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Differences of functions in locally convex spaces and applications to almost periodic and almost automorphic functions

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Abstract. Let f be a function from a topological group G to an abelian group A and the difference $\Delta_\gamma f(t) = f(t\gamma) - f(t)$ be continuous at unity e of G for each γ belongs to a neighbourhood E of e. We proved that if A is a Fréchet space which does not contain a subspace isomorphic to the Banach space c_0, then the boundedness of f on E implies its continuity at e. Similar results are obtained when A is locally convex space and the range $\{y: y = f(x), x \in E\}$ is weakly compact. Applications of these results are performed to the case of almost periodic and almost automorphic functions.

1. Introduction. Let f be a function from a topological group G to an abelian topological group A and let the difference $\Delta_\gamma f(t) = f(t\gamma) - f(t)$ be continuous at the unity e of G for each γ belonging to a neighbourhood E of e. We are interested in conditions implying the continuity of f at e and generally of f on G. Different cases of this problem have already been studied [7], [8], [12]. However, our conditions are weaker even in the simplest case $G = A = R$. These conditions are necessary and sufficient in the case where G is totally bounded (see Remark 3.3). The obtained results have applications in the study of almost periodic (almost automorphic) solutions of differential equations in locally convex spaces. As in Banach spaces, these applications are based on the Bohl–Bohr theorem about the integration of almost periodic functions [1], [5], [6], [9], [12], [22]. It is well known that this theorem is a direct consequence of the difference problem [2], [11], [12], [15], [19].

Our treatement reveals the importance of the property that a locally convex space X does not contain any subspace isomorphic to the Banach space c_0 of convergent to zero complex sequences ($X \not\supset c_0$). The relation of this property to the Bohl–Bohr theorem in the case of Banach spaces was first pointed out by Kadets [14].

The main results of this paper are obtained in Section 2 for the case when $A = X$ is a Fréchet space, $X \not\supset c_0$ (or when X is a general locally convex space but stronger conditions are imposed on the function f) and G is any topological group. The application of the obtained results to the case of almost automorphy (a.a.) and almost periodicity (a.p.) is performed in

Section 3. This application is the restriction of the results of Section 2 to the case where G is a totally bounded topological group. We obtain also a result related to the problem of the differences for left (or right) uniformly continuous functions, even when G is not necessarily totally bounded.

We conclude the present section by an abbreviation that will often be used throughout the discussion. Let $f\colon G \to X$ be a function from the topological group G to the locally convex space X. We shall say that the triple $\{f, G, X\}$ *satisfies condition* (α) on a subset E of G if one of the following holds: (i) X is a Fréchet space, $X \not\supset c_0$ and $f(E)$ is a bounded subset of X, or (ii) X is a locally convex space and the weak closure $\overline{f(E)}^w$ is a weakly compact subset of X.

2. Continuous differences. Let X be a locally convex space and let X^* be its topological dual (i.e. the space of all linear continuous functionals on X). The series $\sum_{n=1}^{\infty} x_n$ of elements of X is said to be *unconditionally convergent* if for each rearrangement of the terms x_n the resulting series converges. The series $\sum_{n=1}^{\infty} x_n$ is said to be *weakly unconditionally convergent* if for each $x^* \in X^*$, the series $\sum_{n=1}^{\infty} |x^*(x_n)|$ is convergent. Pełczyński and Bessaga [4] proved that in a Banach space B the two notions of unconditional convergence are equivalent if and only if $B \not\supset c_0$. In [10] and [20] the same result was proved for sequentially complete locally convex spaces. We note that weakly sequentially complete, semi-reflexive and reflexive locally convex spaces also do not contain subspaces isomorphic to c_0.

The following result is due to the first author.

THEOREM 2.1. *Let the triple $\{f, G, X\}$ satisfy condition (α) on a neighbourhood E of the unity e of G and let the right difference $\Delta_\gamma f(t) = f(t\gamma) - f(t)$ be continuous at e for each $\gamma \in E$. Then f is continuous at e.*

Proof. Without restricting generality we may take $f(e) = 0$ and assume that E is open. Now assume that f is not continuous at e. We shall construct a sequence $\{U_n(0)\}$ of open neighbourhoods of 0 of A and a sequence $\{t_n\}$ of points of E such that

(2.1) $$f(t_n) \notin U_1, \quad n = 1, 2, \ldots,$$

(2.2) $$t_n t_{n_k} t_{n_{k-1}} \ldots t_{n_1} \in E, \quad 1 \leqslant n_1 < n_2 < \ldots < n_{k-1} < n_k \leqslant n-1;\ k = 1, 2, \ldots,$$

(2.3) $$f(t_n t_{n_k} t_{n_{k-1}} \ldots t_{n_1}) - f(t_n) - f(t_{n_k} t_{n_{k-1}} \ldots t_{n_1}) \in U_{n-1}, \quad 1 \leqslant n_1 < n_2 < \ldots < n_k \leqslant n-1,\ n = 2, 3, \ldots;\ k = 1, 2, \ldots,$$

(2.4) $$U_{n+1}(0) + U_{n+1}(0) \subset U_n(0), \quad n = 0, 1, 2, \ldots$$

For each neighbourhood $U(0)$ of 0 of A we write

$$F_U = \{h \in G\colon f(h) \in U(0)\} \cap E,$$
$$E_\gamma(U) = \{h \in G\colon \Delta_\gamma f(h) - \Delta_\gamma f(e) \in U(0)\} \cap E.$$

By assumption we can find $U_0(0)$ and $t_1 \in E$ such that $t_1 \notin F_{U_0}$. Now, let $U_1(0)$ be such that $U_1(0) + U_1(0) \subset U_0(0)$. From the continuity of $\Delta_{t_1} f$ at e it follows that the set $E_{t_1}(U_1) \cap E t_1^{-1} \subset E$ is an open neighbourhood of e. Hence there exists $t_2 \in E_{t_1}(U_1) \cap E t_1^{-1}$ such that $t_2 \notin F_{U_0}$. Clearly we have: $t_2 t_1 \in E$ and $f(t_2 t_1) - f(t_2) - f(t_1) \in U_1(0)$. The construction can be continued by induction. If $t_1, t_2, \ldots, t_{n-1}$ are found, then t_n can be taken such that $t_n \notin F_{U_0}$ and

$$t_n \in \bigcap [E_{t_{n_k} t_{n_{k-1}} \ldots t_{n_1}}(U_{n-1}) \cap E t_{n_1}^{-1} t_{n_2}^{-1} \ldots t_{n_k}^{-1}],$$
$$1 \leqslant n_1 < n_2 < \ldots < n_k \leqslant n-1$$

and clearly (2.1)–(2.4) will be satisfied.

Let $\pi = (n_1, n_2, \ldots)$ be an increasing subsequence of the natural numbers and consider the subseries $\sum_{i=1}^{\infty} f(t_{n_i})$ of the series $\sum_{i=1}^{\infty} f(t_i)$. We denote the partial sums by

$$s_m = s_m(\pi) = \sum_{i=1}^{m} f(t_{n_i})$$

and let

$$\tilde{s}_m = \tilde{s}_m(\pi) = f(t_{n_m} t_{n_{m-1}} \ldots t_{n_1}), \qquad \sigma_m = s_m - \tilde{s}_m.$$

Now we assume that $\{f, G, X\}$ satisfies (i) of condition (α). The neighbourhoods $U_n(0)$ can be defined by

$$U_n(0) = \{x \in X\colon d(x, 0) < \varepsilon_0/2^n\}, \qquad \varepsilon_0 > 0,\ n = 0, 1, 2, \ldots$$

We prove that $\{\sigma_m\}$ is a Cauchy sequence. Indeed, from the equality

$$\sigma_m - \sigma_{m+p} = s_m - s_{m+p} + \tilde{s}_{m+p} - \tilde{s}_m = -\sum_{l=m+1}^{m+p} f(t_{n_l}) + f(t_{n_{m+p}} \ldots t_{n_1}) - f(t_{n_m} \ldots t_{n_1})$$
$$= [f(t_{n_{m+p}} \ldots t_{n_1}) - f(t_{n_{m+p}}) - f(t_{n_{m+p-1}} \ldots t_{n_1}) + \ldots$$
$$\ldots + f(t_{n_{m+1}} \ldots t_{n_1}) - f(t_{n_m}) - f(t_{n_m} \ldots t_{n_1})]$$

and from (2.3) it follows that $d(\sigma_m - \sigma_n, 0) < \varepsilon_0/2^{n-1}$ which shows that $\{\sigma_n\}$ is a Cauchy sequence; hence it is bounded. From the equality

$$s_n(\pi) = \sigma_n + \tilde{s}_n(\pi)$$

we conclude that $\{s_n(\pi)\}$ is bounded.

Thus for each subseries $\sum_{i=1}^{\infty} f(t_{n_i})$, the sequence $\{s_n(\pi)\}$ of partial sums is bounded. This means that each subseries $\sum_{i=1}^{\infty} f(t_{n_i})$ is weakly convergent. It easily follows that $\sum_{\infty=1}^{\infty} f(t_n)$ is weakly unconditionally convergent. Since X is complete and $X \not\supset c_0$, the series is unconditionally convergent [10], [20]. This contradicts (2.1) and proves the theorem when (i) of condition (α) is satisfied.

Now assume that $\{f, G, X\}$ satisfies (ii) of condition (α) on E. The neighbourhoods $U_n(0)$ can then be defined by

$$U_n(0) = \{x \in X:\ p_0(x) < \varepsilon_0/2^n\}, \qquad \varepsilon_0 > 0,\ p_0 \in \mathscr{P};\ n = 0, 1, 2, \ldots,$$

where $\mathscr{P}$ is the family of semi-norms of X. In this case from the weak compactness of $\overline{f(E)}^{w}$ we conclude that the sequence $\{\tilde{s}_n\}$ contains a subnet which weakly converges to an element $x_\pi \in X$. Let J be the natural mapping from X to the quotient space $X_0 = X/p_0^{-1}(0)$ which is normed by $\|J(x)\| = p_0(x)$. As in the first part it easily follows that $\{J(\sigma_n)\}$ is a Cauchy sequence in X_0. Therefore there exists an element $\bar{x}_\pi$ of the completion $\bar{X}_0$ such that $\|J(\sigma_n) - \bar{x}_\pi\| \to 0$ as $n \to \infty$. By an argument similar to that in the first part, the series $\sum_{n=1}^{\infty} J(f(t_n))$ is weakly unconditionally convergent in $\bar{X}_0$. Therefore the sequence $J(\tilde{s}_n) = J(s_n) - J(\sigma_n)$ is a weakly Cauchy sequence in X_0. Since $J(x_\pi)$ is a weak cluster point of $\{J(\tilde{s}_n)\}$, it is in fact its limit. This means that $\{J(s_n)\}$ weakly converges to an element of $\bar{X}_0$. By Orlicz's theorem [4], it follows that $\sum_{n=1}^{\infty} J(f(t_n))$ is unconditionally convergent in $\bar{X}_0$. This contradicts (2.1) and completes the proof of the theorem.

Remark 2.1. Theorem 2.1 is true in the case when $G = E = S$ is a topological semi-group with unity and the same proof is valid. We need only a slight change in the construction of $\{t_n\}$. We take $t_1 \notin F_{U_0}$; $t_2 \in E_{t_1}(U_1)$, $t_2 \notin F_{U_0}$ and proceed by induction. If $t_1, t_2, \ldots, t_{n-1}$ are thus chosen, we may take t_n such that $t_n \notin F_{U_0}$ and $t_n \in \bigcap E_{t_{n_k} t_{n_{k-1}} \ldots t_{n_1}}(U_{n-1})$, $1 \leqslant n_1 < n_2 < \ldots \ldots < n_{k-1} < n_k \leqslant n-1$.

LEMMA 2.1. *Let f: $G \to A$ be a function from the topological group G to the abelian topological group A. If f is continuous at e, then each of the following conditions implies that f is continuous on G:*

(i) *for each $u \in G$, the difference $\Delta_u f$ is continuous at e,*

(ii) *for each γ belonging to a dense subset $\Gamma \subset G$, the difference $\Delta_\gamma f$ is continuous on G.*

Proof. Let $U(0)$ be an arbitrary neighbourhood of 0 of A.

(i) Take a neighbourhood $V(0)$ of 0 of A such that $V(0)+V(0) \subset U(0)$, continuity of f at e implies that $E_1(e) = \{h: f(h)-f(e) \in V(0)\}$ is a neighbourhood of e of G. For arbitrary $t \in G$ the continuity of $\Delta_t f$ at e implies that $E_2^t(e) = \{h: \Delta_t f(h) - \Delta_t f(e) \in V(0)\}$ is also a neighbourhood of e. Hence for each $h \in E_1(e) \cap E_2^t(e)$

$$f(ht)-f(t) = \Delta_t f(h) - \Delta_t f(e) + f(h) - f(e) \in V+V \subset U(0)$$

so that f is continuous at t.

(ii) Take the neighbourhood $V(0)$ such that $V(0)+V(0)-V(0) \subset U(0)$; and let $E_2(e)$ be a symmetric neighbourhood of e of G such that $E_2(e) \cdot E_2(e) \subset E_1(e)$, where $E_1(e)$ is as in (i). Since Γ is dense in G, then for each $t \in G$ we can find $\tau \equiv \tau_t \in E_2$ such that $\tau t \in \Gamma$. Now continuity of $\Delta_{\tau t} f$ at τ^{-1} implies that the set $E_3^t(e) = h\colon \Delta_{\tau t} f(h\tau^{-1}) - \Delta_{\tau t} f(\tau^{-1}) \in V(0)$ is a neighbourhood of e. Hence for each $h \in E_2(e) \cap E_3^t(e)$

$$\begin{aligned} f(ht)-f(t) &= \Delta_{\tau t} f(h\tau^{-1}) - \Delta_{\tau t} f(\tau^{-1}) + f(h\tau^{-1}) - f(e) - f(\tau^{-1}) + f(e) \\ &\in V+V-V \subset U(0) \end{aligned}$$

so that f is continuous at t and hence on G.

Combining this lemma with Theorem 2.1, we obtain

THEOREM 2.2. *Let the triple $\{f, G, X\}$ satisfy condition* (α) *on some neighbourhood E of e. If for each $u \in G$ the difference $\Delta_u f$ is continuous at e, then f is continuous on G.*

Remark 2.2. Conditions (i) and (ii) in Lemma 2.1 can be replaced by:

(i') for each $\gamma \in \Gamma$ and each $u \in E$ the difference $\Delta_\gamma f$ is continuous at e and $\Delta_u f$ is continuous on Γ;

(ii') for each $\gamma \in \Gamma$, the difference $\Delta_\gamma f$ is continuous on E, where E is some fixed neighbourhood of e. This enables to obtain Theorem 2.2 with the corresponding variations of the condition on the differences.

Remark 2.3. Consider the compact multiplicative group $T = \{e^{it}\colon 0 \leqslant t < 2\pi\}$ and the function $f\colon T \to R$, where $f(e^{it}) = 1$ if t is rational and $f(e^{it}) = 0$ if t is irrational. Here the differences $f(e^{i(t+\gamma)}) - f(t) \equiv 0$ for each rational γ, while f is discontinuous everywhere. This shows that the condition on the differences in Theorem 2.2 cannot be replaced by (ii) of Lemma 2.1.

Remark 2.4. Theorem 2.2 remains true if the condition on the differences is replaced by the following one: for some non-negative integer n the differences $\Delta_{\gamma_n, \gamma_{n-1}, \ldots, \gamma_1, u} f \equiv \Delta_{\gamma_n} \Delta_{\gamma_{n-1}} \ldots \Delta_{\gamma_1} \Delta_u f$ are continuous at e for any choice of $u \in G$ and $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ from a neighbourhood E of e.

3. Almost periodic and almost automorphic differences. We shall adopt the following definition of *almost automorphy* (a.a.) and *almost periodicity* (a.p.). A function $f\colon G \to A$ from a topological group (G, J) to an abelian topological group A is said to be a.a. (a.p.) if it is continuous (uniformly continuous) on (G, J_B), where J_B (the Bohr topology) is the maximal totally bounded topology on G which is weaker than J. According to this definition a.a. (a.p.) functions are also continuous (uniformly continuous on (G, J). In the case when $A = X$ is a locally convex space, a proof of the equivalence of this definition to the usually used ones ([5], [6], [21]) is quite similar to that given in [3], [17], [18]. To fix the notations, a function $f\colon G \to A$ will be called *left uniformly continuous* (l.u.c.) on G if for each neighbourhood $U(0)$ of 0 of A, a neighbourhood $E(e)$ of e of G can be found such that $f(th) - f(t) \in U(0)$ for each $t \in G$ whenever $h \in E(e)$. *Right uniform continuity* (r.u.c.) is defined similarly. A function which is l.u.c. and r.u.c. is called *uniformly continuous* (u.c.). If G is totally bounded, then the two notions of l.u.c. and r.u.c. are equivalent. Indeed, let f be r.u.c. on G and let $U(0)$ and $V(0)$ be neighbourhoods of 0 of A, where $U(0)$ is arbitrary and $V(0)+V(0)-V(0) \subset U(0)$. The set $E(e) = \{h\colon f(ht)-f(t) \in V(0)\}$ is a neighbourhood of e of G. From the total boundedness of G we can find $\{u_1, u_2, \ldots, u_n\} \subset G$ such that $G = \bigcup_{i=1}^{n} E(e)u_i$. Right u.c. of f implies its continuity, so that the set $F(e) = \{h\colon f(u_i h)-f(u_i)-f(u_i) \in V(0);\ i = 1, 2, \ldots, n\}$ is a neighbourhood of e. Now each $t \in G$ can be written as $t = \tau u_i$, where $\tau \in E$ and hence $h \in F(e)$ implies that

$$f(th)-f(t) = [f(\tau u_i h)-f(u_i h)]+[f(u_i h)-f(u_i)]-[f(\tau u_i)-f(u_i)] \in V(0)+V(0)-V(0) \subset U(0).$$

Therefore f is l.u.c. The converse is proved similarly.

LEMMA 3.1. *Let the function $f\colon G \to A$ from the totally bounded topological group G to the abelian topological group A be such that for each γ belonging to a dense subset $\Gamma \subset G$, the right difference $\Delta_\gamma f(t) = f(t\gamma)-f(t)$ is u.c. on G. If f is continuous at e, then it is u.c. on G.*

Proof. Let $U(0)$ be an arbitrary neighbourhood of 0 of A and let $V_1(0)$, $V_2(0)$ be such neighbourhoods that $V_1(0)+V_2(0) \subset U(0)$, $V_2(0)-V_2(0) \subset V_1(0)$. Continuity of f at e implies that the set $E_1(e) = \{h\colon f(h)-f(e) \in V_2(0)\}$ is a neighbourhood of e. We take the neighbourhoods $E_2(e)$ and $E_3(e)$ of e such that $E_2(e)\cdot E_2(e) \subset E_1(e)$ and $E_3(e)\cdot E_3(e) \subset E_2(e)$. From the total boundedness of G we find $\{u_1, u_2, \ldots, u_n\} \subset G$ such that $G = \bigcup_{i=1}^{n} E_3(e)u_i$. Since Γ is dense in G, for each u_i we can find $\tau_i \in E_3(e)$ such that $\tau_i u_i = \gamma_i \in \Gamma$. This gives $G = \bigcup_{i=1}^{n} E_2(e)\gamma_i$. The uniform continuity of $\Delta_{\gamma_i} f$ on G implies that the set $E_4(e) = \{h\colon \Delta_{\gamma_i} f(ht)-\Delta_{\gamma_i} f(t) \in V_1(0);\ i = 1, 2, \ldots, n;\ t \in G\}$ is a neighbourhood of e. Hence for each $h \in E_2(e) \cap E_4(e)$:

$$\begin{aligned} f(ht)-f(t) &= f(h\tau\gamma_i)-f(\tau\gamma_i) \\ &= [\Delta_{\gamma_i} f(h\tau)-\Delta_{\gamma_i} f(\tau)]+[f(h\tau)-f(e)]-[f(\tau)-f(e)] \\ &\in V_1+V_1-V_2 \subset U, \quad t \in G. \end{aligned}$$

Therefore f is r.u.c. and hence u.c. on G.

The following theorem contains a variant of Lemma 3.1 which is valid when (G, J) is not necessarily totally bounded.

THEOREM 3.1. *Let G and A be topological groups of which A is abelian. If $f\colon G \to A$ is an a.a. function for which the differences $\Delta_\gamma f$ are r.u.c. for each γ belonging to a dense subgroup $\Gamma \subset G$, then f is r.u.c. on G.*

Proof. Let $U(0)$ and $V(0)$ be neighbourhoods of 0 of A such that $U(0)$ is arbitrary and $V(0)+V(0)-V(0) \subset U(0)$. Continuity of f on (G, J_B) implies that the set $O(e) = \{h \in G\colon f(h)-f(e) \in V(0)\}$ is a neighbourhood of e in J_B. We choose a neighbourhood $O_1(e)$ of e in J_B such that $O_1(e)\cdot O_1(e) \subset O(e)$. Since J_B is weaker than J, there exists a neighbourhood $E_1(e)$ of e in J such that $E_1(e) \subset O_1(e)$. Since Γ is dense in the totally bounded group (G, J_B), we can find $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subset \Gamma$ such that $G = \bigcup_{i=1}^{n} O_1(e)\gamma_i$. The r.u.c. of $\Delta_{\gamma_i} f$ on (G, J) implies that the set $E_2(e) = \{h\colon \Delta_{\gamma_i} f(ht)-\Delta_{\gamma_i} f(t) \in V(0),\ t \in G,\ i = 1, 2, \ldots, n\}$ is a neighbourhood of e in J. Since each $t \in G$ can be written as $t = \tau\gamma_i$, where $\tau \in O_1(e)$, it follows that for each $h \in E_1(e) \cap E_2(e)$

$$\begin{aligned} f(ht)-f(t) &= f(h\tau\gamma_i)-f(\tau\gamma_i) \\ &= [\Delta_{\gamma_i} f(h\tau)-\Delta_{\gamma_i} f(\tau)]+[f(h\tau)-f(e)]-[f(\tau)-f(e)] \\ &\in V+V-V \subset U, \quad t \in G. \end{aligned}$$

Therefore f is r.u.c. on (G, J).

COROLLARY 3.1. *Let $f\colon G \to A$ be a.a. function, for which the differences $\Delta_\gamma f$ are a.p. for each γ belonging to a dense subgroup $\Gamma \subset G$. Then f is a.p. on G.*

Proof. Consider on G the Bohr topology J_B. Then f is continuous on (G, J_B) and $\Delta_\gamma f$ is u.c. on (G, J_B) for each $\gamma \in \Gamma$. From Theorem 3.1 or Lemma 3.1, the function f is u.c. on (G, J_B). This means that f is a.p. on G.

Remark 3.1. In Theorem 3.1 the u.c. of the differences alone is not sufficient to ensure the u.c. of f even in the case $G = A = R$ (see the example in [19]).

Returning to functions with values in locally convex spaces we directly notice that a combination of Lemma 3.1 (for $\Gamma = G$) and Theorem 2.1 yields the following analogue for totally bounded groups of Theorem 2.2.

THEOREM 3.2. *Let the triple $\{f, G, X\}$ satisfy condition (α) on some neighbourhood E of e of the totally bounded topological group G. If for each $u \in G$, the difference $\Delta_u f$ is u.c. (f is no longer assumed continuous), then f is u.c. on G.*

We notice again that the example in Remark 2.3 shows that the condition on the differences in Theorem 3.2 cannot be replaced by the corresponding one in Lemma 3.1 if $\Gamma \neq G$.

Rewriting Theorems 3.2 and 2.2 in the terms of a.a. (a.p.) we obtain

THEOREM 3.3. *Let the triple $\{f, G, X\}$ satisfy condition (α) on the group G (no longer assumed totally bounded). If for each $u \in G$, the difference $\Delta_u f$ is a.a. (a.p.), then f itself is a.a. (a.p.).*

Remark 3.2. Theorem 3.3 is also true for Maak a.p. on semi-groups [16]. The proof follows from Remark 2.1, using a treatment similar to that in [2].

Using the fact that a continuous function on a topological group is a.a. (a.p.) if the restrictions of all its translates to a dense subgroup are a.a. (a.p.) [3] we obtain

THEOREM 3.4. *Let the triple $\{f, G, X\}$ satisfy condition (α) on a dense subgroup $\Gamma \subset G$. If for each $\gamma \in \Gamma$ the difference $\Delta_\gamma f$ is a.a. (a.p.) on G and f is continuous, then f is a.a. (a.p.) on G.*

Remark 3.3. The differences of the bounded function $f(t) = \{\sin(t/n)\}_{n=1}^{\infty}$ from R to c_0 are a.p. while f is not a.p. ([1], p. 53). This proves the necessity of condition (α) in Theorems 3.2–3.4.

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