as write

$$a_n = x_n + iy_n\,, \quad n = 1, 2, \ldots$$

Then we have the following

THEOREM 1. *If the functional H assumes the extremal value for a function of class $\mathfrak{G}_m$, this function satisfies the equation*

$$\frac{g'(z)}{g(z)}\cdot\frac{\mathfrak{L}(g(z))}{\mathfrak{R}(g(z))}=\frac{1}{z}\cdot\frac{\mathfrak{L}(z)}{\mathfrak{R}(z)}\quad in\ 0<|z|<1,$$

where $\mathfrak{L}(\zeta)$ and $\mathfrak{R}(\zeta)$ are rational functions having a pole at point $\zeta=0$, and the equation

$$\frac{g'(z)}{g(z)}\cdot\frac{\hat{\mathfrak{L}}(g(z))}{\hat{\mathfrak{R}}(g(z))}=\frac{1}{z}\cdot\frac{\hat{\mathfrak{L}}(z)}{\hat{\mathfrak{R}}(z)}\quad in\ 0<|z|<1,$$

where, as before, $\hat{\mathfrak{L}}(\zeta)$ and $\hat{\mathfrak{R}}(\zeta)$ are rational functions having a pole at point $\zeta=0$.

THEOREM 2. *The extremal value of the functional $H(x_2, \ldots, y_N)$ is obtained in class $\mathfrak{G}$ for a function belonging to class $\mathfrak{G}_m$, where $k \leqslant N-1$.*

It is interesting to know when an extremal quasi-starlike function of class $\mathfrak{G}_m$ is a starlike function. The answer is given in the following

THEOREM 3. *A function $g(z)$ which belongs to $\mathfrak{G}_m$ is starlike with respect to $z=0$ if and only if*

$$\beta_k = 2/m \qquad (k=1,2,\ldots,m),$$

$$\sigma_k = e^{i\frac{2\pi k}{m}}\ \text{if } m \text{ is an odd number} \qquad (k=1,2,\ldots,m)$$

$$\begin{cases}\sigma_1 = e^{i\varphi}\,(^1), \\ \sigma_2 = e^{i\left(\frac{4\pi}{m}-\varphi\right)}, \\ \sigma_3 = e^{i\left(\frac{4\pi}{m}+\varphi\right)}, \\ \cdots\cdots\cdots\cdots \\ \sigma_m = e^{i(2\pi-\varphi)}, \qquad \text{if } m \text{ is an even number}.\end{cases}$$

Using Theorems 1 and 2 we can obtain sharp bounds for the coefficients a_n $(n=2,3,4)$ of the function $g(z)$ belonging to class $\mathfrak{G}$.

THEOREM 4. *If $g(z)$ belongs to class $\mathfrak{G}$, then*

$$|a_2| \leqslant (2/M)(1-1/M) \qquad \text{if} \quad 1<M<\infty,$$

$$|a_3| \leqslant \begin{cases}(1/M)(1-1/M^2) & \text{if} \quad 1<M\leqslant 3, \\ (1/M)(1-1/M)(3-5/M) & \text{if} \quad 3\leqslant M<\infty,\end{cases}$$

(1) φ is an independent real parameter and $\varphi \in (0, 2\pi)$.

$$|a_4| \leqslant \begin{cases} (2/3M)(1-1/M^3) & \text{if} \quad 1 < M \leqslant M_0, \\ (2/3\sqrt{3}M)(2-3/M^2)^{3/2} & \text{if} \quad M_0 \leqslant M \leqslant M_1, \\ (2\sqrt{6}/27M^3)(1-1/M)\left[\dfrac{(4M^2-2M-11)^3}{-M^2+11M-16}\right]^{1/2} & \text{if} \quad M_1 \leqslant M \leqslant \dfrac{34+3\sqrt{34}}{10}, \\ (2/M)(1-1/M)\big(2-(8/M)+(7/M^2)\big) & \text{if} \quad \dfrac{34+3\sqrt{34}}{10} \leqslant M < \infty, \end{cases}$$

where $x_0 = 1/M_0$ *and* $x_1 = 1/M_1$ *are the roots of the equations*

$$(2-3x^2)^3 = 3(1-x^3)^2$$

and

$$9(2-3x^2)^3(-1+11x-16x^2) = (1-x)^2(4-2x-11x^2)^3$$

in the interval $(0,1)$, *respectively.*

1. IX. 1966

On the determination of univalent functions from their initial coefficients

by G. S. Goodman (London)

We prove the following

THEOREM. *If the n-th partial sum $h_n(z)$ of a power series in S has real coefficients, then there is a real function in S having $h_n(z)$ for its n-th partial sum. If the original function, of which $h_n(z)$ is the n-th partial sum, is bounded on the unit disk, then the real function to which $h_n(z)$ is assigned may be taken to satisfy the same bound.*

A corollary is:

The inequality

$$|a_m| \leqslant m \qquad (m = 2, \ldots, n),$$

must hold whenever the coefficients $a_2, \ldots, a_n$ of a function

$$z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

in S are real.

By similar reasoning, we can prove an analogous theorem for odd or for k-fold symmetric functions.

The detailed version will appear in the Archive for Rational Mechanics and Analysis.

3. IX. 1966

Some sharp estimations of $|a_5 - ta_2^4|$ in the class S

by J. Górski (Kraków)

The presented results were obtained by J. Maj-Kluskowa, J. T. Poole and J. Górski.

Let $f(z)$ belong to the class S. To any function $f(z) \in S$ corresponds a bounded continuum E with the capacity 1 and the coordinate system with the origin 0 in E. The coefficients of $f(z) \in S$ can be expressed as polynomials with respect to the moments

$$s_k = \int z^k d\mu_z$$

where μ is the natural mass-distribution on E [1]. If $\bar{0}$ denotes the mass center of the μ-distribution, then

$$a_5 = -s_4(\bar{0})/4 + 3/4 s_2(\bar{0})^2 + 2a_2 a_4 - a_2^4/2 - a_3^2/2 .$$

The following sharp estimations are obtained:

$$|a_5 - 2a_2 a_4 + a_2^4| \leqslant 5 \quad \text{for} \quad f(z) \in S ,$$

$$|a_5 - t a_2^4| \leqslant 16t - 5 \quad \text{for} \quad t \geqslant 1 ,$$

$$|a_5 - t a_2^4| \leqslant -16t + 5 \quad \text{for} \quad t \leqslant -1;$$

if $2a_2 a_4 - |a_2^4/2 - a_3^2/2| \leqslant 7/2$, then $|a_5| \leqslant 5$.

All moments $|s_k|$ are sharply estimated by $\binom{2k}{k}$.

The detailed version will appear in the Journal of Mathematics and Mechanics, Bloomington, Indiana, USA.

2. IX. 1966

Sur quelques problèmes extrémaux dans la famille des fonctions univalentes et symétriques

par Z. J. Jakubowski (Łódź)

Soient a et M deux nombres arbitraires, mais fixes, choisis dans les intervalles

$$a \geqslant 0 , \quad M > 1 ,$$

R_M — la famille de toutes les fonctions holomorphes univalentes dans le cercle $|z| < 1$ de la forme

$$F(z) = z + A_2 z^2 + \ldots, \quad \operatorname{re} A_n = A_n \quad (n = 2, 3, \ldots)$$

assujeties à la condition $|F(z)| < M$.

Je démontre les théorèmes suivants:

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[1] F. Leja, *Sur les coefficients des fonctions analytiques univalentes dans le cercle et les points extrémaux des ensembles*, Ann. Soc. Polon. Math. 23 (1953), p. 69.

1. *Pour les fonctions de la famille R_M on a l'inégalité*

$$A_3+aA_2 \leqslant \begin{cases} 1-M^{-2}+\frac{1}{4}a^2\frac{\log M}{1-\log M} & \textit{pour} \quad 1 < M \leqslant M_0 \quad \textit{et}\ 0 \leqslant a < 4\,, \\ 1+M^{-2}(\frac{1}{8}u^2-u+1)+\frac{1}{4}a(M^{-1}u-a) & \\ & \textit{pour} \quad \begin{cases} M_0 < M < 4a^{-1} & \textit{et}\ 0 < a < 4\,, \\ \quad M > M_0 & \textit{et}\ a = 0\,, \end{cases} \\ 3+2a-2(4+a)M^{-1}+5M^{-2} & \\ & \textit{pour} \quad M > 1 \quad \textit{et}\ 4M^{-1} < a\,, \end{cases}$$

où $4 < u \leqslant 4M$ et $0 < M_0 \leqslant e$ sont respectivement des racines des équations

$$4M^{-1}+uM^{-1}\log\frac{u}{4M} = a\,,$$

$$4M^{-1}(1-\log M) = a\,,$$

et les limites sont atteintes.

2. *Pour les fonctions de la famille R_M on a l'inégalité*

$$A_3+aA_2 \geqslant \begin{cases} M^{-2}-1-\frac{1}{4}a^2 & \textit{pour} \quad M > 1 \ \textit{et}\ 0 \leqslant a \leqslant 4(1-M^{-1})\,, \\ 5M^{-2}+2(a-4)M^{-1}+3-2a & \textit{pour} \quad M > 1 \ \textit{et} \quad a > 4(1-M^{-1})\,, \end{cases}$$

et les limites sont atteintes.

3. *Pour les fonctions de la famille R_M on a l'inégalité*

$$A_4-3A_2A_3+aA_2 \leqslant 16M^{-3}-48M^{-2}+(46-2a)M^{-1}+2a-14$$

pour $a > 27$ et $M > 1$, et les limites sont atteintes.

Bibliographie

[1] Z. J. Jakubowski, *Maksimum funkcjonalu A_3+aA_2 w rodzinie funkcji jednolistnych o wspólczynnikach rzeczywistych*, Zeszyty Naukowe UŁ, Nauki Mat.-Przyr., Ser. II, 20 (1966), pp. 43-61.

[2] — *Sur les coefficients des fonctions univalentes dans le cercle unité*, Ann. Polon. Math. 19 (1967), pp. 207-233.

[3] — *Sur les coefficients des fonctions univalentes et symétriques en le cercle unitaire*, Bull. Acad. Polon. Sci., Ser. Sci. math., astr. et phys., 14 (1966), pp. 641-644.

1. IX. 1966

Sur une certaine famille de fonctions univalentes

par W. Janowski (Łódź)

Soit S_c une famille de fonctions $f(z), f(0) = 0, f'(c) = 1$, holomorphes et univalentes dans le cercle $|z| < 1$, où c, $|c| < 1$, un nombre fixe. Pour le ζ, $|\zeta| < 1$, arbitraire fixe considérons la fonctionnelle

$$(1) \qquad F(f) = F\big(f(\zeta), \overline{f(\zeta)}, \dots, f^{(n)}(\zeta), \overline{f^{(n)}(\zeta)}\big)\,, \quad f(z) \in S_c\,,$$

où $F(f, \bar{f}, \ldots, f^{(n)}, \overline{f^{(n)}})$ est une fonction analytique de $2n+2$ variables complexes $f, \bar{f}, \ldots, f^{(n)}, \overline{f^{(n)}}$ dans le domaine $\{|f^{(k)}| < \infty\}$ où $k = 0, 1, \ldots \ldots, n, f^{(0)} \equiv f, \overline{f^{(0)}} \equiv \bar{f}$. Nous présentons les résultats des recherches concernant les propriétés du domaine $D_n(\zeta)$ des valeurs de la fonctionnelle (1) et de désignation effective du $D_n(\zeta)$ dans certains cas particuliers.

2. IX. 1966

Sur la borne supérieure d'une fonctionnelle dans la famille de fonctions univalentes bornées

par J. KACZMARSKI (Łódź)

On considère la famille $S(M)$ de fonctions de la forme

$$F(z) = z + A_2 z^2 + \ldots$$

holomorphes et univalentes dans le cercle $|z| < 1$ et assujetties à la condition $|F(z)| < M$, où $M \geqslant 1$ est un nombre librement choisi. La valeur complexe de z, $|z| < 1$ et la valeur réelle a étant fixées, on considère la fonctionnelle

$$(1) \qquad K_a(F) = |F'(z)|/|F(z)|^a, \qquad F \in S(M).$$

A l'aide de l'équation général de fonctions extrémales par rapport aux fonctionnelles dérivables au sens de M. Fréchet, on obtient l'équation différentielle-fonctionnelle verifiée par les fonctions réalisant le maximum de la fonctionnelle (1). En utilisant cette équation on obtient la borne supérieure de la fonctionnelle considérée. La borne s'exprime par quelques formules assez compliquées. Dans le cas limite $M = \infty$ on obtient les résultats connues pour la famille $S(\infty)$.

2. IX. 1966

On Koebe domains for some classes of univalent functions

by J. KRZYŻ (Lublin) and M. O. READE (Ann Arbor, Mich.)

Let S denote the set of all normalized univalent functions defined in the unit disc K, and let S_0 denote a subclass of S. Then the Koebe domain (S_0) is defined as the set

$$\bigcap_{f \in S_0} f(K).$$

Jenkins determined the Koebe domain for the set S_R of S consisting of these functions having real coefficients (Ann. Math. 71 (1960), pp. 1-15).

McGregor determined the Koebe domains for the subclasses of S_R which consist of (a) the star-shaped functions, (b) the convex functions, and (c) the functions convex in the direction of the imaginary axis (Journ. London Math. Soc. 39 (1964), pp. 42-50). The present authors determine the Koebe domains for the following subclasses of S_R: (i) the circularly symmetric functions, (ii) the close-to-convex functions, (iii) the odd convex functions, and (iv) the odd star-like functions. Analogous results are obtained for certain bounded functions and for "exterior" univalent functions.

2. IX. 1966

A theorem on distortion for univalent p-symmetrical functions founded in the disc $|z| > 1$

by L. MIKOŁAJCZYK (Łódź)

Let $\Sigma_p(m)$ denote the class of p-symmetrical functions of the form

(1) $$W_p(z) = z + a_{p-1}/z^{p-1} + a_{2p-1}/z^{2p-1} + \ldots$$

which are holomorphic, univalent in the unit disc and such that

$$|W_p(z)| > m, \quad 0 < m < 1, \quad p \geqslant 1.$$

In the class $\Sigma_p(m)$ for a fixed $z_0 \neq \infty$, $|z_0| > 1$ and a fixed $|W_p(z_0)|$ the following sharp estimation holds:

(2) $$\frac{d^{p+1}(b^{2p}-1)}{b^{p+1}(d^{2p}-m^{2p})} \leqslant |W_p'(z_0)| \leqslant \frac{b^{p-1}(d^{2p}-m^{2p})}{m^p d^{p-1}(b^{2p}-1)} \exp\{-L(T/p;\, b, c)\},$$

where $b = |z_0|$, $d = |W_p(z_0)|$, $T/p = \log(1/m)$, $c = d/m$,

$$L(T/p;\, b, c) = \begin{cases} pQ(x;\, b, c) & \text{for} \quad T/p = P(x;\, b, c), \\ (T/p)^{-1} p\big(\log(c/b)\big)^2 & \\ pP(x;\, b, c) & \text{for} \quad T/p = Q(x;\, b, c); \end{cases}$$

$$P(x;\, b, c) = \frac{x^p-1}{x^p+1}\int_b^x \frac{d\varrho}{\varrho} + \int_x^c \frac{\varrho^p-1}{\varrho^p+1}\cdot\frac{d\varrho}{\varrho}, \quad b \leqslant x \leqslant c,$$

$$Q(x;\, b, c) = \frac{x^p+1}{x^p-1}\int_b^x \frac{d\varrho}{\varrho} + \int_x^c \frac{\varrho^p+1}{\varrho^p-1}\cdot\frac{d\varrho}{\varrho}, \quad b \leqslant x \leqslant c.$$

If $p = 1$, by the elimination of d from estimation (2) we have

(3) $$\frac{d_1^2(b^2-1)}{b^2(d_1^2-m^2)} \leqslant |W_0'(z_0)| \leqslant \exp\left\{\log\frac{d_0^2-m^2}{m(b^2-1)} + \frac{d_0^2}{d_0^2-m^2}\log\frac{bm}{d_0}\right\},$$

where d_1 and d_0 are the roots of the equations

$$\frac{d_1}{(d_1-m)^2} = \frac{b}{(b-1)^2},$$

$$\frac{d_0^2-m^2}{d_0^2}\log\frac{mb}{d_0} = \log m,$$

respectively.

The proof is based on Löwner's theory and on a method due to R. Robinson.

References

[1] L. Mikołajczyk, *Théorème sur la déformation pour les fonctions univalentes, p-symétriques et bornées inférieurement dans le domaine* $|z| > 1$, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., 14 (1966), pp. 245-250.

[2] — *A theorem on distortion for univalent p-symmetrical functions bounded in the disc* $|z| > 1$, Prace Mat. (to appear).

1. IX. 1966

On a class of typically real functions

by B. PIŁAT (Lublin)

Let T_1 be a class of functions regular in the disk $|z| < 1$ which satisfy

(1) $$f(0) = 0,$$

(2) $$\operatorname{im} f(z)\cdot \operatorname{im} z > 0 \quad \text{for any } z,\ |z|<1,\ \text{with } \operatorname{im} z \neq 0,$$

(3) $$\lim_{x\to 1-} f(x) = 1,$$

and let S_1 be the corresponding subclass of univalent functions.

We have the following

THEOREM. *If $f \in T_1$, then there exists a non-negative, non-decreasing function $\mu(t)$ satisfying*

(4) $$\int_0^1 d\mu(t) = 1$$

and

(5) $$\lim_{t\to 0+} \mu(t) = \mu(0) = 0$$

such that

(6) $$f(z) = \int_0^1 \frac{4tz}{z^2+1+2z(2t-1)}\, d\mu(t).$$

Conversely, if $\mu(t)$ satisfies conditions (4) *and* (5), *then the function defined by the right-hand side of* (6) *belongs to* T_1.

This theorem enables us to find the set $\Omega(z)$ of all possible values $f(z)$ where z is fixed in the unit circle and f ranges over the class T_1. We also obtain the exact bounds of $|f(z)|$, $|f'(z)|$ and $|\mathrm{im} f(z)|$ in this class.

In particular, the corresponding results for the class S_1 are generalizations of some theorems of V. Singh [2].

References

[1] M. S. Robertson, *On the coefficients of a typically real function*, Bull. Amer. Math. Soc. 41 (1939), pp. 565-572.

[2] V. Singh, *Some extremal problems for a new class of univalent functions*, Journ. Math. Mech. 7 (1958), pp. 811-821.

[3] B. Piłat, *On typically real functions with Montel's normalization*, Annales UMCS, Sectio A, 18 (1964), pp. 53-72.

1. IX. 1966

On the coefficients of exterior univalent functions

by Ch. Pommerenke (London)

Let $g(z) = z + b_0 + b_1 z^{-1} + \ldots$ be univalent in $|z| > 1$. We define the Clunie constant as the smallest number γ such that, for every $\varepsilon > 0$,

$$|b_n| \leqslant A(\varepsilon) n^{\gamma-1} \quad (n = 1, 2, \ldots)$$

where $A(\varepsilon)$ depends only on ε. It is trivial that $0 \leqslant \gamma \leqslant 0.5$. The upper bound follows at once from the area theorem. J. Clunie (Ann. Math. 69 (1959), pp. 511-519) has given an example showing that $\gamma > 0.02$. In the other direction, nothing non-trivial was known. It is shown that

$$0.139 < \gamma < 0.497 .$$

The upper bound was obtained jointly with J. Clunie (Michigan Math. Journ., to appear). The lower bound is obtained by composition of univalent functions (Journ. London Math. Soc., to appear).

2. IX. 1966

A method of variations in the class of univalent functions with bounded boundary rotation

by O. Tammi (Helsinki)

Consider the class S_k of normalized, regular and univalent functions

$$f(z) = z + \sum_{v=2}^{\infty} a_v z^v \quad (|z| < 1)$$

which map the unit circle onto a domain with a boundary rotation bounded by

$$k\pi \quad (2 \leqslant k \leqslant 4) .$$

For any given $f(z) \in S_k$ is constructed a family of comparison functions. This is done by aid of a special type of interior variation leaving the boundary rotation of the image domain unchanged. The result can be applied to extremum problems, e.g. to give a necessary condition for $f(z)$ maximizing $a_n > 0$. Especially the problem $\max|a_4|$ in S_k is solved. The works in question were done during 1966, together with M. Schiffer.

The detailed version was published in the Journal d'Analyse Mathématique 17 (1966), pp. 109-144.

1. IX. 1966

II. QUASICONFORMAL MAPPINGS

Definitions for a class of plane quasiconformal mappings

by F. W. GEHRING (Ann Arbor, Mich.)

This lecture is a survey of some of the many different ways of characterizing a class of plane quasiconformal mappings. This class was considered by Ahlfors in his treatment of the Teichmüller problem, and it has been studied rather extensively in the last ten years.

The paper will appear in the volume of the Nagoya Mathematical Journal, which is being dedicated to Professor K. Noshiro on his 60th birthday.

5. IX. 1966

Quasiconformal mappings of the unit disc with two invariant points

by J. KRZYŻ (Lublin) and J. ŁAWRYNOWICZ (Łódź)

Let $S_Q^{z_0}$ be the class of Q-quasiconformal mappings of the unit disc onto itself with $f(0) = 0$, $f(z_0) = z_0$ $(0 < |z_0| < 1)$. If $Q = 1$, then $S_Q^{z_0}$ contains only the identity mapping.

The natural question arises how far $f \in S_Q^{z_0}$ can depart from the identity. Using a parametric method analogous to that introduced by Shah Tao-Shing we obtain the following inequalities for Q near 1:

$$(1)\qquad |f(z)-z| \leqslant \frac{4|z_0|}{\pi}\left|\frac{z}{z_0}\left(1-\frac{z}{z_0}\right)\left\{K\left(\sqrt{\frac{z}{z_0}}\right)K'\left(\sqrt{\frac{\bar z}{\bar z_0}}\right)+ + K\left(\sqrt{\frac{\bar z}{\bar z_0}}\right)K'\left(\sqrt{\frac{z}{z_0}}\right)\right\}\mu(z_0)\log Q\,\{1+o(1)\}\right| \qquad (|z| \leqslant |z_0|,\ z \neq z_0),$$

$$(2)\quad |f(z)-z| \leqslant \frac{4|z|}{\pi}\left|\left(1-\frac{z_0}{z}\right)\left\{K\left(\sqrt{\frac{z_0}{z}}\right)K'\left(\sqrt{\frac{\bar z_0}{\bar z}}\right)+ + K\left(\sqrt{\frac{\bar z_0}{\bar z}}\right)K'\left(\sqrt{\frac{z_0}{z}}\right)\right\}\mu(z_0)\log Q\{1+o(1)\}\right| \quad (|z|\geqslant|z_0|,\ z\neq z_0),$$

where K and K' are complete elliptic integrals, $\mu(z_0) \leqslant \frac{1}{2}$ and $\mu(z_0)\to\frac{1}{2}$ as $|z_0|\to 1-$. Inequality (1) can be written in a weaker but simpler form

$$(3)\qquad |f(z)-z| \leqslant (1/4\pi^2)\{\Gamma(1/4)\}^4|z_0|\log Q\{1+o(1)\}.$$

The result appeared in the Michigan Math. Journ. 14 (1967), pp. 487-492.

5. IX. 1966

Harmonic forms with given boundary behavior in Riemannian spaces

by K. LARSEN (Santa Monica, Calif.), M. NAKAI (Nagoya) and L. SARIO (Santa Monica, Calif.)

The principal function problem has recently been considered by F. Browder [1] for solutions of elliptic systems of partial differential equations in Euclidean space, by G. Weill [4], M. Schiffer and M. Glasner [5] for harmonic functions in Riemannian spaces, and by M. Nakai [3] for harmonic fields in Riemannian spaces. The problem has remained open for harmonic forms in Riemannian spaces. I would like to report on some joint work [2] with K. Larsen and M. Nakai on this problem.

Let V be a Riemannian space and V_0 a regular subregion with boundary, Γ and let $V_1 = V-\bar V_0$. Let σ be in the class $H(\bar V_1)$ of harmonic forms in $\bar V_1$ defined by $\Delta\sigma = 0$, with $\Delta = d\delta+\delta d$. The problem is to find a ϱ in the class $H(V)$ of harmonic forms on all of V that imitates the behavior of σ in V_1. More precisely, let E be the space of square integrable differential forms on V, D the subspace of C^∞ forms with compact supports, and E_Δ the closure in E of the space ΔD of forms $\Delta\varphi$ with $\varphi \in D$. We extend σ as C^∞ to all of V and require that $\sigma-\varrho \in E_\Delta$. This means two things. First, $\|\sigma-\varrho\| < \infty$. Second, in view of $E_\Delta = \overline{\Delta D}$, the form $\sigma-\varrho$ has, in a sense, zero data on the ideal boundary of V, i.e., ϱ "solves" the Dirichlet problem for harmonic forms with boundary values σ. We shall call ϱ the *principal harmonic form* corresponding to σ.

The complete solution of the problem is as follows:

THEOREM. *Given $\sigma \in H(\bar U_1) \cap C^\infty(V)$ there exists a principal harmonic form $\varrho \in H(V)$ characterized by $\sigma-\varrho \in E_\Delta$ if and only if for all $\varphi \in D$*

$$\int_\Gamma \sigma\wedge *d\varphi+\delta\sigma\wedge *\varphi-\varphi\wedge *d\sigma-\delta\varphi\wedge *\sigma = O(\|\Delta\varphi\|).$$

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3. IX. 1966

On certain functional equations for quasiconformal mappings

by J. Ławrynowicz (Łódź)

The author obtains and discusses the system of functional equations

$$w_z = ww_z(1) + \frac{1}{2\pi}\iint\limits_{R\leqslant|\zeta|\leqslant 1} w \sum_{\nu=-\infty}^{+\infty}\Bigg\{\frac{\varphi_1(\zeta)}{\zeta^2}\left(\frac{w+R^{2\nu}\zeta}{w-R^{2\nu}\zeta}-\frac{1+R^{2\nu}\zeta}{1-R^{2\nu}\zeta}\right)- -\frac{\overline{\varphi_2(\zeta)}}{\bar\zeta^2}\left(\frac{1+R^{2\nu}\bar\zeta w}{1-R^{2\nu}\bar\zeta w}-\frac{1+R^{2\nu}\bar\zeta}{1-R^{2\nu}\bar\zeta}\right)\Bigg\}d\xi\,d\eta\,,$$

$$w_{\bar z} = ww_{\bar z}(1) + \frac{1}{2\pi}\iint\limits_{R\leqslant|\zeta|\leqslant 1} w \sum_{\nu=-\infty}^{+\infty}\Bigg\{\frac{\varphi_1(\zeta)}{\zeta^2}\left(\frac{w+R^{2\nu}\zeta}{w-R^{2\nu}\zeta}-\frac{1+R^{2\nu}\zeta}{1-R^{2\nu}\zeta}\right)- -\frac{\overline{\varphi_2(\zeta)}}{\bar\zeta^2}\left(\frac{1+R^{2\nu}\bar\zeta w}{1-R^{2\nu}\bar\zeta w}-\frac{1+R^{2\nu}\bar\zeta}{1-R^{2\nu}\bar\zeta}\right)\Bigg\}d\xi\,d\eta$$

$$(\zeta = \xi + i\eta)$$

for a certain sufficiently regular subclass of quasiconformal mappings $w = f(z)$, $f(1) = 1$, of an annulus $r \leqslant |z| \leqslant 1$ onto $R \leqslant |w| \leqslant 1$, where

$$\varphi_1(w) = \{1/(1-|q(f^{-1}(w))|^2)\}\times \times\{q_z(f^{-1}(w))-(1/f^{-1}(w))q(f^{-1}(w))\}\exp(-2i\arg f_w^{-1}(w))\,,$$

$$\varphi_2(w) = \{1/(1-|q(f^{-1}(w))|^2)\}\times \times\{q_{\bar z}(f^{-1}(w))+(1/f^{-1}(w))q(f^{-1}(w))\}\exp(-2i\arg f_{\bar w}^{-1}(w))\,,$$

and q denotes the complex dilatation of $w = f(z)$. In the limiting case $r = R = 0$ one obtains the corresponding results for quasiconformal mappings of the unit disc onto itself.

The detailed version of this result appeared in the Annales Polonici Mathematici 20 (1968), pp. 153-165.

5. IX. 1966

Some examples for the Koebe conjecture

by R. J. Sibner (Stanford, Calif.)

Koebe has conjectured that every plane domain is conformally equivalent to a circle domain (a domain bounded by circles and points). The author has obtained the result that this is true for every domain which is quasiconformally equivalent to a circle domain [Symposium on Quasiconformal Mappings, Tulane University, May 1965]. The following remarks are then useful in obtaining many new examples of domains for which the Koebe conjecture is true.

Remark 1. Suppose that the domains $D_1, \ldots, D_n$ have disjoint complements and each is conformally equivalent to a circle domain. Then their intersection $\bigcap D_j$ is also conformally equivalent to a circle domain.

Remark 2. Suppose that a domain D is "quasisymmetric" with respect to a Jordan curve γ which intersects each boundary component of D. Then D is conformally equivalent to a circle domain.

5. IX. 1966

III. FUNCTIONS OF SEVERAL COMPLEX VARIABLES

On some applications of potentials in investigations of extremal functions

by W. Bach (Kraków)

Let R^n be the n-dimensional Euclidean space, $\omega(p_1, \ldots, p_a)$ a non-negative continuous and symmetric function of $a \geqslant 2$ variables $p_1, \ldots, p_a \in R^n$, and μ a positive Radon measure. Using the potentials $U(\mu: p)$ defined by the formula

$$U(\mu: p) = \int \ldots \int \log \frac{1}{\omega(p, p_2, \ldots, p_a)} d\mu(p_2) \ldots d\mu(p_a),$$

the author either solves or gives a method of solving some problems of Leja (see F. Leja, *Sur certaines suites de fonctions extrémales de plusieurs variables complexes*, Ann. Polon. Math. 12 (1962), pp. 105-114).

The results are published in the paper: W. Bach, *Własności potencjałów o jądrach dwu lub więcej zmiennych i ich zastosowanie do badania funkcji ekstremalnych*, Wyższa Szkoła Pedagogiczna w Krakowie 25, Prace Mat. 4 (1966), pp. 5-52.

6. IX. 1966

Interior distinguished sets and their applications in pseudo-conformal transformations in the theory of two complex variables

by S. BERGMAN (Stanford, Calif.)

Continuing the previous work (Journal d'Analyse Mathématique 13 (1964), p. 317, Proc. of Conference on Complex Analysis, Minneapolis, Springer (1965), p. 30) the author investigates the invariant metric $ds = (\sum T_{mn} - dz_m d\bar{z}_n)^{-1}$ and the invariant $J^{(\varkappa)}$, $\varkappa = 1, 2$ (see (1) Mémor. Sci. Math. 106 and 108, pp. 52 and 55). [Invariant = invariant with respect to PCT's (pseudo-conformal transformations).] In the case of Reinhardt circular domains R, $J^{(\varkappa)}(z_1, z_2)$ (if not constant) has an isolated critical point at the center P of R or a critical hypersurface passes through P. (See (1), p. 55, and G. Springer, Duke Math. Journ. 18 (1951), p. 411.) These properties are invariant in PCT's and, except for certain cases, are sufficient to characterize the image in D of the center P of R. We assume that $D = T(R)$, where T is a PCT of R. The representative domain $\boldsymbol{R}(D, t)$ with respect to t is the circular domain R. In this way a procedure is obtained which permits one to decide whether a domain, say D, is a pseudo-conformal image of R or not. Our approach can be generalized to the case of rigid (¹) domains B. Under some additional hypotheses one can construct an (α, β)-hull $h^3_{\alpha\beta}$ in a domain B as follows. Suppose that ∂B consists of points described on pp. 8-9 of (1, 108), sub 1°. Let

$$d\sigma = \left|\frac{9\pi^2}{2} J^{(1)}_B(z_1, z_2, \bar{z}_1, \bar{z}_2) - 1\right| ds$$

be the length of the line element of an (invariant) metric defined along the interior normal at Q, $Q \in \partial B$. This metric will be called σ-metric. We determine to every Q, $Q \in \partial B$, the point q which lies on the interior normal at Q and has the distance α from Q using the σ-metric. The points q form a closed hypersurface g^3_α. Around every point M of g^3_α we draw the hypersphere of a radius β (measured using the s-metric, i.e., the metric with the line element ds). The envelope of these hyperspheres forms the hull $h^3_{\alpha\beta}$. Under some additional hypotheses $h^3_{\alpha\beta}$ lies completely in B. The hypersurface $h^3_{\alpha\beta}$ is invariant with respect to PCT's of B. Using properties of distinguished sets (critical sets of $J^{(\varkappa)}_B$) lying in $h^3_{\alpha\beta}$, one obtains conditions in order that two domains B_1 and B_2 be pseudo-conformally mapped onto each other.

3. IX. 1966

(¹) Rigid means that B does not admit PCT's onto itself with an interior fixed point.

Quelques formules intégrales $(n+p)$-dimensionnelles pour les fonctions analytiques des plusieurs variables complexes

par F. Bierski (Kraków)

1. Soit D un domaine borné de l'espace C^n de n variables complexes $z = (z_1, \ldots, z_n)$, $f(z)$ une fonction analytique dans D, continue avec ses dérivées jusqu'à l'ordre j dans $D+\partial D$, où j est un des nombres $0, 1, \ldots, n-1$. Il existe plusieurs formules intégrales pour $f(z)$ comme celle de Cauchy [5], de Bergman [2], de Weil [10], de Martinelli [7], de Fantappiè [6], de Temliakow [9], de Aizenberg [1], de Opial et Siciak [8] etc. Toutes ces formules sont de la forme:

$$f(z) = \int_S \Phi\big(f(s)\big) I(s, z) ds \tag{1}$$

où S est une variété à m dimensions $(m = n, \ldots, 2n-1)$ contenue dans $D+\partial D$, ds est un élément de S au point $s \in S$, Φ est un opérateur linéaire, $I(s, z)$ est une fonction de deux points s et z, dite noyau de l'intégrale (1). La formule (1) sera dite m-dimensionnelle au noyau analytique (non analytique), si la variété S est à m dimensions et le noyau $I(s, z)$ est une fonction analytique (non analytique) des variables $z = (z_1, \ldots, z_n)$.

2. L'objet de ma communication concerne quelques formules intégrales pour les fonctions analytiques des plusieurs variables complexes [3], [4].

Soit S_p le simplexe à p dimensions $(p = 1, \ldots, n-1)$:

$$S_p = \{\tau = (\tau_1, \ldots, \tau_p):\ 0 < \tau_1 < \tau_2 < \ldots < \tau_p < 1\}\,, \tag{2}$$

et Δ le policylindre à $2n$ dimensions

$$\Delta = \{z = (z_1, \ldots, z_n):\ |z_k| < r_k\,,\ r_k > 0\,,\ k = 1, \ldots, n\}\,. \tag{3}$$

Désignons par Z_{n+p} une variété à $(n+p)$ dimensions contenue dans Δ et définie par les équations parametriques:

$$Z_{n+p} = \{\zeta = (\zeta_1, \ldots, \zeta_n):\ \zeta_k = \varrho_k(\tau, r) e^{i\varphi_k}\,,\ k = 1, \ldots, n\} \tag{4}$$

où τ parcourt le simplexe (2), $0 \leqslant \varphi_k \leqslant 2\pi$, $r = (r_1, \ldots, r_n)$ est constant et $\varrho_k(\tau, r) = \varrho_k(\tau_1, \ldots, \tau_p, r_1, \ldots, r_n)$ est une fonction continue, positive pour $\tau \in S_p$ et telle que

$$\begin{aligned} &\varrho_k(0, \ldots, 0, r_1, \ldots, r_n) = \begin{cases} 0 & \text{pour} \quad k = 1, \ldots, p\,, \\ r_k & \text{pour} \quad k = p+1, \ldots, n\,, \end{cases} \\ &\varrho_k(0, \ldots, 0, \tau_j = 1, \ldots, \tau_p = 1, r_1, \ldots, r_n) = \begin{cases} 0 & \text{pour} \quad k \neq j\,, \\ r_k & \text{pour} \quad k = j\,. \end{cases} \end{aligned} \tag{5}$$

Désignons encore par $u_{pk} = u_{pk}(\zeta, \tau, z)$ l'expression

$$u_{pk} = \frac{\tau_1}{\zeta_1} z_1 + \frac{\tau_2 - \tau_1}{\zeta_2} z_2 + \dots + \frac{\tau_p - \tau_{p-1}}{\zeta_p} z_p + \frac{1 - \tau_p}{\zeta_k} z_k , \tag{6}$$

pour $k = p+1, \dots, n$.

Soit $z = (z_1, \dots, z_n)$ un point quelconque mais fixe du domaine Δ et Z_{n+p} une variété telle que, en chaque point ζ de Z_{n+p}, on ait

$$|u_{pk}(\zeta, \tau, z)| < 1 \qquad \text{pour} \qquad k = p+1, \dots, n . \tag{7}$$

THÉORÈME 1. *Soit $f(z)$ une fonction analytique dans le domaine Δ, continue avec ses dérivées jusqu'à l'ordre j dans $\bar{\Delta}$ où j est un des nombres $0, 1, \dots, p$. Alors en chaque point $z \in \Delta$ remplissant la condition* (7) *on a pour $p = 1, \dots, n-1$, $j = 0, 1, \dots, p$ la formule:*

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{Z_{n+p}} \left\{ \frac{\partial^j}{\partial t^j} [t^{n-1} f(\zeta t)] \right\}_{t=1} I_{p-j} \frac{d\zeta_1}{\zeta_1} \wedge \dots \wedge \frac{d\zeta_n}{\zeta_n} \wedge d\tau_1 \wedge \dots \wedge d\tau_p \tag{8}$$

où $f(\zeta t) = f(\zeta_1 t, \dots, \zeta_n t)$ et I_{p-j} est donné par la formule

$$I_{p-j} = (1-\tau_p)^{n-p-1} \left\{ \frac{\partial^{p-j}}{\partial t^{p-j}} \left[t^{n-j-1} \prod_{k=p+1}^{n} (1 - u_{pk} t)^{-1} \right] \right\}_{t=1} . \tag{9}$$

THÉORÈME 2. *Soit $f(z)$ une fonction analytique dans un domaine n-cerclé D de centre $z = 0$, continue avec ses dérivées jusqu'à l'ordre j dans $D + \partial D$, où j est un des nombres $0, 1, \dots, p$. Alors à chaque point $z \in D$ correspond* 1° *un point frontière $\zeta^0 \in \partial D$, pour lequel $|z_k| < |\zeta_k^0|$, $k = 1, \dots, n$,* 2° *une variété* (4) *où $r_k = |\zeta_k^0|$, $k = 1, \dots, n$, remplissant les conditions* (7) *telle que pour chaque $p = 1, \dots, n-1$, et chaque $j = 0, 1, \dots, p$ on a la formule:*

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{Z_{n+p}} \left\{ \frac{\partial^j}{\partial t^j} [t^{n-1} f(\zeta t)] \right\}_{t=1} I_{p-j} \frac{d\zeta_1}{\zeta_1} \wedge \dots \wedge \frac{d\zeta_n}{\zeta_n} \wedge d\tau_1 \wedge \dots \wedge d\tau_p \tag{10}$$

ou I_{p-j} est donné par la formule (8).

3. En conservant les notations D et Z_{n+p} du théorème 2, désignons par D^+, $(\partial D)^+$ et Z_{n+p}^+ les images de D, ∂D et Z_{n+p} dans l'octant positif de l'espace réel R_n d'après la transformation $z \to |z|$. Lorsque le point $(|\zeta_1|, \dots, |\zeta_n|)$ parcourt la variété Z_{n+p}^+ la demi-droite $0|\zeta|$ décrit sur $(\partial D)^+$ une variété S_{n+p}^+ à p dimensions. Désignons par S_{n+p} la variété à $(n+p)$ dimensions, située dans ∂D, dont la projection $\zeta \to |\zeta|$ sur $(\partial D)^+$ est la variété S_{n+p}^+. Les équations de S_{n+p} ont la forme:

$$\zeta_k = \lambda_k(\tau, r) e^{i\varphi_k} , \qquad k = 1, \dots, n , \tag{11}$$

et dépendent (comme celles de Z_{n+p}) de $(n+p)$ parametres réels: $\tau_1, \tau_2, \dots, \tau_p$, $\varphi_1, \dots, \varphi_n$.

THÉORÈME 3. *Soit $f(z)$ une fonction analytique remplissant les conditions du théorème 2. Alors pour $p = 1, \ldots, n-1$, $j = 0, 1, \ldots, p$ on a les formules*

$$(12)\quad f(z) = \frac{1}{(2\pi i)^n} \int\limits_{S_{n+p}} \left\{\frac{\partial^j}{\partial t^j}[t^{n-1}f(\zeta t)]\right\}_{t=1} I_{p-j}\frac{d\zeta_1}{\zeta_1}\wedge \ldots \wedge \frac{d\zeta_n}{\zeta_n}\wedge d\tau_1\wedge \ldots \wedge d\tau_p$$

où ζ_k sont donnés par (11) *et le noyau I_{p-j} est donné par* (9).

En tenant compte des formules (11) *on peut donné à* (12) *la forme suivant:*

$$(13)\quad f(z) = \frac{1}{(2\pi)^n} \int\limits_0^{2\pi} d\varphi_1 \ldots \int\limits_0^{2\pi} d\varphi_n \int\limits_{S_p} \left\{\frac{\partial^j}{\partial t^j}\left[t^{n-1}f\left(\lambda(\tau, r)e^{i\varphi}t\right)\right]\right\}_{t=1} J_{p-j}\, d\tau_1 \ldots d\tau_p$$

où S_p est donné par (2) *et le noyau J_{p-j} prend la forme:*

$$(14)\quad \begin{gathered} J_{p-j} = (1-\tau_p)^{n-p-1}\left\{\frac{\partial^{p-j}}{\partial t^{p-j}}\left[t^{n-j-1}\prod_{k=p+1}^{n}(1-v_{pk}t)^{-1}\right]\right\}_{t=1}, \\ v_{pk} = \frac{\tau_1}{\lambda_1(\tau, r)}e^{-i\varphi_1}z_1 + \frac{\tau_2-\tau_1}{\lambda_2(\tau, r)}e^{-i\varphi_2}z_2 + \ldots + \frac{\tau_p-\tau_{p-1}}{\lambda_p(\tau, r)}e^{-i\varphi_p}z_p + \frac{1-\tau_p}{\lambda_k(\tau, r)}e^{i\varphi_k}z_k \end{gathered}$$

pour $k = p+1, \ldots, n$.

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6. IX. 1966

Analytic continuation for functions of several variables and the Feynman integral

by R. H. CAMERON and D. A. STORVICK (Minneapolis, Minn.)

Theorems relating analyticity and separate analyticity for functions of several variables are proved. The following theorem is an example:

Let Ω_1 be a simply connected domain of the z_1-plane and let Ω_j denote the z_j-plane, $|z_j| < \infty$, for $j = 2, \ldots, n$. Let L_1 be a line segment, $L_1 \subset \Omega_1$, and let L_j be a complete line contained in Ω_j for $j = 2, \ldots, n$. Let $f(z_1, \ldots, z_n)$ be defined and continuous on

$$S = (L_1 \times \Omega_2 \times \ldots \times \Omega_n) \cup (\Omega_1 \times L_2 \times \Omega_3 \times \ldots \times \Omega_n) \cup \ldots \cup (\Omega_1 \times \ldots \times \Omega_{n-1} \times L_n).$$

It is proved that if $f(z_1, \ldots, z_n)$ is analytic in Ω_1 for $(z_2, \ldots, z_n) \in (L_2 \times \ldots \times L_n)$ and if $f(z_1, \ldots, z_n)$ is analytic in Ω_2 for $(z_1, z_3, \ldots, z_n) \in (L_1 \times L_3 \times \ldots \times L_n), \ldots,$ and if $f(z_1, \ldots, z_n)$ is analytic in Ω_n for $(z_1, \ldots, z_{n-1}) \in (L_1 \times \ldots \times L_{n-1})$, then there exists a function $g(z_1, \ldots, z_n)$ analytic in $(\Omega_1 \times \ldots \times \Omega_n)$ such that

$$g(z_1, \ldots, z_n) = f(z_1, \ldots, z_n) \quad \textit{on } S.$$

These theorems are used to develop properties of the Feynman integral.

Conditions on the function F are given under which the following translation formula for analytic Feynman integrals (see Journ. Analyse Math. 10 (1962/63), pp. 287-361) holds. It is shown that the analytic Feynman integral of $F(x+x_0)$ is equal to the product of

$$\exp\Big\{i/2 \int_a^b [x_0'(t)]^2 dt\Big\}$$

with the analytic Feynman integral of $F(x)\exp\big\{i\int_a^b x_0'(t)\,dx(t)\big\}$.

6. IX. 1966

Sur une généralisation de certains lemmes

par F. LEJA (Kraków)

Soit $\omega(p_1, p_2, \ldots, p_a)$ une fonction non négative, continue et symétrique de $a \geqslant 2$ points $p_1, p_2, \ldots, p_a$, variables dans un espace topologique T, et E un ensemble compact de points de T. Désignons par $p^{(n)}$ un système de $n+1 \geqslant a$ points $p_0, p_1, \ldots, p_n$ quelconques de E et posons pour $j = 0, 1, \ldots, n$

$$\Delta^{(j)}(p^{(n)}) = \prod_{\substack{k=0\\(k\neq j)}}^{n} \prod_{\substack{0\leqslant l_3<\ldots<l_a\leqslant n\\(l_\nu\neq j,k)}} \omega(p_j, p_k, p_{l_3}, \ldots, p_{l_a}),$$

$$\Phi^{(j)}(z, p^{(n)}) = \prod_{\substack{k=0\\(k\neq j)}}^{n} \prod_{\substack{0\leqslant l_3<\ldots<l_a\leqslant n\\(l_\nu\neq j,k)}} \frac{\omega(z, p_k, p_{l_3}, \ldots, p_{l_a})}{\omega(p_j, p_k, p_{l_3}, \ldots, p_{l_a})}$$

où z est un point variable dans T. On suppose que les valeurs $\omega(p_j, p_k, p_{l_1}, \ldots, p_{l_a})$ sont positives. Formons les bornes

$$\Delta_n(E) = \sup_{p^{(n)}\in E} \{\min_j \Delta^{(j)}(p^{(n)})\}\,,$$

$$\Phi_n(z, E) = \inf_{p^{(n)}\in E} \{\max_j \Phi^{(j)}(z, p^{(n)})\}\,.$$

On sait que dans le cas $a = 2$ les suites $\{\Delta_n(E)\}$ et $\{\Phi_n(z, E)\}$ satisfont aux inégalités

$$\Delta_{\mu+\nu} \leqslant \Delta_\mu \Delta_\nu\,, \quad \Phi_{\mu+\nu} \geqslant \Phi_\mu \Phi_\nu\,, \quad (\mu, \nu = 1, 2, \ldots)\,,$$

(voir: F. Leja, *Teoria funkcji analitycznych*, Warszawa 1957, pp. 259 et 262) ce qui entraîne la convergence des suites $\{\sqrt[n]{\Delta_n(E)}\}$ et $\{\sqrt[n]{\Phi_n(z, E)}\}$. Si $a > 2$ les inégalités précédentes deviennent fausses. Dans le cas général $a \geqslant 2$ les quantités $\Delta_n(E)$ et $\Phi_n(z, E)$ satisfont aux inégalités

$$\Delta_{\mu+\nu}^{a_{\mu+\nu}} \leqslant \Delta_\mu^{a_\mu} \Delta_\nu^{a_\nu}\,, \quad \Phi_{\mu+\nu}^{a_{\mu+\nu}} \geqslant \Phi_\mu^{a_\mu} \Phi_\nu^{a_\nu}$$

où $a_n = 1/\binom{n-1}{a-2}$, $n = 2, 3, \ldots$ Si $a = 2$, on a $a_n = 1$ pour $n = 2, 3, \ldots$ et les inégalités dernières se reduisent aux précédentes.

La travail sera inséré dans les Ann. Polon. Math.

7. IX. 1966

Boundary value problems and the Poincaré-Lefschetz isomorphism

by H. Röhrl (San Diego, Calif.)

An extension of the Poincaré-Lefschetz isomorphism to coefficients in sheaves of non-abelian groups is given. This isomorphism is used to classify certain boundary value problems and to give a solution theory for these boundary value problems.

6. IX. 1966

Some uniqueness theorems for analytic functions of several complex variables

by L. A. Rubel (Urbana, Ill.)

Using an extension by Arsove of a theorem of Paley and Wiener on perturbation of basis in a topological vector space, we prove in this joint work by B. A. Taylor and the author a number of results. The following is typical.

Theorem. *Suppose* $\{w^{(m)}\}$, $m \in \underline{Z}_+^n$, *is a sequence of elements of* $\underline{C}^n$ *such that*

$$\limsup_{\|m\|\to\infty} \|w^{(m)}\|\,\|m\| < \infty\,.$$

If $f\colon \underline{C}^n \to \underline{C}$ *is an entire function such that* $f^{(m)}(w^{(m)}) = 0$ *for all* $m \in \underline{Z}_+^n$, *then* f *must be the null function* $f = 0$.

A paper with detailed proofs will be published as *Some uniqueness theorems for analytic functions of one and of several variables*, by B. A. Taylor and myself, in the Quarterly Journal of Mathematics, Oxford.

6. IX. 1966

A generalization of a theorem of Pringsheim for functions of several variables

by H. Zahorska (Łódź)

The functions and sets considered in this paper satisfy the following conditions.

Let A denote a closed and bounded set in a real n-dimensional Euclidean space such that any function of n variables defined in it and having the total differential at any point of it has an unambiguous total differential. Moreover, let A be locally star-like (i.e. for any point $p \in A$ let there exist a neighbourhood U such that for any $q \in AU$ the segment joining p and q belongs to A).

The sets A possessing the above properties will be called *sets of the type T*. If a function g has the total differential at a point $p \in A$ (A — a set of the type T), then the corresponding coefficients of the total differential will be called *partial derivatives of the function g at the point p*.

By means of this definition we may define partial derivatives of the functions in question at points at which the usual definition cannot be applied; e.g. by changing one of the variables we may reach a point beyond the set in which g is defined. By the definition of the sets of the type T those derivatives are unambiguously defined. Consider a function f of the class C_∞ defined in A, i.e. a function whose partial derivatives at any point $p \in A$ are all continuous and finite, and its Taylor series with the centre in p.

Let $r = r(p) > 0$ denote a number such that the multiple Taylor series of the function f converges absolutely for increases of the variables absolutely smaller than r.

This number is called the *radius of absolute convergence.*

I shall prove the following

THEOREM. *If for any point p belonging to a set A of the type T and for a function f from C_∞ defined in A the radius of absolute convergence of the Taylor series of the function f with the centre in p is $r(p) \geqslant \delta > 0$ (δ being constant independent of p), then the function f is analytic in A, i.e. the function f is represented in a neighbourhood of any point $p \in A$ by its Taylor series with the centre at p.*

The function f may obviously be analytically extended on a real n-dimensional or even $2n$-dimensional (complex) neighbourhood of A; if, however, A is not connected, it need not be the same function or even different branches of the same function on various components of A. It can easily be proved that a set of type T contains a finite number of components which are continua and all of which are of the type T.

This result was published by Pringsheim in 1893 for functions of one variable defined on a closed segment.

The proof given by Pringsheim was simple and short but it contained an error. The first correct proof was given by Boas in 1935. Obviously, for functions of one variable the only continua of the type T are closed segments.

Using sets which are not of the type T (e.g. sets without the local starlikeness) we may find examples in which the Pringsheim theorem is not true.

3. IX. 1966

IV. OTHER TOPICS IN COMPLEX FUNCTIONS

Transforms of isolated singular points

by Z. Charzyński (Łódź)

Let

$$T:\ \xi = U(x, y)\,,\ \ \eta = V(x, y)$$

be a continuous transformation defined in an open plane set and having in it a reversible differential of the first order everywhere, except perhaps at points of an isolated exceptional set.

Then T is an interior mapping in Stoïlov's sense, i.e. for every point (x_0, y_0) the image of any neighbourhood of this point under T covers a certain neighbourhood of $(\xi_0, \eta_0) = \big(U(x_0, y_0), V(x_0, y_0)\big)$.

7. IX. 1966

A condition equivalent to Riemann's conjecture

by J. Chądzyński (Łódź)

Let $\{s_n\}$ denote a sequence of non-trivial zeros of the Riemann Zeta function, their multiplicity being regarded. It can be proved that

$$\sqrt{\pi}\sum_{n=1}^{\infty}\exp\big(c(s_n-\tfrac{1}{2})^2\big) = \frac{1}{2\sqrt{c}}(\gamma+\log\pi) - \sqrt{\pi}\big(1-\Phi(\tfrac{1}{2}\sqrt{c})\big)\exp(\tfrac{1}{4}c) + \sum_{n=1}^{\infty}\left\{\frac{1}{2n\sqrt{c}} - \sqrt{\pi}\exp\big(4c(n+\tfrac{1}{4})^2\big)\big[1-\Phi\big(2(n+\tfrac{1}{4})\sqrt{c}\big)\big]\right\} + \left\{2\sqrt{\pi}\exp(\tfrac{1}{4}c) - \frac{1}{\sqrt{c}}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\exp\left(-\frac{\log^2 n}{4c}\right)\right\}$$

for $\operatorname{re} c > 0$, where γ denotes the Euler constant, $\Phi(z)$ is the probability integral, and $\Lambda(n)$ is the Dirichlet function in the theory of numbers. Basing on this formula we can prove the following

THEOREM. *All the non-trivial zeros of the Riemann Zeta function lie on the straight line* $s = \frac{1}{2}+it$ *if and only if for every number* ε, $0 < \varepsilon < \pi/2$, *the expression*

$$\Omega(c) = 2\sqrt{\pi}\exp(\tfrac{1}{4}c) - \frac{1}{\sqrt{c}}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\exp\left(-\frac{\log^2 n}{4c}\right)$$

converges uniformly to 0 *as* $c\to\infty$ *in the angle* $|\operatorname{Arg} c| < \pi/2-\varepsilon$.

The proof of this theorem is based on the following

LEMMA. *If the zeros* s_n *do not all lie on the straight line* $s = \frac{1}{2}+it$, *then there exists a number* φ^*, $0 < \varphi^* < \frac{1}{2}\pi$, *such that*

$$\lim_{\substack{c\to\infty\\ \operatorname{Arg} c = a}}\left|\sum_{n=1}^{\infty}\exp\left(c(s_n-\tfrac{1}{2})^2\right)\right| = \begin{cases} 0 & \textit{for} \quad |a| < \varphi^*, \\ \infty & \textit{for} \quad \varphi^* < |a| < \tfrac{1}{2}\pi \ \textit{a.e.} \end{cases}$$

Moreover,

$$\varphi^* = 2\{\min_n|\operatorname{Arg}(s_n-\tfrac{1}{2})|\} - \frac{\pi}{2}.$$

3. IX. 1966

On Tsuji functions

by W. K. HAYMAN (London)

In a recent paper Collingwood and Piranian [1], following Tsuji [4], considered the class of functions $f(z)$ meromorphic in $|z| < 1$ and such that

$$\overline{\lim_{r\to 1}}\int_0^{2\pi} f^*(re^{i\theta})r\,d\theta < +\infty,$$

where

$$f^*(z) = \frac{|f'(z)|}{1+|f(z)|^2}$$

is the spherical derivative of $f(z)$. They defined a segment of Julia as a rectilinear segment s lying except for one end-point $e^{i\theta}$ in $|z| < 1$ and such that, in every Stolz angle containing s, $f(z)$ assumes infinitely often all values in the closed plane with at most two exceptions. If every segment s with end-point $e^{i\theta}$ is a segment of Julia, then $e^{i\theta}$ is a Julia point. Collingwood and Piranian provided some interesting examples showing for instance that for a meromorphic function every point on $|z| = 1$ may be a Julia point. They also made the following conjectures.

I. A regular Tsuji function can have at most finitely many points $e^{i\theta}$ which are end-points of segments of Julia.

II. More strongly such a function can have only finitely many segments of Julia.

III. A regular normal Tsuji function can have no segments of Julia.

We shall disprove I and II by proving [2]

THEOREM I. *There exists a regular Tsuji function with infinitely many Julia points.*

We can also prove III and rather more [3]

THEOREM II. *A normal (meromorphic) Tsuji function remains continuous in* $|z| \leqslant 1$.

Thus such a function cannot have any segments of Julia.

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3. IX. 1966

Uniformization of a class of algebraic functions of the third degree by the method of differential equations

by J. JANIKOWSKI (Łódź)

I extend the method of differential equations used by Charzyński to functions of a complex variable. I prove the following

THEOREM. *Let $f(z)$ and $g(z)$ be two polynomials of the third degree. Consider the equation*

$$f(z) = g(w).$$

Assume that if $f(z_0) = g(w_0)$ for a pair (z_0, w_0) of complex numbers, then either $f'(z_0) \neq 0$ or $g'(w_0) \neq 0$. Under these assumptions there exist two functions $z(t)$ and $w(t)$ defined over the whole open complex plane such that

1. *$z(t)$ and $w(t)$ are meromorphic,*

2. *we have the equation*

$$f(z(t)) = g(w(t))$$

in the whole open plane.

7. IX. 1966

On Hardy-Orlicz spaces

by R. LEŚNIEWICZ (Poznań)

We denote by N the class of functions F analytic in the disc $\Delta = \{|z| < 1\}$ such that

$$\sup_{0 \leqslant r < 1} \int_0^{2\pi} \log^+ |F(re^{it})|\, dt < \infty .$$

Nevanlinna has shown that functions of this class have the non-tangential limits $F(e^{it}) = \lim_{z \to e^{it}} F(z)$ almost everywhere on the circle $\{|z| = 1\}$. We denote by N' the subclass of N made up of functions $F \in N$ for which the integrals

$$\int_0^{x} \log^+ |F(re^{it})|\, dt , \qquad 0 \leqslant r < 1 ,$$

are uniformly absolutely continuous. The classes N and N' are linear sets ([6], Ch. VII/7).

Now, a function non-decreasing and continuous for $u \geqslant 0$, vanishing only at $u = 0$ and tending to ∞ as $u \to \infty$ will be called a *φ-function*. Let φ_1 and φ_2 be two φ-functions. We say that φ_1 *is not weaker than* φ_2 and write $\varphi_2 \prec \varphi_1$ if

$$\varphi_2(u) \leqslant a\varphi_1(ku) \quad \text{for} \quad u \geqslant u_0$$

holds with some constants $a, k > 0$ and $u_0 \geqslant 0$. We say that φ_1 and φ_2 are *equivalent* and write $\varphi_1 \sim \varphi_2$ if $\varphi_1 \prec \varphi_2$ and $\varphi_2 \prec \varphi_1$. We say that a φ-function φ *satisfies the Δ_2-conditions* if

$$\varphi(2u) \leqslant d\varphi(u) \quad \text{for} \quad u \geqslant u_0$$

holds with some constants $d > 0$ and $u_0 \geqslant 0$ ([3]).

A φ-function φ which can be represented in the form

$$\varphi(u) = \Phi(\log u) \quad \text{for} \quad u > 0 ,$$

where Φ is convex on the whole real axis and satisfies

$$\lim_{x \to \infty} \frac{\Phi(x)}{x} = \infty ,$$

will be called a *log-convex φ-function*. The log-convex φ-functions are increasing for $u \geqslant 0$. Every function $\psi(u^s)$, where $s > 0$ and ψ is a convex φ-function, is a log-convex φ-function.

In the sequel the letter φ will be used only for log-convex φ-functions.

Let F be an analytic function in Δ. We shall use the notation:

$$\mu_\varphi(F) = \sup_{0\leqslant r<1} \int_0^{2\pi} \varphi[|F(re^{it})|]\,dt\,.$$

We denote by $H^{*\varphi}$ and K^φ the sets of functions F analytic in Δ such that $\mu_\varphi(\lambda F) < \infty$ for some and for all $\lambda > 0$, respectively. The classes $H^{*\varphi}$ and K^φ are linear sets and $K^\varphi \subset H^{*\varphi} \subset N'$.

Then the following theorems can be proved:

1. *Every function of the class N' belongs to some class $H^{*\varphi}$.*

2. *The equality*

$$\mu_\varphi(F) = \int_0^{2\pi} \varphi[|F(e^{it})|]\,dt$$

holds for $F \in N'$ (in particular for $F \in H^{*\varphi}$ or $F \in K^\varphi$).

3. *The inequality*

$$|F(z)| \leqslant \varphi_{-1}\left\{\frac{\mu_\varphi(F)}{\pi(1-|z|)}\right\}$$

holds for $F \in N'$ and $z \in \Delta$. Here, φ_{-1} denotes the inverse function of φ.

4. *The inclusion $H^{*\varphi_1} \subset H^{*\varphi_2}$ (also $K^{\varphi_1} \subset K^{\varphi_2}$) holds if and only if $\varphi_2 \precsim \varphi_1$. The equality $H^{*\varphi_1} = H^{*\varphi_2}$ (also $K^{\varphi_1} = K^{\varphi_2}$) holds if and only if $\varphi_1 \sim \varphi_2$.*

5. *The equality $H^{*\varphi} = K^\varphi$ holds if and only if φ satisfies the Δ_2-condition.*

In $H^{*\varphi}$ an F-norm can be defined as follows:

$$\|F\|_\varphi = \inf\{\varepsilon > 0:\ \mu_\varphi(F/\varepsilon) \leqslant \varepsilon\}\,.$$

The class $H^{*\varphi}$ with the norm $\|\cdot\|_\varphi$ is a Fréchet space. This space will be called a *Hardy-Orlicz space* and will be denoted by $[H^{*\varphi}, \|\cdot\|_\varphi]$.

We have the following theorems:

6. *If $\|F_n\|_\varphi \to 0$, then $F_n(z) \to 0$ uniformly on every compact subset of Δ.*

7. *The class K^φ is identical with the closed linear hull in $[H^{*\varphi}, \|\cdot\|_\varphi]$ of the set of functions analytic in Δ and continuous in $\bar{\Delta} = \{|z| \leqslant 1\}$. Moreover, the equality*

$$\lim_{r\to 1-} \|F - T_rF\|_\varphi = 0\,, \quad \textit{where} \quad T_rF(z) = F(rz)\,,$$

holds for $F \in K^\varphi$.

8. *K^φ is a separable space with the norm $\|\cdot\|_\varphi$. Polynomials with rational coefficients form a countable set everywhere dense in K^φ.*

9. *The space* $[H^{*\varphi}, \|\cdot\|_\varphi]$ *is separable if and only if* φ *satisfies the* Δ_2*-condition.*

In the case $\varphi(u) = \psi(u^s)$, where $0 < s \leqslant 1$ and ψ is a convex φ-function, we can define an *s-homogeneous norm in* $H^{*\varphi}$ by means of the formula:

$$\|F\|_{s\varphi} = \inf\{\varepsilon > 0:\ \mu_\varphi(F/\varepsilon^{1/s}) \leqslant 1\}\,.$$

The norms $\|\cdot\|_\varphi$ and $\|\cdot\|_{s\varphi}$ are equivalent.

The following result is interesting:

10. *There exists in* $H^{*\varphi}$ *an s-homogeneous norm* $\|\cdot\|^0$ $(0 < s \leqslant 1)$ *with the property:* $\|F_n\|^0 \to 0$ *implies* $F_n(z) \to 0$ *uniformly on every compact subset of* Δ *if and only if* $\varphi(u) \sim \psi(u^s)$, *where* ψ *is a convex* φ*-function.*

In the proof of this theorem the arguments of [2] and [4] are applied (also [1] and [5]).

Full proofs of the above theorems will appear in Studia Mathematica.

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7. IX. 1966

Analytic continuability of wave functions and its applications

by J. Ławrynowicz and L. Wojtczak (Łódź)

1. Let us consider the general form of a wave equation

$$\Delta\psi + 2m[E + V(E, \vec{r})]\psi = 0\,. \tag{1}$$

Here m denotes the mass of a particle,

$$E = E'[1 + (1/2m)\varepsilon^2 E']\,, \tag{2}$$

$$V = V' + (1/2m)\varepsilon^2 V'(E' + V') + \varepsilon^2[1 + (1/2m)(E' + V)]\nabla\nabla\mu + \\ + \eta\varepsilon^2[1 + (1/2m)(E' + V')]\sigma[\nabla \times \nabla\mu]\,, \tag{3}$$

E' and $V' = V'(E', \vec{r})$ denote the energy parameter and the potential of the particle, respectively (in the sense of the Schrödinger equation), where $\vec{r}$ is a radius vector of this particle. The parameter σ denotes the spin operator, $\mu = \varepsilon^2/[2m+\varepsilon^2(E'+V')]$, and ε, η are parameters depending on the character of the equation. In particular, for $\varepsilon = 0$, $\eta = 0$ the equation (1) represents a Schrödinger equation, for $\varepsilon = 1/137$, $\eta = 0$ a Klein-Gordon equation, and for $\varepsilon = 1/137$, $\eta = 1$ a Dirac equation.

Generally the complicated character of a hamiltonian causes difficulties in an effective solution of the wave equation. For different problems there are different ways of seeking solutions, but they may be applied in particular cases only. The paper [3] here summarized gives a uniformization of the way of obtaining eigenfunctions, eigenvalues, and other properties which belong to a given hamiltonian. These properties are in themselves a set of properties which are comparable with experiments and are a starting point for the calculation of other properties by means of the quantum mechanics.

We discuss the analytic continuability of wave equation solutions considered as meromorphic functions from the discrete onto the continuous side of the energy spectrum in the plane of the complex energy parameter k. We prove two theorems giving necessary and sufficient conditions for such continuability. Next, we transfer the results obtained to the solutions of certain systems of differential equations, namely to the wave functions corresponding to the individual quantum numbers. As a result we obtain general expressions for the scattering amplitude and scattering matrix elements depending on the discrete wave functions and on their analytic continuations. The result obtained contributes to those presented by Sasakawa [4] and it widens their range to the relativistic case.

We remark here that the functions ψ_{nls} (n, l, s — quantum numbers) belonging to the discrete spectrum a_{nls} can be expressed as

$$(4)\qquad \psi_{nls}(a_{nls}, \vec{r}) = N_{nls} \sum_{\nu\lambda\sigma} (2\lambda+1)^{1/2} A^{nls}_{\nu\lambda\sigma} \psi^{0}_{\nu\lambda\sigma}(a^{0}_{\nu\lambda\sigma}, r)\, Y_{\lambda\sigma}(\vartheta, \varphi) ,$$

where N_{nls} are normalization constants, r, ϑ, φ denote the polar coordinates of the particle under consideration, $\psi^{0}_{\nu\lambda\sigma}$ are the solutions of the equations

$$(5)\qquad [\Delta_r - a^{0\,2}_{\nu\lambda\sigma} - \lambda(\lambda+1)r^{-2} + 2mL(\lambda, \sigma, \lambda, \sigma;\, a^{0}_{\nu\lambda\sigma}, r)]\psi^{0}_{\nu\lambda\sigma} = 0$$

which fulfil the usually assumed conditions, and L is described in [3]. The coefficients $A^{nls}_{\nu\lambda\sigma}$ and a_{nls} can be determined by the formulae

$$(6)\qquad A^{nls}_{nls}(a^{0\,2}_{nls} - a^{2}_{nls}) + \sum_{nls \neq \nu\lambda\sigma} A^{nls}_{\nu\lambda\sigma} \langle nls | 2mL(l, s, \lambda, \sigma;\, a^{0}_{nls}, r) | \nu\lambda\sigma\rangle = 0 ,$$

which can be obtained by applying a known method due to Titchmarsh. In consequence, the analytic continuability of wave functions permits us to reduce the problem of solving (1) to the problem of solving (5).

2. We illustrate our results by some examples interesting from the point of view of physics. First we apply our results in the case of a potential $V = V_0\mu e^{-\mu r}/(1-e^{-\mu r})$, which is a fairly good approximation of the Yukawa potential for properly chosen real constants V_0 and μ. Next we calculate the probability of the deuteron desintegration in a proton-proton collision. This is related to an earlier result of Eriksson, Hulthen and Johansons [2]. It concerns the problem of molecular dissociation of diatomic molecules. Now we present the main result.

Let us assume that dissociation is caused by the action of an external oscillating force with a frequency ω and an amplitude f. Let p_{vE} denote the probability of transition of a molecule from the bound state E_v to the domain of energy $E_v + D_e$, where D_e is the dissociation energy, i.e. to the domain where the molecule is dissociated. Let M denote the reduced mass of the molecule and δ — the Dirac delta function. We discuss the vibrational levels of molecules characterized by a potential curve

$$V(r) = D_e[1 + x(ax+b)/(1+x)^2], \tag{7}$$

where $x = c\exp(-\gamma r)$, $a = \eta(2-\eta)/(\eta-1)^2$, $b = 2(2-\eta)/(\eta-1)$, $c = (\eta-1)\exp(\gamma r_e)$, $\gamma = (2\hat{\Delta}^{1/2} - \hat{\Gamma}^{1/2})/r_e$, $\eta = (\hat{\Gamma}/\hat{\Delta})^{1/2}$, $\hat{\Gamma} = (1 + a_\omega \omega_e/6B_e^2)^2$, $\hat{\Delta} = k_e\gamma_e^2/2D_e$, and the parameters $B_e, D_e, k_e, \gamma_e, a_\omega, \omega_e$ are spectroscopic constants of molecules for which the above notation is commonly used in the theory of molecules (cf. [5]).

It can be verified that in case of the potential in question our conditions of continuability are fulfilled, so that at the side of continuous energy spectrum we can already determine directly the wave function, namely

$$\psi_E(-k, r) = (1-x)^p\big[e^{ikr}{}_2F_1\big(A(ik), B(ik); C(ik); x\big) + e^{ikr}{}_2F_1\big(A(-ik), B(-ik); C(-ik); x\big)\big], \tag{8}$$

where k is the continued "complex" energy parameter, $p = \frac{1}{2}\{1-[1+ 8MD_e(a+b)]^{1/2}\}$, $A(-ik) = p + [p(p-1) + 2MD_e b/\gamma^2 - k^2/\gamma^2]^{1/2} - ik/\gamma$, $B(-ik) = p - [p(p-1) + 2MD_e b/\gamma^2 - k^2/\gamma^2]^{1/2} + ik/\gamma$, and $C(-ik) = 1 + 2ik/\gamma$.

Let us consider now only the case of dissociation which has occurred under the influence of interaction caused by scattering particles on a diatomic molecule. We assume that a particle with momentum p collides with a diatomic molecule in vibrational level v. During the collision it transfers the energy $\omega = (p^2 - p'^2)/2\mu = q^2/2\mu$, where p' is the momentum of the scattered particle, and μ — the reduced mass of the particle-molecule system. If we assume additionally that the molecule has been in the ground state $v = 0$ and the dissociated molecule is described by a wave s, then

$$(9) \qquad dp_{0E}/dE = (M/2E)^{1/2}|\langle E_0|\exp(i\mu qr/M)|E\rangle|^2\delta(\omega - D_e - \tfrac{1}{2}\omega_e - E)\,.$$

Hence, by direct calculation,

$$(10) \qquad \begin{aligned} p_{0E} &= (M/2E)^{1/2} N_0^2 |J_p^+(A, B, C) + J_p^-(A, B, C)|^2\,, \\ N_0(a) &= [(1/2a)\,{}_2F_1(-2p, 2a/\gamma;\ 1+2a/\gamma;\ c)]^{1/2}\,, \end{aligned}$$

where, in the case $c = 1$,

$$(11) \qquad \begin{aligned} J_p^{\pm}(A, B, C) = (1/\gamma)[\Gamma(2p+1)\Gamma(1+a/\gamma)/\Gamma(2p+2+a/\gamma)] \times \\ \times {}_3F_2(A^{\pm}, B^{\pm}, 1+a/\gamma;\ C^{\pm}, 2p+2+a/\gamma;\ 1)\,, \end{aligned}$$

$$a = a_0 - \gamma + i(\tau \pm a)\,, \quad \tau = \mu q/M\,, \quad \operatorname{re} p > -\tfrac{1}{2}\,, \quad \operatorname{re}(a/\gamma) > -1\,,$$
$$\operatorname{re}(C^{\pm} + 2p + 1 - A^{\pm} - B^{\pm}) > 0\,.$$

Analogous but more complicated results can be obtained in the case $c \neq 1$.

From the point of view of physics the resulting probability of transition may also be interpreted as the probability of dissociation of a molecule by quanta of the falling light, where molecules are considered as linear oscillators. Our result corresponds to that obtained by Askarian [1]. Since he applies the potential of Morse, it can be proved that our result gives a much better approximation of reality. Notice that our result can be used in the investigations of the structure of molecules by means of the lasser light, similarly to the case of the above-mentioned formula of Askarian.

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7. IX. 1966

Sur la géométrie da la représentation conforme

par P. T. MOCANU (Cluj)

Soit C_a: $z = \varphi(t, a)$, $t \in \langle t_0, t_1\rangle$, $a \in \langle 0, +\infty)$, $\varphi(t, 0) = 0$, une famille de courbes de Jordan isotopes qui se contractent à l'origine. Considèrons deux fonctions f et g holomorphes et univalentes dans un domaine qui contient ces courbes et notons $\Gamma_a^f = f(C_a)$, $\Gamma_a^g = g(C_a)$. Supposons que $f(0) = g(0) = 0$ et que pour chaque a et $\boldsymbol{a}$, $a \leqslant \boldsymbol{a}$, nous avons $\Gamma_a^g \subset (\Gamma_{\boldsymbol{a}}^f)$. On dit que la courbe $\Gamma_{\boldsymbol{a}}^f$ est étoilée par rapport à Γ_a^g, où $a \leqslant \boldsymbol{a}$, si l'angle fait avec l'axe réel par chaque tangente à la courbe Γ_a^g menée du point $f(z) \in \Gamma_{\boldsymbol{a}}^f$, $z = \varphi(t, \boldsymbol{a})$, varie d'une manière monotone avec le paramètre t.

La condition nécessaire et suffisante pour que la courbe $\Gamma_{\boldsymbol{a}}^f$ soit étoilée par rapport à Γ_a^g est

$$\operatorname{im}\frac{(dz/dt)f'(z)}{f(z)-g(\zeta)} \geqslant 0 \tag{1}$$

pour chaque $z = \varphi(t, \boldsymbol{a})$ et chaque $\zeta = \varphi(\tau, a)$ où τ vérifie l'équation

$$\operatorname{im}\frac{(d\zeta/d\tau)g'(\zeta)}{f(z)-g(\zeta)} = 0\,. \tag{2}$$

Si l'inégalité (1) est stricte, alors on dit que $\Gamma_{\boldsymbol{a}}^f$ est strictement étoilée par rapport à Γ_a^g.

Pour a fixé, on introduit une notion de rayon généralisé d'étoilement (c'est la borne supérieure des nombres $\boldsymbol{a}$, $\boldsymbol{a} \geqslant a$), tel que $\Gamma_{\boldsymbol{a}}^f$ soit strictement étoilée par rapport à Γ_a^g et on déduit le système qui détermine ce rayon. On étudie quelques cas particuliers.

5. IX. 1966

Degree of convergence of some sequences in the conformal mapping theory

by J. SICIAK (Kraków)

Let E be a compact plane set each component of which has the ordinary diameter $\geqslant 2r$, $r = \text{const} > 0$. Then:

1° $G(z) \leqslant (\pi+\sqrt{\delta/r})\sqrt{\delta/r}$, if $\operatorname{dist}(z, E) < \delta$, $\delta > 0$, where $G(z)$ is the Green function of the unbounded component of $C\setminus E$ with pole at ∞.

2° The degrees of convergence of standard number sequences (sequences of functions) converging to the logarithmic capacity of E (to $G(z)$) are estimated by expressions of the form $O\big((1/n)\log(n+1)\big)$.

3° If E is a compact subset of the unit circle, then a result analogous to 2° is valid also for the hyperbolic capacity of E.

2° and 3° are generalizations of recent results due to W. Kleiner and Ch. Pommerenke, who considered only the case of the logarithmic capacity of a connected set.

The paper has been published in Coll. Math. 16 (1967), pp. 45-59.

7. IX. 1966

Sur l'approximation de courbes arbitraires par les lignes de niveau des fonctions rationnelles

par J. ŚLADKOWSKA (Łódź)

L'objet de ce communiqué est l'approximation de la courbe arbitraire à points multiples par la courbe sur laquelle une certaine fonction rationnelle et symétrique est réelle. Plus précisement, je démontre que, si

$$C\colon\ w = w(\tau)\,,\quad 0 \leqslant \tau \leqslant 1\,,$$

alors pour $\varepsilon > 0$ arbitraire, il existe une fonction rationnelle et symétrique $\Phi(w)$ et une courbe régulière

$$L\colon\ w = \omega(\tau)\,,\quad 0 \leqslant \tau \leqslant 1\,,$$

telles que

$$\operatorname{im}\Phi(w) = 0 \quad \text{pour} \quad w \in L$$

et

$$|w(\tau) - \omega(\tau)| < \varepsilon \quad \text{pour} \quad 0 \leqslant \tau \leqslant 1\,.$$

7. IX. 1966

Über die Verschiebung der Wurzeln algebraischer Gleichungen bei Abänderung der Koeffizienten im Zusammenhang mit dem Minimum-Prinzip

von W. TUTSCHKE (Berlin)

In der Mitteilung wird eine Abschätzung behandelt, die angibt, um wieviel höchstens sich die Nullstellen eines Polynoms verschieben können, wenn die Koeffizienten in bestimmter Weise abgeändert werden. Eine solche Abschätzung von O. Zaubek in Math. Nachr. 25 (1963), S. 319-329 angegeben. Ebenso wie sich der Fundamentalsatz der Algebra auf den Satz von Rouche zurückführen läßt, gewinnt diese Abschätzung aus einer Verschärfung der Aussage des Satses von Rouché. Diese Verschärfung der Aussage des Satzes von Rouché gibt an, wie weit die Nullstellen von $f(z)+g(z)$ von denen von $f(z)$ entfernt sind, wenn auf dem Rand $\sup|g(z)|/\inf|f(z)|$ bekannt ist (Math. Nachr. 25 (1963), S. 331-333; Arch. d. Math. (im Druck)).

3. IX. 1966

PART TWO. PROBLEMS

I. SEMINAR ON EXTREMAL PROBLEMS IN ANALYTIC FUNCTIONS

Problem proposed by Z. CHARZYŃSKI (Łódź)

Suppose that a function $z = \Psi(w)$ is holomorphic and invertible in a certain neighbourhood of 0, and satisfying $\Psi'(0) = 1$. Clearly, the inverse $w = \Phi(z)$ is also holomorphic in a neighbourhood of 0 and it satisfies $\Phi'(0) = 1$.

Expanding Ψ and Φ as power series in w and z, respectively, we have

$$\Psi(w) = w + \mu_2 w^2 + \dots , \qquad \Phi(z) = z + \lambda_2 z^2 + \dots ,$$

where

$$\lambda_n = U_n(\mu_2, \dots, \mu_n) , \qquad n = 2, 3 \dots ,$$

U_n being "universal" polynomials, independent of Ψ and Φ, e.g. $U_2(\mu_2) = -\mu_2$, $U_3(\mu_2, \mu_3) = -\mu_3 + \mu_2^2$, etc.

Let S be the class of all normalized univalent functions of the form

$$F(z) = z + A_2 z^2 + \dots , \qquad |z| < 1 .$$

It can be proved with the help of the notion and properties of univalent polynomials [1] that the well known Bieberbach conjecture is equivalent to the following "Supremum Minimorum" conjecture:

$$\text{(C)} \qquad \sup_Q |U_n(D_2, \dots, D_n)| \varrho_Q^{n-1} \leqslant n ,$$

where the supremum is taken over all polynomials of the form

$$Q(\zeta) = \zeta + D_2 \zeta^2 + \dots + D_L \zeta^L , \qquad L \geqslant n .$$

Here ϱ_Q is defined by

$$\varrho_Q = \min_{1 \leqslant j \leqslant L-1} |Q(\zeta_j)|$$

where ζ_j, $j = 1, \dots, L-1$, denote all zeros of the derivative $Q'(\zeta)$.

An analogous procedure can be applied to other, more general expressions, which depend on the coefficients in S.

The problem is to find a method of calculating the expressions which appear in (C), and more general expressions of a similar type.

2. IX. 1966

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[1] Z. Charzyński, *Fonctions univalentes inverses. Polynômes univalentes*, Bull. Soc. Sci. Lettres Łódź 9, 7 (1958), pp. 1-21.

Problems proposed by G. S. GOODMAN (London)

Let Σ denote the class of all one-to-one conformal mappings

$$g(z) = z + c_1/z + c_2/z^2 + \dots$$

of $|z| > 1$. To each function $g \in \Sigma$ there is associated the set of Faber polynomials $F_n(\lambda)$, $n = 1, 2, \dots$, by means of the generating function

$$\frac{zg'(z)}{g(z) - \lambda} = \sum_{n=0}^{\infty} \frac{F_n(\lambda)}{z^n} .$$

Denote by $\lambda_{n\nu}$, $\nu = 1, \dots, n$, the zeroes of the nth Faber polynomial $F_n(\lambda)$.

1. What is the radius σ of the smallest disc centred at the origin which contains all the zeroes $\lambda_{n\nu}$, $\nu = 1, \dots, n$; $n = 1, 2, \dots$, independently of the function $g \in \Sigma$? The example $g(z) = z + 1/z$ shows that $\sigma \geqslant 2$, while Grunsky's inequalities can be used to prove that $\sigma < 2.04^+$. We conjecture that $\sigma = 2$.

2. Let E denote the complement of the image of $|z| > 1$ under the mapping $g \in \Sigma$, and let $\operatorname{co} E$ denote the convex hull of E. Conjecture: *all* the zeroes of Faber polynomials associated with g belong to $\operatorname{co} E$. Pommerenke has already proved this when E itself is convex.

3. Let Σ^* denote the class of functions g^* which come from functions $g \in \Sigma$ by replacing the coefficients in the Laurent expansion of g by their absolute values, thus:

$$g^*(z) = z + |c_1|/z + |c_2|/z^2 + \dots ,$$

and let σ^* be defined for the Faber polynomials F_n^* associated with functions in Σ^* in just the same way that σ was defined for Σ.

We can then prove that

$$\sigma \leqslant \sigma^* = \sup_{\Sigma^*}[\inf_{r>1} g^*(r)] < 2.25 .$$

We conjecture that $\sigma^* = 2$, with $g^*(z) = z + 1/z$ extremal in Σ^*.

2. IX. 1966

Problems proposed by W. K. HAYMAN (London)

Let $T_1(l)$ be the class of functions $w = f(z)$ meromorphic in $|z| < 1$ for which there exists a sequence of continua C_n surrounding $z = 0$ and lying in $|z| < 1$, such that

$$\varlimsup_{n \to \infty} L(C_n) \leqslant l < +\infty ,$$

where $L(C_n)$ is the length of the image of C_n by $f(z)$ in the metric of the Riemann sphere. Then every point of $\{C: |z| = 1\}$ is either a point of continuity or a Picard for $f(z)$.

1. On the additional hypothesis that the endpoints of asymptotic paths, i.e. paths along which $f(z)$ approaches some limit, are dense on C, the set E of points of C near which $f(z)$ assumes all values with exactly two exceptions infinitely often is at most countable, and the set where two fixed points are omitted is finite. Is the additional hypothesis necessary?

2. Can the set of points E of problem 1 be in fact infinite?

3. How many distinct asymptotic values can a function of our class have at a given point of C, a) if f is regular, b) if f is meromorphic? In case a) it is not difficult to see that the set can be at most countable.

4. If ζ is a point of the set E of problem 1, can $f(z)$ have infinitely many segments of Julia at ζ, i.e. segments s lying except for one endpoint ζ in $|z| < 1$, and such that $f(z)$ assumes in any angle with vertex ζ and containing s all values with two exceptions infinitely often?

For the positive statements and background see the author's *The boundary behaviour of Tsuji functions*, to be published in the Michigan Mathematical Journal.

2. IX. 1966

Problèmes proposés par W. JANOWSKI (Łódź)

1. La detérmination du domaine de variabilité de la fonctionnelle:

(a)
$$F(f) = F\left(\zeta\frac{f'(\zeta)}{f(\zeta)}, \bar{\zeta}\overline{\frac{f'(\zeta)}{f(\zeta)}}\right),$$

(b)
$$F(f) = F\left(\zeta\frac{f''(\zeta)}{f'(\zeta)}, \bar{\zeta}\overline{\frac{f''(\zeta)}{f'(\zeta)}}\right),$$

(c)
$$F(f) = F\left(\frac{f''(\zeta)}{f'(\zeta)}, \frac{f'''(\zeta)}{f'(\zeta)}, \frac{f^{\mathrm{IV}}(\zeta)}{f'(\zeta)}\right).$$

Si $f(z) \in S_M$, S_M — la famille des fonctions $f(z)$, $f(0) = 0$, $f'(0) = 1$, holomorphes et univalentes dans le cercle $|z| < 1$ et bornées: $|f(z)| < M$, $M > 1$, ζ, $|\zeta| < 1$, — le point arbitraire fixé, $F(f, \bar{f}, f', \overline{f'}, \ldots, f^{\mathrm{IV}}, \overline{f^{\mathrm{IV}}})$ — la fonction analytique de variables complexes $f, \bar{f}, f', \overline{f'}, \ldots, f^{\mathrm{IV}}, \overline{f^{\mathrm{IV}}}$ dans le domaine

$$\{|f^{(k)}| < \infty, \ |\overline{f^{(k)}}| < \infty\} \quad \text{où} \quad k = 0, 1, \ldots, 4, \quad f^{(0)} = f, \quad \overline{f^{(0)}} = f.$$

2. La detérmination des bornes intérieure et superieure de la courbure de la représentation de la circonference $|z| = \varrho < 1$, à l'aide des fonctions des familles R^*, R^0 et $\tilde{R}$, où R^* — la famille des fonctions $w = f(z)$, $f(0) = 0$, $f'(0) = 1$ holomorphes et univalentes dans le cercle $|z| < 1$, étoillées par

rapport au point $w = 0$, dont les coefficients du developpement sont reélles; $R^0 \subset R^*$ — la famille des fonctions convèxes; $\tilde{R}$ — la famille des fonctions $f(z)$, $f(0) = 1$, $\mathrm{re} f(z) > 0$, holomorphes et univalentes dans le cercle $|z| < 1$, dont les coefficients sont reélles.

3. Si on considère la fonctionnelle

$$F(f) = F\big(f(\zeta)\,,\,\overline{f(\zeta)}\,,\,f'(\zeta),\,\ldots,\,\overline{f^{(n)}(\zeta)}\big) \qquad \text{où} \qquad w = f(z) \in S^*$$

($S^* \subset S$ — la famille des fonctions étoillés par rapport au point $w = 0$), on démontre que les fonctions frontales sont de la forme:

$$f^*(z) = z/\prod_{k=1}^{p} (1-a_k z)^{a_k} \tag{1}$$

où

$$a_k > 0\,, \quad a_0 + \ldots + a_p = 2\,, \quad |a_k| = 1\,,\ a_k \neq a_l \quad \text{si } k \neq 1 \text{ et } p \leqslant n+1.$$

Caracteriser les fonctionnelles $F(f)$, pour lesquelles les fonctions frontales sont de la forme (1), où $p = 1$ où $p = 2, \ldots, p = n+1$. La même question si $f(z) \in R^*$. Dans ce cas les fonctions frontales sont de la forme (1) et si $\mathrm{im}\, a_k \neq 0$, le produit $\prod_{k=1}^{p} (1-a_k z)^{a_k}$ contient aussi le facteur $(1-\bar{a}_k z)^{a_k}$.

2. IX. 1966

Problems proposed by J. Krzyż (Lublin)

1. *Coefficient problem for bounded non-vanishing functions* (J. Krzyż and M. O. Reade).

Let B be the class of functions $f(z) = \sum_{n=0}^{\infty} b_n(f) z^n$ regular and non-vanishing in the unit disk Δ and such that $|f(z)| \leqslant 1$ in Δ for each $f \in B$.

(a) Find B_n, $n = 3, 4, \ldots$, where $B_n = \sup_{f \in B} |b_n(f)|$.

(b) Find $\overline{\lim\limits_{n\to+\infty}}\, B_n$.

(c) Prove (or disprove) that $\overline{\lim\limits_{n\to+\infty}}\, B_n < 1$.

Remarks: for each individual $f \in B$ we have $\lim\limits_{n\to\infty} b_n(f) = 0$; $B_0 = 1$, $B_1 = B_2 = 2/e$, which is an unpublished result of J. Krzyż and M. O. Reade, also obtained by T. H. Mac Gregor.

2. *Isoperimetric defect.* Let $f(z)$ be a function univalent in the disk $|z| < R$, let Γ_r $(r = |z| < R)$ be the level-lines and let $L(r)$ be the length of Γ_r and $A(r)$ the area enclosed by Γ_r. Prove (or disprove) that $\delta(r) = L^2(r)/4\pi A(r)$ increases as r runs over $(0, R)$ and $f(z)$ is a convex function.

Remark: a weaker result (monotonic behaviour of $L^2(r) - 4\pi A(r)$) has been proved by M. Biernacki and J. Krzyż. The counterexample for the monotoneity of $L^2(r)/4\pi A(r)$ constructed by J. Krzyż and K. Radziszewski for the general case involves non-convex mappings.

3. *Univalent functions with fixed second coefficient.* Let $S(a)$ be the class of functions $f(z) = z + az^2 + \ldots$ with $0 \leqslant a < 2$ regular and univalent in the unit disk Δ.

(a) Determine the set $K(a) = \bigcap_{f \in S(a)} f(\Delta)$.

(b) Find $f \in S(a)$ yielding the minimum of the area of $f(\Delta)$.

Remark: the problem 3(b) is due to H. S. Shapiro.

4. *Biernacki's integral.* Let S be the class of univalent functions with the usual normalization.

Find the radius of univalence r_0 of

$$F(z) = \int_0^z \frac{f(\xi)}{\xi} d\xi, \quad f \in S.$$

Remark: $0.92 < r_0 < 0.98$ according to J. Krzyż and Z. Lewandowski.

5. *Circular symmetrization and subordination.* Let D and Δ be simply connected domains of hyperbolic type containing the origin and satisfying $D \subset \Delta$. Let D^* be the domain obtained by circular symmetrization of D w.r.t. the positive real axis. Let

$$f(z) = a_1 z + a_2 z^2 + \ldots,$$

$$f^*(z) = a_1^* z + a_2^* z^2 + \ldots,$$

$$F(z) = A_1 z + A_2 z^2 + \ldots$$

be univalent functions with the positive derivative at $z = 0$ mapping one-to-one the unit disk onto D, D^*, Δ, respectively. Put

$$M(r,f) = \sup_{0 \leqslant \theta \leqslant 2\pi} |f(re^{i\theta})|,$$

$$I_\lambda(r,f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\lambda d\theta, \quad \lambda > 0,$$

and suppose that $A(r,f)$ is the area enclosed by the level line $w = f(re^{i\theta})$. It is well-known that

(i) $a_1 \leqslant A_1$;

(i*) $a_1 \leqslant a_1^*$ [Pólya-Szegö];

(ii) $M(r,f) \leqslant M(r,F)$;

(ii*) $M(r,f) \leqslant M(r,f^*)$ [J. Krzyż];

(iii) $I_\lambda(r,f) \leqslant I_\lambda(r,F)$;
(iv) $A(r,f) \leqslant A(r,F)$.

Prove (or disprove) that

(iii*) $I_\lambda(r,f) \leqslant I_\lambda(r,f^*)$;
(iv*) $A(r,f) \leqslant A(r,f^*)$.

2. IX. 1966

Problems proposed by P. T. MOCANU (Cluj)

1. Let $f(z)$ be a regular function in a convex domain D, and z_1, z_2 two fixed points in D. Then there is a point ζ on the straight line segment $z_1 z_2$ and there is a complex number λ, $|\lambda| \leqslant 1$, such that

$$f(z_1) - f(z_2) = \lambda f'(\zeta)(z_1 - z_2)$$

(this is the well known Darboux formula).

If the function f is univalent in D, then $\lambda \neq 0$.

Let F be a compact class of functions regular and univalent in D. The problem is to find the minimum value of $|\lambda|$ for all $f \in F$.

2. Let $g(z)$ be a function regular in the unit disc and let r, $0 < r < 1$, be a fixed number. The problem is to find the best approximation of g by functions $f \in S$ in the disc $\{|z| \leqslant r\}$, i.e. to find

$$\mu(r) = \min_{f \in S} \max_{|z| \leqslant r} |f(z) - g(z)| .$$

$\mu(r) = 0$ will be a necessary and sufficient condition in order that $g \in S$.

2. IX. 1966

Problems proposed by CH. POMMERENKE (London)

Let E be a compact set in the complex plane and

$$\Delta_n(E) = \max_{w_1, \ldots, w_n \in E} \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^{n} \prod_{\nu=1}^{n} |w_\mu - w_\nu| .$$

Let $\operatorname{cap} E = 1$, that is, the transfinite diameter is normalized to 1.

1. Is it true that always

$$\Delta_n(E) \geqslant n^n ?$$

(Remark. This is true for the case of a continuum.)

2. Let E be a continuum. Is it true that

$$\Delta_n(E) \leqslant K^n n^n$$

for some constant K?

(Remark. Probably, this is false. It is known that $\Delta_n < (4e^{-1}\log n + + 4)^n n^n$.)

3. Let E be an analytic arc. Is there $\eta = \eta(E) > 0$ such that

$$\Delta_n(E) > (1+\eta)^n n^n \qquad (n > n_0(E))?$$

(Remark. This is false if E is a closed analytic curve, in which case $\Delta_n < K(E)n^n$.)

2. IX. 1966

Problems proposed by M. O. Reade (Ann Arbor, Mich.)

1. Given a real harmonic function $H(z)$, $|z| < 1$, subject to the conditions

(1) $$H(0) = 1\,,$$

(2) $$\lim_{\lambda \to 1} \int_0^{2\pi} |H(\lambda e^{i\theta})|\, d\theta = 2\pi M\,, \qquad M > 1\,.$$

Given $a > 0$, $1 < a < M$. Then find $R = R(a)$ such that

$$\int_0^{2\pi} |H(\lambda e^{i\theta})|\, d\theta \leqslant 2\pi a$$

holds for $0 \leqslant \lambda < R(a)$ (and holds for all such $H(z)$).

2. Given $f(z) = z + a_2 z^2 + \ldots$, univalent in the disc D: $|z| < 1$, with the property $\mathrm{re} f'(z) > 0$. Give a *geometric* characterization of the domain $f(D)$ similar to the geometric characterizations of convex, star and close-to-convex domains.

3. Given $f(z) = z + a_2 z^2 + \ldots$, with $\mathrm{re}\{zf'(z)/f(z)\} \geqslant \frac{1}{2}$ for $|z| < 1$. Characterize *geometrically* the domain $f(D)$.

4. Given a real functional $F(p(z))$ defined on the set P of Carathéodory functions $p(z) = 1 + p_1 z + \ldots$ in the disc D: $|z| < 1$. Let $S_n(z) = 1 + p_1 z + + \ldots + p_n z^n$, and let $z_0 \in D$. Is it true that

$$\operatorname*{Max}_{p \in P} F(S_n(z_0))$$

is attained for $S_n^*(z_0) = 1 + az_0 e^{i\theta} + az_0^2 e^{2i\theta} + \ldots + az_0^n e^{in\theta}$ for some real θ?

5. Let $f(z) = z + a_2 z^2 + \ldots$ be analytic for $|z| < 1$, $f'(z) \neq 0$ there, and such that

$$\lim_{\lambda \to 1} \int_0^{2\pi} \left| \mathrm{re}\left(1 + z\frac{f''(z)}{f'(z)}\right)\right| d\theta \leqslant k\pi\,, \qquad \text{where} \qquad k > 2\,.$$

Find the radius of univalence and the radius of starlikeness of $f(z)$. (Paatero, 1931-1933, has found the radius of convexity for $f(z)$.)

6. Find the domain of values for $f(z_1)/f(z_2)$, where $f(z)$ has the properties given in Problem 5 and where

$$0 < |z_1| < |z_2| < 1\ , \quad z_1, z_2 \text{ are fixed}\ .$$

7. Given $f(z) = z + a_0 + a_1/z + \ldots$, $|z| > 1$ with $\operatorname{re} f'(z) \geqslant 0$ for $|z| > 1$. Find the radius of univalence of $f(z)$.

2. IX. 1966

II. Seminar on quasiconformal mappings

Problems proposed by F. W. GEHRING (Ann Arbor, Mich.)

A. *Problems on plane quasiconformal mappings*

1. In a recent paper (Ann. Acad. Sci. Fenn. 388 (1966), pp. 1-15), F. W. Gehring and E. Reich have proved that there exist a constant a and a function $b(K)$, where $1 \leqslant a \leqslant 40$, $b(K) > 0$, and $b(K) = 1 + O(K-1)$ as $K \to 1$, such that

$$\frac{m(f(E))}{\pi} \leqslant b(K)\left(\frac{m(E)}{\pi}\right)^{K^{-a}} \tag{1}$$

for each K-quasiconformal mapping f of the unit disk B^2 onto itself with $f(0) = 0$ and each measurable set $E \subset B^2$. Show that (1) holds with $a = 1$ or produce a counterexample.

2. For each K, $1 \leqslant K < \infty$, let $p(K)$ be the upper bound of the numbers p for which all K-quasiconformal mappings f possess generalized L^p-derivatives. B. Bojarski has shown (Mat. Sbornik, 43 (1957), pp. 451-503) that $p(K) > 2$ while the simple example

$$f(z) = z\,|z|^{1/K-1}$$

shows that $p(K) \leqslant 2K/(K-1)$. Prove that $p(K) = 2K/(K-1)$ or give a counterexample.

3. Suppose that J is a non negative function defined almost everywhere in D. Try to characterize the functions J which are the Jacobians of quasiconformal mappings f of D. Next, given such a function J, what can one say about the extremal quasiconformal mappings f of D which have J as their Jacobian and have minimal maximal dilatation?

4. Suppose that f is a homeomorphism of a domain D. Then f is a K-quasiconformal mapping if

$$\operatorname{mod} f(Q) \leqslant K \operatorname{mod} Q \tag{2}$$

for all rectangles Q with $\bar{Q} \subset D$ (Comm. Math. Helv. 36 (1961), pp. 19-32) or if (2) holds for all quadrilaterals Q with $\mathrm{mod}\,Q = 1$ and $\bar{Q} \subset D$ (Ann. Acad. Sci. Fenn. 368 (1965), pp. 1-16). Is it true that f is K-quasiconformal if (2) holds for all squares Q with $\bar{Q} \subset D$, that is for all rectangles Q with $\mathrm{mod}\,Q = 1$?

B. *Problems on quasiconformal mappings in n-space R^n*

1. Suppose that f is a quasiconformal mapping of R^{n-1} onto R^{n-1} where $n > 3$. Show that f has a quasiconformal extension f^* of R^n onto R^n. Ahlfors has established this result for the case where $n = 3$ (Proc. Nat. Acad. Sci. USA 51 (1964), pp. 768-771).

2. Suppose that f is a quasiconformal mapping of R^n onto R^n with maximal dilatation $K(f) > 1$, and suppose that $n \geqslant 3$. Do there exist quasiconformal mappings f_1 and f_2 such that $f = f_1 \circ f_2$ and such that

$$K(f_1) = K(f_2) = K(f)^{1/2}?$$

What about the situation where f is a quasiconformal mapping of the unit ball B^n onto itself?

5. IX. 1966

Problems proposed by J. Krzyż (Lublin)

1. Let S_Q be the class of Q-quasiconformal mappings of the unit disc onto itself with $f(0) = 0$, $f(1) = 1$ for each $f \in S_Q$.

Given z_0 $(0 < |z_0| < 1)$ find the region $\Omega(z_0)$ of variability of $f(z_0)$ (z_0 being fixed and f running over S_Q).

2. Let T_Q be the class of Q-quasiconformal mappings of the closed unit ball $\bar{B}$ in R^3 onto itself of the form $y = f(x)$, $x \in \bar{B}$; $f(0) = 0$, $f(e_1) = e_1$, $f(e_2) = e_2$ $(e_1, e_2 \in \partial B)$.

Given x_0, e_1, e_2 find the region of variability of $f(x_0)$ for f running over T_Q (x_0, e_1, e_2 being fixed).

3. Find a parametric representation for the class T_Q defined above which would be an analogue of the representation found by Shah Tao-shing (Science Record, 1959).

5. IX. 1966

Problems proposed by J. Ławrynowicz (Łódź)

We consider the following classes of Q-quasiconformal mappings:

S_Q — of the closed unit disc onto itself, 0 being fixed,

S_Q^1 — of the closed unit disc onto itself, 0 and 1 being fixed,

$S_Q^{z_0}$ — of the closed unit disc onto itself, 0 and z_0 being fixed, $0 < |z_0| < 1$,

$S_Q^{r,R}$ — of an annulus $\{z\colon r \leqslant |z| \leqslant 1\}$ onto $\{w\colon R \leqslant |w| \leqslant 1\}$, 1 being fixed.

1. The following precise estimates are known for $f \in S_Q$ (see Wang Chuan-fang [8], and Lehto and Virtanen [6]):

$$4^{1-Q}|z|^Q \leqslant |f(z)| \leqslant 4^{1-1/Q}|z|^{1/Q}\,, \tag{1}$$

$$16^{1-Q}|z_1-z_2|^Q \leqslant |f(z_1)-f(z_2)| \leqslant 16^{1-1/Q}|z_1-z_2|^{1/Q}\,. \tag{2}$$

Obviously, they are best possible in S_Q^1, and can be improved in $S_Q^{z_0}$ and $S_Q^{r,R}$. Find the precise estimates of the type (1) and (2) in $S_Q^{z_0}$ and $S_Q^{r,R}$.

2. Suppose a continuous curve L emanating from $z_0 = \exp(i\theta_0)$ is contained in a Stolz angle $L(\alpha)$: $|\arg(1-z/z_0)| < \frac{1}{2}\pi-\alpha$, $0<\alpha<\frac{1}{2}\pi$. It is well known (Gavrilov [3], Agard [1]) that the image of L under $w = f(z)$, $f \in S_Q$, is contained in the Stolz angle $L(\beta)$, vertex at $w_0 = f(z_0)$, where

$$Q^{-1}\nu(\tan^2\tfrac{1}{4}\alpha) \leqslant \nu(\tan^2\tfrac{1}{4}\beta) \leqslant Q\nu(\tan^2\tfrac{1}{4}\alpha)\,. \tag{3}$$

Here, for $0<r<1$, $\nu(r)$ is the modulus of the unit disc slit along the real axis from 0 to r. Find the analogues of (3) in $S_Q^{z_0}$ and $S_Q^{r,R}$.

3. (i) Shah Tao-shing [7] has found the exact upper bound of

$$|f(z)-z|/\log Q\,, \tag{4}$$

taken over $|z| \leqslant 1$, $f \in S_Q^1$, $Q>1$, which is equal to $M = (1/4\pi^2)\{\Gamma(\frac{1}{4})\}^4$.

(ii) It is also known that $|f(z)-z| \leqslant \varrho(|z|, Q)-|z|$ in S_Q^1, where $\varrho(r, Q)$ satisfies

$$QK'(\varrho^*)/K(\varrho^*) = K'(r^*)/K(r^*)\,, \quad \varrho^* = (1+1/\varrho)^{-1/2}\,, \quad r^* = (1+1/r)^{-1/2}\,,$$

and K, K' are complete elliptic integrals (Belinskii [2]).

(iii) An estimate of (4) in $S_Q^{z_0}$, can be obtained in terms of $|z|$, $|z_0|$ and $|z-z_0|$, so that it tends to the precise estimate as $|z_0| \to 1-$ (Krzyż and Ławrynowicz [5]).

Find the upper bound of (4) as a function 1° of $|z|$ in S_Q^1, $Q>1$, 2° of $\arg z$ in S_Q^1, $Q>1$, 3° of $|z|$, $|z_0|$ and $|z-z_0|$ in $S_Q^{z_0}$, $Q>1$.

4. Krushkal [4] has given two methods of studying the functional (4) in $S_Q^{r,R}$; one of them can be applied to more general functionals. These methods are connected with a general variational method for quasiconformal mappings which is due to Belinskii [2]. The methods of Krushkal show that the exact upper bound $M(r, R)$ of (4), taken over $r \leqslant |z| \leqslant 1$, $f \in S_Q^{r,R}$, $Q>1$, and even $M(r) = \sup\limits_R M(r, R)$, must probably be considered as a new special function, because it is a solution of a differential equation connected with Weierstrass' $\wp$-function. It is also natural to study the analogous upper bounds $M(r, R, |z|)$ and $M(r, R, \arg z)$ taken over the lines $|z| = \text{const}$ and $\arg z = \text{const}$.

Study the properties of $M(r)$, $M(r, R)$, $M(r, R, |z|)$ and $M(r, R, \arg z)$.

5. Let $G(z_0, z_1, z_2; f) = \{f(z_1)-f(z_0)\}/\{f(z_2)-f(z_0)\}$. Belinskii [2] has proved that each of the expressions

$$\sup|G(z_0, z_1, z_2; f)| - \inf|G(z_0, z_1, z_2; f)|, \tag{5}$$

$$\sup|\arg G(z_0, z_1, z_2; f) - \arg(z_1-z_0) + \arg(z_2-z_0)| \tag{6}$$

is less than $M^*\varepsilon$, where $M^* = (8/\pi)\int_0^1 K(r^2)\,dr = (1/4\pi^2)\{\Gamma(\frac{1}{4})\}^4$, and the sup and inf are taken over $|z_0| \leqslant 1$, $|z_1| \leqslant 1$, $|z_2| \leqslant 1$, $|z_1-z_0| = |z_2-z_0|$, $f \in S_Q$, $1 \leqslant Q \leqslant 1+\varepsilon$. The constant M^* is the best possible.

Find analogous estimates of (5) and (6), i.e. of the form $M_1^*(r, R)\varepsilon$ and $M_2^*(r, R)\varepsilon$, respectively, in the case of $f \in S_Q^{r,R}$, $1 \leqslant Q \leqslant 1+\varepsilon$. My conjecture is that $M_1^*(r, R) = M_2^*(r, R) = M(r, R)$. It is probably an easier problem to prove or disprove this conjecture.

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5. IX. 1966

Problem proposed by Ch. Pommerenke (London)

(a) Let $f(z, t) = e^t z + a_2(t)z^2 + \ldots$ be analytic and univalent in $|z| < 1$ for each $t \in \langle 0, +\infty)$, and let

$$\{f(z, t)\colon |z| < 1\} \subset \{f(z, \tau)\colon |z| < 1\} \quad (0 \leqslant t < \tau < +\infty).$$

Then it is known that, for almost all t,

$$\frac{\partial}{\partial t} f(z, t) = zf'(z, t)p(z, t)$$

with $p(0, t) \equiv 1$, $\operatorname{re} p(z, t) > 0$; $|z| < 1$.

(b) Suppose now that

$$p(z, t) = \frac{1+\psi(z, t)}{1-\psi(z, t)}, \qquad |\psi(z, t)| \leqslant \varrho < 1$$

with fixed ϱ. It can be proved that $f(z, t)$ is then continuous in $|z| \leqslant 1$, and the function

$$w(z) = \begin{cases} f(z, 0) & \text{for} \quad |z| \leqslant 1\,, \\ f(e^{i\theta}, \log r) & \text{for} \quad z = re^{i\theta},\ r > 1\,, \end{cases}$$

is a $(1+\varrho)/(1-\varrho)$-quasiconformal mapping of the entire plane onto itself. Thus, $w(z)$ is a quasiconformal extension of $f(z, 0)$.

(c) Ahlfors gave necessary and sufficient conditions under which the function $f(z) = z + \ldots$, analytic and univalent in $|z| < 1$, can be quasiconformally extended onto the entire plane (not uniquely!).

Problem. Suppose that $f(z)$ can be quasiconformally extended to $|z| < \infty$ (see (c)). Is it possible to find a function $f(z, t)$ (see (a) and (b)) for which $f(z, 0) = f(z)$?

5. IX. 1966

Problems proposed by D. A. STORVICK (Minneapolis, Minn.)

1. The nature of those domains in 3-space which can be mapped quasiconformally onto the unit ball has been the object of study of many authors (cf. Gehring, Väisälä). It has been proved by Reade (Bull. Amer. Math. Soc. 63 (1957), p. 193) that if a simply connected domain of finite volume and connected complement can be mapped quasiconformally onto the unit ball, then necessarily there must correspond to those boundary points of the domain which are accessible by a finite path, a set of measure 4π on the unit sphere.

Conjecture: The exceptional set of measure zero in the above result is actually of logarithmic capacity zero. This is the case in the plane, where the result follows from Beurling's theorem concerning the conformal mapping of a simply connected domain of finite area onto the unit disk.

2. [This problem was posed to me originally by E. Calabi and has been studied by several others.]

Is it possible for two functions $f(z)$, $g(z)$ to be defined and analytic in the disk D: $|z| < 1$ and to possess the following three properties:

1) for each point $z \in D$; $|f'(z)| + |g'(z)| \neq 0$,

2) for each pair of points $\{z_1, z_2\} \subset D$,

$$|f(z_1) - f(z_2)| + |g(z_1) - g(z_2)| \neq 0\,,$$

3) for each ζ, $|\zeta| = 1$,

$$\lim_{z\to\zeta} \{|f(z)| + |g(z)|\} = \infty ?$$

5. IX. 1966

III. SEMINAR ON FUNCTIONS OF SEVERAL COMPLEX VARIABLES

Problem proposed by P. T. Mocanu (Cluj)

Let $f(z_1, z_2, \ldots, z_n)$, $f(0, 0, \ldots, 0) = 0$, be a regular function in the polydisc $\{|z_k| < r_k\}$, $k = 1, 2, \ldots, n$. The problem is to find necessary and sufficient conditions in order that the image of the polydisc $\{|z_k| < \varrho_k\}$ $\varrho_k < r_k$, $k = 1, 2, \ldots, n$, under the transformation $w = f(z_1, z_2, \ldots, z_n)$ be a convex domain (or a domain star-like with respect to the origin).

6. IX. 1966

Problèmes proposés par F. Norguet (Strasbourg)

1. Poursuivre l'extension entreprise dans [1], [2], [3] de la théorie des espaces de Stein aux espaces analytiques strictement q-pseudoconvexes. Exemple de problème non résolu: recherche d'espaces universels de plongement. Il serait utile d'étudier aussi certains espaces ayant un type de pseudoconvexité mixte: exemple: $V - W$, où V et W sont deux domaines d'holomorphie.

2. Démontrer, dans le cas de la d''-cohomologie, un théorème, analogue au théorème 1 de [6]; un tel théorème fournirait le développement d'une classe de cohomologie en série de Laurent.

3. Le théorème 2 de [6] admet pour généralisation une formule établie dans [7] qui exprime le comportement des homomorphismes résidus composés définis dans [4], [5], [8] vis-à-vis de la structure multiplicative de la cohomologie; cette formule, qui n'est pas une conséquence du théorème 2 de [6], généralise le lemme p. 135 de [4]; est-il possible de mieux comprendre son origine et sa signification en la rapprochant de formules analogues, connues éventuellement dans d'autres domaines des mathématiques: algèbre, topologie algébrique?

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6. IX. 1966

Problems proposed by J. SICIAK (Kraków)

Let E be a compact set in C^n. Put $L(z, E) = \sup\{|P_\nu(z)|: P_\nu(z)$ is an arbitrary polynomial of n complex variables of degree $\leqslant \nu$ $(\nu = 0, 1, \ldots)$ such that $|P_\nu(z)| \leqslant 1$ on $E\}$, $z = (z_1, \ldots, z_n) \in C^n$ (compare with [1], p. 335, formula (13)).

1. Characterize sets E for which $L(z, E) < \infty$ in C^n.

2. Characterize sets E for which $L^*(z, E) := \limsup_{z' \to z} L(z', E)$ is continuous in C^n.

3. Given an arbitrary system of $\nu_* = \binom{\nu+n}{n}$ points in C^n

$$p_i = (z_{1i}, \ldots, z_{ni}) \quad (i = 1, \ldots, \nu_*),$$

put

$$V(p_1, \ldots, p_{\nu_*}) := \det [z_{1i}^{k_{1l}} \ldots z_{ni}^{k_{nl}}] \quad (i, l = 1, \ldots, \nu_*),$$

$\{k_{1l}, \ldots, k_{nl}\}$ $(l = 1, \ldots, \nu_*)$ denoting the sequence of all solutions in non-negative integers of the inequality $k_1 + \ldots + k_n \leqslant \nu$. Let

$$V_\nu := \sup_{p_i \in E} |V(p_1, \ldots, p_{\nu_*})|, \quad v_\nu(E) := V^{1/n\binom{\nu+n}{n+1}}, \quad \nu = 1, 2, \ldots$$

It is known [2] that if $E = E_1 \times \ldots \times E_n$ is a Cartesian product of plane sets $E_1, \ldots, E_n$, then

$$v_\nu(E) \to [d(E_1) \ldots d(E_n)]^{1/n},$$

where $d(E_k)$ denotes the transfinite diameter of E_k.

Does $\{v_\nu(E)\}$ converge for every compact $E \subset C^n$?

4. Is it true that $L(z, E) < \infty$ in C^n iff $\liminf v_\nu(E) > 0$?

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6. IX. 1966

Probleme vorgeschlagene von W. Tutschke (Berlin)

Es sei $F = F(z_1, \dots, z_n)$ eine Komplexwertige Funktion von n Komplexen Veränderlichen $z_j = x_j + iy_j$, $j = 1, \dots, n$. Weiter sei $p(z_1, \dots, z_n)$ eine reelwertige, positive Funktion. Dann heißt F p-analytische Funktion, wenn F dem DGl-System

$$\partial F/\partial \bar{z}_j = \{(1-p)/(1+p)\}(\partial F/\partial z_j)$$

genügt. Die n Komplexen Ableitungen $\partial F/\partial \bar{z}_j$, $j = 1, \dots, n$, sind damit bereits durch die n Ableitungen $\partial F/\partial z_j$, $j = 1, \dots, n$, festgelegt. Integrierbarkeitsbedingungen für die Komplexen Ableitungen sind ein System von Gleichungen, deren vorgegebene Funktionen F_j genügen müssen, damit eine Funktion F existiert, für die $\partial F/\partial z_j = F_j$ ist. Für p-analytische Funktionen lauten diese Bedingungen so (die Bedingungen sind notwendig und bei geeigneten topologischen Voraussetzungen auch hinreichend):

$$\text{(I)} \qquad \frac{\partial}{\partial z_\lambda}\left(\frac{1}{1+p}F_\mu\right) = \frac{\partial}{\partial z_\mu}\left(\frac{1}{1+p}F_\lambda\right),$$

$$\text{(II)} \qquad \frac{\partial}{\partial \bar{z}_\lambda}\left(\frac{1}{1+p}F_\mu\right) = a_{\lambda\mu} + ib_{\lambda\mu}\,.$$

Die Bestimmung von $a_{\lambda\mu}$ $(= a_{\mu\lambda})$ und $b_{\lambda\mu}$ $(= -b_{\mu\lambda})$ ergibt dabei

$$\text{(III)} \qquad F_\lambda\left(\frac{\partial p}{\partial x_\mu} - \frac{\partial p}{\partial y_\mu}\right) = F_\mu\left(\frac{\partial p}{\partial x_\lambda} - \frac{\partial p}{\partial y_\lambda}\right).$$

(Math. Nachr., im Druck).

Bemerkung: im Fall holomorpher Funktionen ist $p \equiv 1$. Die Bedingung (III) wird dann gegenstandslos, (II) besagt, daß F selbst holomorph sein muß. (I) nimmt die Form

$$\partial F_\mu/\partial z_\lambda = \partial F_\lambda/\partial z_\mu$$

an.

Problem. Es sind die Komplexen Integrierbarkeitsbedingungen für den Fall anzugeben, daß F Lösung eines allgemeineren Systems der Form

$$\partial F/\partial \bar{z}_j = \Phi_j(z_1, \dots, z_n, F, \partial F/\partial z_1, \dots, \partial F/\partial z_n)\,, \qquad j = 1, \dots, n\,,$$

ist.

6. IX. 1966

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