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An ordinary differential equation homeomorphic with an essential control system

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The main result of this paper is the existence of the ordinary differential equation

$$x' = f(t, x) \tag{1}$$

with a family of solutions equivalent with respect to homeomorphisms $R^2 \to R^2$ (but obviously not with respect to diffeomorphisms) with the family of all trajectories of the (essential) control system

$$x' = u, \quad |u| \leqslant 1 \tag{2}$$

(Theorem T).

This problem and many ideas of the paper have been suggested by A. Pliś.

If $f(t, x)$ is a bounded function on R^2 uniformly Lipschitz-continuous in x, then the family of solutions for the ordinary differential equation (1) is homeomorphic with the family of parallel straight lines. Obviously if the solutions of the initial problem for (1) are non-unique (as is the case in our example) such a homeomorphism does not exist. In the general case it is difficult to find the topological structure of solutions. Therefore our construction of (1) is not straightforward.

Denote by $S(f)$ the family of all solutions of (1) and by L a family of Lipschitz-continuous functions mapping R in R with the Lipschitz constant equal to 1. The set L is identical with the family of all trajectories of the control system (2).

In this paper we give the following theorem:

THEOREM T. *There exist a continuous function f: $R^2 \to R$ and a homeomorphism h: $R^2 \to R^2$ such that the mapping h^*: $L \to S(f)$ induced by h (i.e. for $v \in L$, $h^*(v) = h(\{v\})$, $v \subset R^2$) is a bijection.*

Theorem T gives an example of an ordinary differential equation possessing distinct maximal and minimal integrals depending continuously on the initial points. The straight lines $x = t + \text{const}$ are mapped by h^*

in the maximal integrals of (1) and the straight lines $x = -t + \text{const}$ in the minimal integrals of (1). Theorem T therefore answers the problem (it is not known to me who was the first to pose the problem) whether the maximal integral can depend continuously on the initial points in the non-trivial case of an ordinary differential equation without the uniqueness property.

The function f constructed in this paper also furnishes such an example: every point of R^2 is a point of non-uniqueness of the solutions of (1); other functions with this property were constructed for instance in [1], [2].

Proof of Theorem. T. Let $r\colon [0, \infty) \to [0, \infty)$ be a continuous increasing function satisfying the following conditions:

(3) $r(0) = 0, \quad \forall t, s \geqslant 0 \ |r(t) - r(s)| \leqslant r(|t-s|), \quad \lim r(t) = \infty \ \text{ as } t \to \infty.$

In constructing of the function f the following well-known lemma will play an essential role.

Lemma 1. *Let A be a compact and non-empty subset of a metric space (X, ϱ). If the function $v\colon A \to R$ satisfies condition*

$$|v(x) - v(y)| \leqslant r\big(\varrho(x, y)\big), \quad \forall x, y \in A, \tag{4}$$

then the functions v^* and v_* given by the formulas

$$v^*(x) = \min\{v(y) + r\big(\varrho(x, y)\big);\ y \in A\}, \tag{5}$$

$$v_*(x) = \max\{v(y) - r\big(\varrho(x, y)\big);\ y \in A\} \tag{6}$$

provide extensions of the function v on the whole space X, with condition (4) *satisfied on X and every extension w of v which satisfies* (4) *on X fulfilling the inequalities*

$$v_* \leqslant w \leqslant v^* \quad \text{on } X. \tag{7}$$

Lemma 2. *Let B be the union of finite number curves on a common compact interval I and let for $t \in I$*

$$B^t = \{x \in R;\ (t, x) \in B\}. \tag{8}$$

Let f be continuous on B and $v(x) = f(t, x)$ $(t \in I)$ satisfy (4); *then the functions*

$$f^*(t, x) = \min\{f(t, y) + r(|x-y|);\ y \in B^t\}, \tag{9}$$

$$f_*(t, x) = \max\{f(t, y) - r(|x-y|);\ y \in B^t\} \tag{10}$$

are continuous on $I \times R$.

Proof. Lemma 1 implies that the functions f^* and f_* are continuous in x uniformly with respect to t. Consider arbitrarily fixed numbers t, s, x. Let $a \in B^t$, $c \in B^s$ be such points that

$$f(t, a) + r(|x-a|) = \min\{f(t, y) + r(|x-y|);\ y \in B^t\},$$

$$f(s, c) + r(|x-c|) = \min\{f(s, y) + r(|x-y|);\ y \in B^s\}.$$

From the definition

$$f^*(t, x)-f^*(s, x)$$
$$= \min\{f(t, y)+r(|x-y|); y \in B^t\}-\min\{f(s, y)+r(|x-y|); y \in B^s\}$$

it follows for any $y \in B^t$ and $z \in B^s$ that

$$f(t, a)+r(|x-a|)-f(s, z)-r(|x-z|) \leqslant f^*(t, x)-f^*(s, x)$$
$$\leqslant f(t, y)+r(|x-y|)-f(s, c)-r(|x-c|);$$

hence

$$f(t, a)-f(s, z)-r(|z-a|) \leqslant f^*(t, x)-f^*(s, x) \leqslant f(t, y)-f(s, c)+r(|y-c|). \tag{11}$$

Let ε be an arbitrarily fixed positive number. The function f being continuous on B and r on $[0, \infty)$, there exists such a positive number δ that if $|\tau-\tau'|$, $|u-u'| < \delta$, (τ, u), $(\tau', u') \in B$, then $|f(\tau, u)-f(\tau', u')| < \frac{1}{2}\varepsilon$ and $r(|u-u'|) < \frac{1}{2}\varepsilon$. For a given t there exists such a positive number $\eta < \delta$ that if $|t-s| < \eta$, then $|k(t)-k(s)| < \delta$ for any curve $k \subset B$. Hence, putting $a = k(t)$, $c = l(s)$, we get $|k(t)-k(s)| < \delta$ and $|l(t)-l(s)| < \delta$ for $|t-s| < \eta$. For $z = k(s)$, $y = l(t)$ inequality (11) implies

$$|f^*(t, x)-f^*(s, x)| < \varepsilon, \tag{12}$$

i.e. f^* depends continuously on t and therefore, in virtue of the uniform continuity in x, it is continuous on $I \times R$. The continuity of f_* is proved similarly.

Remark 1. If outside any neighbourhood of $x = 0$ the function r is uniformly Lipschitz-continuous, then outside any neighbourhood of B the functions (9), (10) are uniformly Lipschitz-continuous in x.

We put $r(s) = s^{1/2}$. This function satisfies conditions (3). Consider curves $x_0^\sim$, $x_1^\sim$, $y_0^\sim$, $y_1^\sim$ defined as follows:

$$x_0^\sim(t) = \begin{cases} \dfrac{(t-2k\sqrt{2})^2}{4}+k, & 2k\sqrt{2} \leqslant t \leqslant (2k+1)\sqrt{2}, \\ & k = 0, \pm 1, \pm 2, \ldots, \\ -\dfrac{(t-2k\sqrt{2})^2}{4}+k, & (2k-1)\sqrt{2} \leqslant t \leqslant 2k\sqrt{2}, \end{cases} \tag{13}$$

$$\begin{matrix} x_1^\sim(t) = x_0^\sim(t-1), \\ y_0^\sim(t) \equiv 0,\ y_1^\sim(t) \equiv 1, \end{matrix} \qquad t \in J = [0, 2\sqrt{2}+1]. \tag{14}$$

Denote by $K' \subset R^2$ the closed set bounded by the curves $x_0^\sim$, $x_1^\sim$, $y_0^\sim$, $y_1^\sim$ and $K = K' \cup y_0^\sim \cup y_1^\sim$, $\partial K = K \setminus \operatorname{int} K$,

$$x_i = x_i^\sim \cap K', \quad y_i = y_i^\sim \cap K' \quad (i = 0, 1). \tag{15}$$

LEMMA 3. *Let the functions h and g and sets $H, G \subset K$ possess the following properties (properties (16)-(24)):*

(16) *h, g are continuous and satisfy (4) (for $v(x) = h(t,x)$ and $v(x) = g(t,x)$) on K.*

(17) *Outside any neighbourhood of H the function h is uniformly Lipschitz-continuous in x.*

(18) $$h(t,x) \leqslant \max\{h(t,y) - r(|x-y|);\, y \in G^t\}.$$

(19) *The set H is such a union of a finite family of solutions minimal to the right* [1] *and simultaneously maximal to the left for the equation*

(19a) $$x' = h(t,x),$$

and the set G is such a union of a finite family of solutions maximal to the right and simultaneously minimal to the left for the equation

(19b) $$x' = g(t,x),$$

that there exists a homeomorphism of the square $\{0 \leqslant x, y \leqslant 1\}$ onto K' mapping a finite number of segments $\{x = \text{const}, 0 \leqslant y \leqslant 1\}$ on the solutions contained in G and a finite number of segments $\{0 \leqslant x \leqslant 1, y = \text{const}\}$ on the solutions contained in H. The segment $x = j$ is mapped on x_j and the segment $y = j$ is mapped on y_j $(j = 0, 1)$ $(x_j \subset G, y_j \subset H)$.

If $l \subset G$ (or H) and l is defined in $[a, b] \subset J$, we define:

$$l^{\sigma} = \begin{cases} l & \text{in } [a,b], \\ 0 \text{ (or } x_0) & \text{in } [0,a], \\ 1 \text{ (or } x_1) & \text{in } [b, 2\sqrt{2}+1]. \end{cases}$$

(20) *If l, m are solutions from G (from H) not separated by any third solution from G (from H), then*

$$|l^{\sigma}(t) - m^{\sigma}(t)| < a,$$

where a is a positive number.

(21) *g is the extention of $h|(H \cup \partial K)$ (the restriction of h to $H \cup \partial K$) on the set K given by (9) for $B = H \cup \partial K$.*

(22) *If l, φ are arbitrary solutions, $l \subset G, \varphi \subset H$ and s is a point such that $l(s) = \varphi(s)$, then there exists a $\delta > 0$ such that the inequality $|t-s| < \delta$ implies*

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[1] A solution $y(t)$ of (1) defined on the interval $a \leqslant t \leqslant b$ is called a *solution of* (1) *minimal to the right* if for any solution $x(t)$ of (1) defined for $c \leqslant t \leqslant d$, where $a \leqslant c \leqslant b$, we have $x(t) \geqslant y(t)$ for $c \leqslant t \leqslant \min(b, d)$. Solutions minimal to the left and maximal to the right or left are defined in an analogous manner.

$$g\big(t, l(t)\big) = h\big(t, \varphi(t)\big) + r\big(|\varphi(t) - l(t)|\big). \tag{23}$$

$$h(t, x) = g(t, x) \Leftrightarrow (t, x) \in H \cup G \cup \partial K. \tag{24}$$

Under these assumptions we have

(i) *For an arbitrary number* a^*, $0 < a^* \leqslant 1$, *such continuous functions* h^*, g^* *and sets* $H^*, G^* \subset K'$ *can be defined that conditions* (16)-(24) *are satisfied, where* a, h, g, H, G *are replaced by* a^*, h^*, g^*, H^*, G^* *and* $H \subset H^*$, $G \subset G^*$.

(ii) *For every solution* $\lambda \subset G^*(H^*)$ *and any two solutions* ν, μ *from* $H^*(G^*)$ *not separated by any third solution from* $H^*(G^*)$, *if* $\nu(t) \leqslant \lambda(t) \leqslant \mu(t)$ *on an interval* I, *then the length of* I *is smaller than* $9r(a^*)$.

Proof. Consider the extension $h^*: K \to R$ of the function $g|(G \cup \partial K)$ on the whole set K given by formula (10) for $B = G \cup \partial K$ and consider the equation

$$x' = h^*(t, x). \tag{25}$$

Property 1. $h^* \geqslant h$.

Proof. Since $h^*(t, x) = \max\{g(t, y) - r(|x-y|);\ y \in (G \cup \partial K)^t\}$, inequality (18) and $g \geqslant h$ imply $h^* \geqslant h$.

Property 2. *If* $b \subset H$ *is a solution of* (19a), *then* b *is a solution of* (25).

Proof. The equality $b'(t) = h\big(t, b(t)\big)$ and $b \subset H$ implies $b'(t) = g\big(t, b(t)\big)$. Hence the inequalities

$$h(t, x) \leqslant h^*(t, x) \leqslant g(t, x), \qquad \forall (t, x) \in K$$

imply $b'(t) = h^*\big(t, b(t)\big)$.

Property 3. $h^*(t, x) = \max\{g(t, y) - r(|x-y|);\ y \in G^t\}$.

Proof.

$$g\big(t, y_i^{\sim}(t)\big) - r\big(|x - y_i^{\sim}(t)|\big) = h\big(t, y_i^{\sim}(t)\big) - r\big(|x - y_i^{\sim}(t)|\big) \leqslant h(t, x)$$

and from (18) there exists such a $y \in G^t$ that

$$h(t, x) \leqslant h(t, y) - r(|x-y|).$$

Hence, in virtue of the inequality $h \leqslant g$,

$$g\big(t, y_i^{\sim}(t)\big) - r\big(|x - y_i^{\sim}(t)|\big) \leqslant \max\{g(t, y) - r(|x-y|);\ y \in G^t\}.$$

Remark 2. Obviously all the solutions of (19b) contained in G are solutions of (25) and, since $g \geqslant h^*$, they are maximal to the right and also minimal to the left.

Property 4. *The family of all solutions* φ *of equation* (25) (*on* K') *minimal to the right is homeomorphic with the family of all segments parallel* *to the x-axis and contained in the square $\{0 \leqslant x, y \leqslant 1\}$ (i.e. there exists a homeomorphism from K' to the square, mapping solutions onto segments), and each solution minimal to the right is maximal to the left.*

Proof. Let l be any solution, $l \subset G$, and suppose at first that $l \neq x_i$ $(i = 0, 1)$; let φ be any solution, $\varphi \subset H$, and suppose at first that $\varphi \neq y_i$. Let s be such a point that $l(s) = \varphi(s)$. It follows from (19) that $l(t) > \varphi(t)$ for $t > s$ and $l(t) < \varphi(t)$ for $t < s$. In virtue of (23) there exists a positive number δ such that $\delta \leqslant \operatorname{dist}(l, G \setminus l)$ and that (23) is satisfied for $|t-s| < \delta$. We shall prove that if $|t-s| < \delta$ and x is between $l(t)$ and $\varphi(t)$, then

$$h^*(t, x) = g\big(t, l(t)\big) - r\big(|x - l(t)|\big). \tag{26}$$

Suppose that $k(t) \leqslant \varphi(t) \leqslant x \leqslant l(t)$ or $l(t) \leqslant x \leqslant \varphi(t) \leqslant k(t)$, $k \subset G$. We have $g\big(t, k(t)\big) \leqslant h\big(t, \varphi(t)\big) + r\big(|k(t) - \varphi(t)|\big)$. Hence

$$\begin{aligned} g\big(t, k(t)\big) - r\big(|x - k(t)|\big) &\leqslant h\big(t, \varphi(t)\big) + r\big(|k(t) - \varphi(t)|\big) - r\big(|x - k(t)|\big) \\ &\leqslant h\big(t, \varphi(t)\big) \leqslant h\big(t, \varphi(t)\big) + r\big(|l(t) - \varphi(t)|\big) - r\big(|x - l(t)|\big) \\ &= g\big(t, l(t)\big) - r\big(|x - l(t)|\big). \end{aligned}$$

Now suppose that $\varphi(t) \leqslant x \leqslant l(t) \leqslant k(t)$ or $k(t) \leqslant l(t) \leqslant x \leqslant \varphi(t)$. Then the properties of the function $r(s) = s^{1/2}$ imply

$$r\big(|k(t) - \varphi(t)|\big) + r\big(|x - l(t)|\big) \leqslant r\big(|k(t) - x|\big) + r\big(|l(t) - \varphi(t)|\big),$$

i.e.

$$\begin{aligned} h\big(t, \varphi(t)\big) &+ r\big(|k(t) - \varphi(t)|\big) - r\big(|k(t) - x|\big) \\ &\leqslant h\big(t, \varphi(t)\big) + r\big(|l(t) - \varphi(t)|\big) - r\big(|x - l(t)|\big), \end{aligned}$$

i.e.

$$g\big(t, k(t)\big) - r\big(|k(t) - x|\big) \leqslant g\big(t, l(t)\big) - r\big(|x - l(t)|\big). \tag{27}$$

Therefore $\max\{g(t, y) - r(|y - x|);\ y \in G^t\} = g\big(t, l(t)\big) - r\big(|x - l(t)|\big)$ for $|t-s| < \delta$, where x is between $l(t)$ and $\varphi(t)$.

If $l = x_i$ or $\varphi = y_i$, then inequality (27) holds for x between l and φ either for $0 \leqslant t-s < \delta$ or for $0 \leqslant s-t < \delta$.

If a and b are two distinct solutions contained in H not separated by any third solution from H, $a^\sigma \leqslant b^\sigma$ and q, s are numbers such that $a(q) = l(q)$ and $b(s) = l(s)$, then for any $t \in (q, s)$ there exists an $\varepsilon(t) > 0$ such that the inequality $|x - l(t)| < \varepsilon(t)$ implies $h^*(t, x) = g\big(t, l(t)\big) - r\big(|x - l(t)|\big)$. Indeed, we have $g\big(t, l(t)\big) = g(t, c) + r\big(|c - l(t)|\big)$ for a fixed $t \in (q, s)$ and for a certain $c \in H$. The inequalities

$$\begin{aligned} g(t, z) - r\big(|l(t) - z|\big) &\leqslant g(t, c) + r(|c - z|) - r\big(|l(t) - z|\big) \\ &< g(t, c) + r\big(|l(t) - c|\big) = g\big(t, l(t)\big) \end{aligned}$$

are satisfied for all $z \in G^t \setminus \{l(t)\}$; therefore the inequality

$$g(t, z) - r(|x - z|) < g\big(t, l(t)\big) - r\big(|x - l(t)|\big) \tag{28}$$

is satisfied for every $z \in G^t \setminus \{l(t)\}$ and $x = l(t)$. The functions appearing in the above inequality are continuous in x and the set G^t is finite; therefore such a function $\varepsilon = \varepsilon(t)$ can be chosen that inequality (28) is satisfied for $|x - l(t)| < \varepsilon(t)$. Moreover, $\varepsilon(t)$ can be chosen so as to be continuous because the functions appearing in (28) are.

Now let f be an arbitrary solution of (25) minimal to the right, $d(t) = f(t) - l(t)$ and $d(c) = 0$ for a number c. We have $d'(t) = f'(t) - l'(t)$. If $c \in (q, s)$ in virtue of the equality $f'(t) = h^*(t, f(t))$ the inequality $|f(t) - l(t)| < \varepsilon(t)$ on a neighbourhood N of c implies

$$f'(t) = g(t, l(t)) - r(|f(t) - l(t)|) = l'(t) - r(|f(t) - l(t)|)$$

or

$$d'(t) = -r(|d(t)|),$$

i.e.

$$f(t) = l(t) \pm \frac{(t-c)^2}{4} \tag{29}$$

on N; i.e., the solution f is maximal to the left on N and, simultaneously, is (locally) the unique solution of (25) crossing l at $t = c$. Condition (26) implies that if $f(t)$ is between $l(t)$ and $a(t)$ on the interval $[q, q+\delta)$ (or between $l(t)$ and $b(t)$ on the interval $(s-\delta, s]$), we have (29) and therefore f is maximal to the left. The solutions a, b being minimal to the right and maximal to the left (Properties 1 and 2), if $f(q) = a(q)$ (or $f(s) = b(s)$), then $f(t) = a(t)$ on the interval $[q, q+\delta)$ (or $f(t) = b(t)$ on the interval $(s-\delta, s]$), i.e., a is a (locally) unique solution crossing l at $t = q$ (analogously b is a locally unique solution crossing l at $t = s$). Now Property 4 of solutions of H is true because the function h^* is Lipschitz-continuous in x outside the set G.

PROPERTY 5. $h^*(t, x) = g(t, x) \Leftrightarrow (t, x) \in H \cup G \cup \partial K$.

Proof. In virtue of (21), the definition of function h^* and Property 3 it follows that there exists a $y \in (H \cup \partial K)^t$ such that $g(t, x) = g(t, y) + r(|x - y|)$ and a $u \in G^t$ such that $h^*(t, x) = g(t, u) - r(|x - u|)$. Therefore $h^*(t, x) - g(t, x) = g(t, u) - g(t, y) - (r(|x-u|) + r(|x-y|)) \leqslant r(|u-y|) - (r(|x-u|) + r(|x-y|)) < 0$ for $x \notin (H \cup G \cup \partial K)^t$. Now suppose that $x \in H^t$. Property 2 implies that $g(t, x) = h^*(t, x)$. If $x \in (G \cup \partial K)^t$, then the definition of h^* implies $h^*(t, x) = g(t, x)$. Thus the proof of Property 5 is complete.

Let a^* be an arbitrary positive number and let $\delta > 0$ be such a number that if c is an arbitrary solution contained in G and d an arbitrary solution of (25), then the inequality $|c(t) - d(t)| \leqslant \delta$ and the equality $c(s) = d(s)$ imply the inequality $|t - s| \leqslant a^*$. Write

$$G_\delta = \{(t, x) \in K; \exists (t, y) \in G, |x - y| \leqslant \delta\}.$$

On the set $K\setminus G_{\frac{1}{2}\delta}$ the function h^* is uniformly Lipschitz-continuous with the Lipschitz-constant $M = M(\delta)$; therefore there exists a constant q such that if α and β are arbitrary solutions of (25) contained in $K\setminus G_{\frac{1}{2}\delta}$, then

$$\frac{\max(\alpha-\beta)}{\min(\alpha-\beta)} < q.$$

Consider the continuous function

$$f(t, x, z) = h^*(t, z)+r(|x-z|)-h^*(t, x).$$

The sets

$$A = \{(t, x, z);\ (t, x)\in \overline{K\setminus G_\delta},\ (t, z)\in G_{\frac{1}{2}\delta}\},$$

$$B = \left\{(t, x, z);\ (t, x)\in \overline{K\setminus G_\delta},\ (t, z)\in \overline{K\setminus G_{\frac{1}{2}\delta}},\ |x-z| \geqslant \frac{1}{8M^2}\right\}$$

are compact, and therefore we can consider the numbers

$$v_1 = \min\{f(u);\ u\in A\}, \qquad v_2 = \min\{f(u);\ u\in B\}.$$

For every t, x there exists such a number $y\in G^t$ that $h^*(t, x) = h^*(t, y)-r(|x-y|)$. Hence

$$f(t, x, z) = h^*(t, z)-h^*(t, y)+r(|x-z|)+r(|x-y|)$$
$$\geqslant r(|x-y|)+r(|x-z|)-r(|y-z|);$$

therefore $f(t, x, z) > 0$ for $x\notin G^t$ and $x \neq z$, i.e. $v_1, v_2 > 0$. Write

(30) $$P = \min\left\{a^*, \frac{v_1^2}{4}, \frac{v_2^2}{4}, \frac{1}{8M^2}, \frac{a^{*2}}{4q}, \frac{[r(\frac{2}{3})-r(\frac{1}{3})]^2}{M^2}, \frac{[r(\frac{1}{3})-\frac{1}{2}]^2}{M^2}\right\}.$$

We define the set H^* in the following manner:

$$H^* = H\cup H',$$

where H' is a set of points belonging to a finite number of solutions of (25) minimal to the right (and maximal to the left). We choose them in such a way that if α, β are solutions of H^* not separated by any third solution of H^*, then

$$\max|\alpha^\sigma(t)-\beta^\sigma(t)| \leqslant P,$$

and if, for any solution b contained in G, $\alpha(t) < b(t) < \beta(t)$ on an interval I, then I is shorter than a^*.

The extention g^* of the function $h^*|(H^*\cup\partial K)$ can be defined on the whole set K by formula (9). We shall consider the equation

(31) $$x' = g^*(t, x).$$

For this equation properties analogous to Properties 1-5 can be obtained:

PROPERTY 1′. $g^* \leqslant g$.

PROPERTY 2′. *If $b \subset G$ is a solution of* (19b), *then b is a solution of* (31).

PROPERTY 3′. $g^*(t, x) = \min\{h^*(t, y)+r(|x-y|);\ y \in (H^* \cup y_0^{\sim} \cup y_1^{\sim})^t\}$.

PROPERTY 4′. *The family of all solutions φ of equation* (31) *maximal to the right is homeomorphic with the family of all segments parallel to the x-axis and contained in the square $\{0 \leqslant x, y \leqslant 1\}$, and each solution maximal to the right is minimal to the left.*

PROPERTY 5′. $g^*(t, x) = h^*(t, x) \Leftrightarrow (t, x) \in G \cup H^* \dot{\cup} \partial K$.

We define the set G^* in the following manner:

$$G^* = G \cup G',$$

where G' is a set of points belonging to a finite number of solutions of (31) maximal to the right (and minimal to the left simultaneously). We choose them in such a way that if α, β are solutions of G^* not separated by any third solution of G, then $\max|\alpha^\sigma(t)-\beta^\sigma(t)| < a^*$ and if, for any solution b contained in H^*, $\alpha(t) < b(t) < \beta(t)$ on an interval I, then I is shorter than a^*.

PROPERTY 6. *If c_1, c_2 are two solutions from H^* not separated by any third solution from H^* and, for a certain t, $c_i(t) \notin G_\delta^t$ $(i = 1, 2)$, then for $x \in [c_1(t), c_2(t)]$ the following formula is true:*

$$g^*(t, x) = \min\{h^*(t, c_1(t))+r(|x-c_1(t)|),\ h^*(t, c_2(t))+r(|x-c_2(t)|)\}. \tag{32}$$

Moreover, if for a number x we have $c_1(t) \leqslant x \leqslant \frac{1}{3}(c_2(t)-c_1(t))$, then

$$g^*(t, x) = h^*(t, c_1(t))+r(|x-c_1(t)|), \tag{33}$$

and if $c_2(t)-\frac{1}{3}(c_2(t)-c_1(t)) \leqslant x \leqslant c_2(t)$, then

$$g^*(t, x) = h^*(t, c_2(t))+r(|x-c_2(t)|). \tag{34}$$

Proof. The definitions of the function h^* and the set H^* imply that for $x \in [c_1(t), c_2(t)]$

$$h^*(t, c_i(t))+r(|x-c_i(t)|)-h^*(t, x) \leqslant 2r(|x-c_i(t)|) \leqslant 2r(P) \qquad (i = 1, 2).$$

We define the functions

$$f_i(t, x, z) = h^*(t, z)+r(|x-z|)-h^*(t, c_i(t))-r(|x-c_i(t)|) \quad (i = 1, 2).$$

The definition of $f(t, x, z)$ implies

$$f_i(t, x, z) = f(t, x, z)-[h^*(t, c_i(t))+r(|x-c_i(t)|)-h^*(t, x)].$$

If $x \in [c_1(t), c_2(t)]$, $(t, z) \in G_{\frac{1}{2}\delta}$ we have $(t, x, z) \in A$ and therefore, in virtue of (30),

$$f_i(t, x, z) \geqslant v_1-2r(P) \geqslant 0. \tag{35}$$

If $(t, z) \notin G_{1\delta}$ and $|x-z| \geqslant 1/8M^2$, we have $(t, x, z) \in B$ and therefore

$$f_i(t, x, z) \geqslant v_2 - 2r(P) \geqslant 0. \tag{36}$$

Now suppose that $\operatorname{dist}(z, [c_1(t), c_2(t)]) < 1/8M^2$. Then

$$f_i(t, x, z) = r(|x-z|) - r(|x-c_i(t)|) + h^*(t, z) - h^*(t, c_i(t)) \tag{37}$$

$$\geqslant |c_i(t)-z| \frac{1}{2r(|x-\xi_i|)} - M|c_i(t)-z| = |c_i(t)-z| \left(\frac{1}{2r(|x-\xi_i|)} - M\right) \geqslant 0$$

for $i = 1$ if $z < c_1(t)$ and for $i = 2$ if $z > c_2(t)$, where ξ_i is between $c_i(t)$ and z. Therefore, from (35), (36) and (37) it follows that, for $z \in H^*$ and $x \in [c_1(t), c_2(t)]$,

$$f_1(t, x, z) \geqslant 0 \quad \text{or} \quad f_2(t, x, z) \geqslant 0. \tag{38}$$

This means that formula (32) holds true. To prove (33) and (34) let us fix x, $c_1(t) \leqslant x \leqslant c_1(t) + \frac{1}{3}(c_2(t) - c_1(t))$. Then

$$\begin{aligned} & h^*(t, c_2(t)) + r(|c_2(t)-x|) - h^*(t, c_1(t)) - r(|x-c_1(t)|) \\ & \quad \geqslant h^*(t, c_2(t)) - h^*(t, c_1(t)) + r\left(\tfrac{2}{3}(c_2(t)-c_1(t))\right) - r\left(\tfrac{1}{3}(c_2(t)-c_1(t))\right) \\ & \quad \geqslant r(c_2(t)-c_1(t)) \left(r(\tfrac{2}{3}) - r(\tfrac{1}{3})\right) - M(c_2(t)-c_1(t)) \\ & \quad = r(c_2(t)-c_1(t)) \left[r(\tfrac{2}{3}) - r(\tfrac{1}{3}) - Mr(c_2(t)-c_1(t))\right] > 0. \end{aligned}$$

Similarly, for x, $c_1(t) + \frac{2}{3}(c_2(t)-c_1(t)) \leqslant x \leqslant c_2(t)$, we have

$$h^*(t, c_1(t)) + r(|x-c_1(t)|) - h^*(t, c_2(t)) - r(|c_2(t)-x|) > 0.$$

Thus the proof of Property 6 is complete.

Now (i) in Lemma 3 is an immediate consequence of Properties 1-6. To prove (ii) let $\lambda \subset G^*$ and $\alpha, \beta \subset G$ be such solutions not separated by any third solution from G that $\alpha(t) \leqslant \lambda(t) \leqslant \beta(t)$. Let $I = [\tau, \sigma]$. From the definitions of H^* and the number δ it follows that

$$\alpha(t) + \delta \leqslant \nu(t) < \mu(t) \leqslant \beta(t) - \delta \tag{39}$$

on the interval $I^{\sim} = [\tau + 2a^*, \sigma - 2a^*]$. We are now going to evaluate the length of $I^{\sim}$ provided $I^{\sim}$ is non-empty. Let us divide $I^{\sim}$ into three sets, I_1, I_2, I_3:

$$I_1 = \{t \in I^{\sim};\ \mu(t) - \lambda(t) \leqslant \tfrac{1}{3}(\mu(t) - \nu(t))\}, \tag{40}$$

$$I_2 = \{t \in I^{\sim};\ \nu(t) + \tfrac{1}{3}(\mu(t)-\nu(t)) \leqslant \lambda(t) \leqslant \mu(t) - \tfrac{1}{3}(\mu(t)-\nu(t))\}, \tag{41}$$

$$I_3 = \{t \in I^{\sim};\ \lambda(t) - \nu(t) \leqslant \tfrac{1}{3}(\mu(t) - \nu(t))\}. \tag{42}$$

Suppose that I_1 is non-empty and $t \in I_1$. From Property 6

$$\begin{aligned} \lambda'(t) &= g^*(t, \lambda(t)) = h^*(t, \mu(t)) + r(\mu(t) - \lambda(t)) \\ &= \mu'(t) + r(\mu(t) - \lambda(t)). \end{aligned} \tag{43}$$

If, for a number $s \in I_1$, $\lambda(s) = \mu(s) - \frac{1}{3}(\mu(s) - \nu(s))$, then (from (43)) $\lambda'(s) > \mu'(s) - \frac{1}{3}(\mu'(s) - \nu'(s))$; therefore the set I_1 is an interval and $\sigma - 2a^* \in I_1$. In an analogous way we find that the set I_3 is an interval and $\tau + 2a^* \in I_3$ or I_3 is empty. Hence I_2 is also an interval. From (43) we have, for $t \in I_1$, $\mu(t) - \lambda(t) = \frac{1}{4}(t - c)^2$ for a certain c. The inequality in (40) implies $(t-c)^2 \leqslant \frac{4}{3}(\mu(t) - \nu(t))$, i.e., $|t - c| \leqslant 2r(\frac{1}{3}a^*)$. Hence

$$|I_1| \leqslant 2r(\tfrac{1}{3}a^*). \tag{44}$$

In an analogous manner the length of the interval I_3 can be evaluated:

$$|I_3| \leqslant 2r(\tfrac{1}{3}a^*). \tag{45}$$

Now suppose that $t \in I_2$. Property 6 implies that

$$\begin{aligned}\lambda'(t) &= g^*(t, \lambda(t)) \\ &= \min\{h^*(t, \nu(t)) + r(\lambda(t) - \nu(t)),\ h^*(t, \mu(t)) + r(\mu(t) - \lambda(t))\} \\ &\geqslant \min\{\nu'(t), \mu'(t)\} + r(\tfrac{1}{3}(\mu(t) - \nu(t))) \\ &\geqslant \nu'(t) - |\mu'(t) - \nu'(t)| + r(\tfrac{1}{3}(\mu(t) - \nu(t))).\end{aligned}$$

Therefore

$$\begin{aligned}\lambda'(t) - \nu'(t) &\geqslant r(\tfrac{1}{3}(\mu(t) - \nu(t))) - |\mu'(t) - \nu'(t)| \\ &\geqslant r(\tfrac{1}{3}(\mu(t) - \nu(t))) - M(\mu(t) - \nu(t)) > \tfrac{1}{2}r(\mu(t) - \nu(t))\end{aligned}$$

in virtue on the inequality

$$r(\mu(t) - \nu(t)) \leqslant \frac{r(\frac{1}{3}) - \frac{1}{2}}{M}.$$

Hence

$$\lambda'(t) - \nu'(t) \geqslant \tfrac{1}{2}\min r(\mu(t) - \nu(t)).$$

This means that if s is an arbitrary point of I_2, then for $h > 0$

$$\max(\mu(t) - \nu(t)) \geqslant \lambda(s+h) - \nu(s+h) \geqslant \tfrac{1}{2}h \min r(\mu(t) - \nu(t)),$$

i.e.

$$\begin{aligned} h \leqslant \frac{2\max(\mu(t) - \nu(t))}{\min r(\mu(t) - \nu(t))} &\leqslant \frac{2r(q)\max(\mu(t) - \nu(t))}{\max r(\mu(t) - \nu(t))} \\ &= 2r(q)\max r(\mu(t) - \nu(t)) \leqslant a^* \end{aligned} \tag{46}$$

Formulas (44), (45), (46) imply that

$$|I^{\sim}| \leqslant 4r(\tfrac{1}{3}a^*) + a^*,$$

and in virtue of the definition of $I^{\sim}$

$$|I| \leqslant 5a^* + 4r(\tfrac{1}{3}a^*) \leqslant 9r(a^*).$$

For $\lambda \subset H^*$ and $\nu, \mu \subset G^*$ it follows from the definition of the set G^* that $|I| \leqslant a^* \leqslant 9r(a^*)$. Thus the proof of Lemma 3 is complete.

Let a_i be such a decreasing sequence of positive numbers that $\lim a_i = 0$, $a_1 = 1$. Write $H_1 = y_0 \cup y_1$, $G_1 = x_0 \cup x_1$. Let $\bar{h}(t, x) = \min\{r(x), r(x-1)\}$ for $(t, x) \in K$. We define the functions $h_1, g_1 \colon K \to R$ by the formulas

$$h_1(t, x) = \max\{\bar{h}(t, y) - r(|x-y|);\, y \in (G_1 \cup \partial K)^t\},$$

$$g_1(t, x) = \min\{h_1(t, y) + r(|x-y|);\, y \in (H_1 \cup \partial K)^t\}.$$

The curves y_0 and y_1 are solutions of the equation

$$x' = h_1(t, x) \tag{47}$$

minimal to the right (and maximal to the left) and the curves x_0 and x_1 are solutions of

$$x' = g_1(t, x) \tag{48}$$

maximal to the right (and minimal to the left). The functions h_1, g_1 and the sets H_1, G_1 possess properties (16)-(24), where $a = a_1 = 1$.

Suppose that the continuous functions h_n, g_n and the subsets H_n, G_n of K satisfy conditions (16)-(24), where $a = a_n$. Lemma 3 implies that the functions $h_{n+1} = h^*$ for $h = h_n$ and $g_{n+1} = g^*$ for $g = g_n$ and the subsets of K, $H_{n+1} = H^*$ for $H = H_n$ and $G_{n+1} = G^*$ for $G = G_n$ satisfy conditions (16)-(24) for $a = a_{n+1}$ and $H_n \subset H_{n+1}, G_n \subset G_{n+1}$. The constructed sequence of the functions $\{h_n\}_{n \in N}$, $\{g_n\}_{n \in N}$ and the sets $\{H_n\}_{n \in N}$, $\{G_n\}_{n \in N}$ has properties (16)-(24). Moreover, this sequence has the following property, resulting from Lemma 3 and the construction.

COROLLARY. *The set* $\bigcup (H_n \cap G_n)$ *is dense on* K'.

LEMMA 4. *The sequences* h_n, g_n *are convergent on* K' *to the same limit.*

Proof. From Properties 2 and 2′ and the definition of h_n and g_n it follows that for every n and $(t, x) \in K'$

$$h_n(t, x) \leqslant h_{n+1}(t, x) \leqslant g_{n+1}(t, x) \leqslant g_n(t, x).$$

Hence both sequence are convergent. We shall show that the convergence is uniform. We have $h_m(t, x) - h_n(t, x) = 0$ $\forall (t, x) \in H_n$, where $n \leqslant m$. For any y there exists an $x \in H_n^t$ such that $|x-y| < a_n$. Therefore

$$|h_m(t, y) - h_n(t, y)| \leqslant |h_m(t, y) - h_m(t, x)| + |h_n(t, x) - h_n(t, y)| \leqslant 2r(a_n)$$

and uniform convergence follows. The proof for g_n is analogous. It is enough to prove that $\lim h_n(t, x) = \lim g_n(t, x)$. The sequence $k_n(t, x) = g_n(t, x) - h_n(t, x) = 0$ for $(t, x) \in H_n \cup G_n$ and for any n. The uniform convergence of k_n implies the continuity of $k = \lim k_n$. Obviously $k = 0$ on $S = \bigcup (H_n \cup G_n)$ and therefore $k = 0$ on K' because S is dense in K'.

We define

$$f = \lim h_n = \lim g_n. \tag{49}$$

The function f is continuous on K'. We have, for any $n \in N$,

$$h_n \leqslant f \leqslant g_n. \tag{50}$$

This implies that each solution contained in $\bigcup H_n$ is a solution of the equation

$$x' = f(t, x) \tag{51}$$

minimal to the right and each solution contained in $\bigcup G_n$ is a solution of (51) maximal to the right.

The constructed function f has the following property:

If $(t, x) \in y_0$, then $(t - 2\sqrt{2}, x+1) \in y_1$ and

$$f(t, x) = f(t - 2\sqrt{2}, x+1) \ (= 0). \tag{52}$$

If $(t, x) \in x_0$, then $(t-1, x) \in x_1$ (formulas (13), (14), (15)) and

$$f(t, x) = f(t-1, x). \tag{53}$$

The whole plane R^2 is the union of the following sets:

$$K(m, n) = \{(s, y);\ s = t+n,\ y = x+m \text{ for some } (t, x) \in K'\} \quad (m, n \in Z).$$

If $(s, y) \in K(m, n)$, we define

$$f(s, y) = f(t, x).$$

In virtue of (52), (53) the function f is continuous on R^2.

LEMMA 5. *The equation*

$$x' = f(t, x) \tag{54}$$

has the following properties:

(I) *The family of solutions maximal to the right on R (they are simultaneously minimal to the left) and also the family of solutions minimal to the right on R (they are simultaneously maximal to the left) are homeomorphic with the family of all straight lines parallel to a given one.*

(II) *Each maximal solution meets any minimal solution at one and only one point.*

Proof. Property (I) results from Properties 4 and 4′, the density of the set $\bigcup H_n$ ($\bigcup G_n$) in K' and the definition of f on R^2. To prove property (II) let φ be an arbitrary solution of (54) minimal to the right and let ψ be an arbitrary solution of (54) maximal to the right. There exist sequences of minimal solutions $v_n, w_n \subset H_n$ and of maximal solutions $m_n, l_n \subset G_n$,

$v_n \leqslant \varphi \leqslant w_n$, $m_n \leqslant \psi \leqslant l_n$. The solutions contained in H_n meet the solutions contained in G_n; therefore the solutions φ and ψ meet at a point of K'. Lemma 3 (ii) implies that they meet at only one point. From property (I) and the definition of the function f it follows that this property also holds on R^2.

Now we are going to define the homeomorphism $h\colon R^2 \to R^2$. Let (t, x) be an arbitrary point of R^2, let $\varphi\colon R \to R$ be the solution of (54) maximal to the right, let $\psi\colon R \to R$ be the solution of (54) minimal to the right, and let φ, ψ pass through the point (t, x). If $\varphi(s) = 0$ for the number s and $\psi(q) = x_0^{\sim}(q)$ for the number q, then we define the mapping $k\colon R^2 \to R^2$ by the formula

$$k(t, x) = \big(s, x_0^{\sim}(q)\big). \tag{55}$$

Lemma 5 implies that the mapping k is a homeomorphism $R^2 \to R^2$ and therefore the mapping $l\colon R^2 \to R^2$, given by the formula

$$l = (p_2+p_1, p_2-p_1)\circ k, \tag{56}$$

where p_i are suitable (coordinate) projections, is a continuous injection. Let $k'(s, y) = \frac{1}{2}(s-y)$, $k''(s, y) = \frac{1}{2}(s+y)$ and

$$h = k^{-1}\circ(k', k''). \tag{57}$$

It is easy to see that the continuous mapping $h\colon R^2 \to R^2$ is an injection. It is enough to show that $h = l^{-1}$, i.e., that $h\circ l = l\circ h = e_{R^2}$. From the definition we have

$$\begin{aligned} l\circ h &= (p_2+p_1, p_2-p_1)\circ k\circ k^{-1}\circ(k', k'') = (p_2+p_1, p_2-p_1)\circ(k', k'') \\ &= (k'+k'', k''-k'), \end{aligned}$$

i.e.

$$(l\circ h)(s, y) = \big(\tfrac{1}{2}(s+y)+\tfrac{1}{2}(s-y), \tfrac{1}{2}(s+y)-\tfrac{1}{2}(s-y)\big) = (s, y).$$

Similarly

$$\begin{aligned} h\circ l &= k^{-1}\circ(k', k'')\circ(p_2+p_1, p_2-p_1)\circ k \\ &= k^{-1}\circ\Big(\tfrac{1}{2}\big(p_2+p_1-(p_2-p_1)\big), \tfrac{1}{2}(p_2+p_1+p_2-p_1)\Big)\circ k = k^{-1}\circ k = e_R. \end{aligned}$$

Therefore the mapping h given by (57) is an automorphism of R^2. It follows from the definition of h (which is inverse to l) that each straight line $t-a$ is mapped by h into an integral of (54) passing through the point $t = \frac{1}{2}a$, $x = 0$ and maximal to the right, and every straight line $b-t$ is mapped into the minimal integral passing through $(\frac{1}{2}b, \frac{1}{2}b)$ and vice versa.

Now let q be an arbitrary function Lipschitz-continuous with the Lipschitz constant 1. It can be approximated by a sequence of Lipschitz-continuous functions q_n satisfying the condition $q_n'(t) = \pm 1$ with the possible exception of at most finite values of t. Then $h(q_n)$ is a piecewise solution of (54) and $h(q_n) \to h(q)$; therefore $h(q)$ is a solution of (54). Similarly, if φ is arbitrary integral of (54), it can be approximated by an integral piecewise maximal or minimal. Therefore $h^{-1}(\varphi) = \lim h^{-1}(\varphi_n)$ and $h^{-1}(\varphi)$ is a limit of continuous functions piecewise linear with the derivatives equal to ± 1, i.e., $h^{-1}(\varphi)$ is a Lipschitz-continuous function with the Lipschitz constant equal to 1. Therefore the mapping $h^*\colon L \to S(f)$ is a bijection.

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