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Multi-valued solutions of a linear functional equation

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Abstract. If Φ_0 is an upper semi-continuous solution of the inequality $H(x, \Phi[f(x)]) \subset \Phi(x)$ with compact and convex values in a Hausdorff topological vector space Y, then there exists a minimal solution $\Phi \subset \Phi_0$ of the equation $H(x, \Phi[f(x)]) = \Phi(x)$ with the same properties. It is assumed that the function H is linear with respect to the second variable. Moreover, if Y is a uniformly convex Banach space, then this minimal solution is almost everywhere single-valued in the sense of category.

Let X be a topological space and Y be a real or complex Hausdorff topological vector space and let

$$n(Y) = \{A \subset Y\colon A \neq \emptyset\},$$
$$c(Y) = \{A \in n(Y)\colon A \text{ is compact}\},$$
$$cc(Y) = \{A \in c(Y)\colon A \text{ is convex}\}.$$

DEFINITION 1. A multifunction $\Phi\colon X \to n(Y)$ is *upper semi-continuous* (u.s.c.) at x_0 iff for every open set G containing $\Phi(x_0)$ there exists an open neighbourhood U of x_0 such that $\Phi(x) \subset G$ for every $x \in U$. A multifunction is u.s.c. iff it is u.s.c. at every point $x \in X$.

We consider the equation

$$\Phi(x) = H(x, \Phi[f(x)]), \tag{1}$$

where $f\colon X \to X$, $H\colon X \times Y \to Y$ are given functions and $\Phi\colon X \to n(Y)$ is an unknown multifunction.

We will need the following

LEMMA 1 (cf. [2]). *Suppose that for every $x \in X$ the function $H(x, \cdot)$ is continuous in Y. Let F be a family of functions $\Phi\colon X \to c(Y)$ such that the following two conditions are fulfilled*:

(I) *if $\Phi \in F$, then $H(\cdot, \Phi \circ f(\cdot)) \in F$;*

(II) *if $\Phi_i \in F$ for every $i \in I$ and $\{\Phi_i\}_{i \in I}$ is a chain, then $\bigcap_{i \in I} \Phi_i \in F$.*

Let $\Phi_0 \in F$ be a multifunction fulfilling the inequality

$$H(x, \Phi[f(x)]) \subset \Phi(x). \tag{2}$$

Then the formulas

(3) $$\Phi_{n+1}(x) = H(x, \Phi_n[f(x)]), \qquad n = 0, 1, 2, \ldots,$$

(4) $$\bar{\Phi} = \bigcap_{n=0}^{\infty} \Phi_n$$

define a solution $\bar{\Phi} \subset \Phi_0$ *of equation* (1) *in* F. *Moreover, there exists a minimal solution* $\underline{\Phi} \subset \Phi_0$ *of equation* (1) *in the family* F.

DEFINITION 2. Equation (1) has *property* (P) in a family F of functions Φ: $X \to n(Y)$ iff the assertion of Lemma 1 is fulfilled, i.e., whenever $\Phi_0 \in F$ is a solution of inequality (2), then formulas (3) and (4) define a solution $\bar{\Phi} \subset \Phi_0$ of equation (1) in F and there exists a minimal solution $\underline{\Phi} \subset \Phi_0$ of (1) in F.

THEOREM 1. *If* f: $X \to X$ *and* H: $X \times Y \to Y$ *are continuous and* H *is linear with respect to the second variable, then equation* (1) *has property* (P) *in the family of all u.s.c. functions* Φ: $X \to cc(Y)$.

Proof. Let Φ_0: $X \to cc(Y)$ be a u.s.c. solution of inequality (2) and let

$$F = \{\Phi\colon \Phi \subset \Phi_0,\ \Phi\colon X \to cc(Y),\ \Phi \text{ is u.s.c.}\}.$$

It is enough to prove that F fulfils conditions (I) and (II) of Lemma 1. We have

$$H(x, \Phi[f(x)]) \subset H(x, \Phi_0[f(x)]) \subset \Phi_0(x)$$

for every $\Phi \in F$. The set $H(x, \Phi[f(x)])$ is compact and convex for every $x \in X$ and $H(\cdot, \Phi \circ f(\cdot))$ is u.s.c. (cf. [2], Lemma 3). The intersection of a chain of non-empty, compact and convex sets is non-empty, compact and convex and the intersection of a family of u.s.c. multifunctions with compact values is u.s.c. Thus the family F fulfils conditions (I) and (II).

DEFINITION 3. We define a multifunction cl Φ: $X \to n(Y)$ by

$$y \in \mathrm{cl}\, \Phi(x) \Leftrightarrow (x, y) \in \mathrm{cl}\, \Gamma_\Phi,$$

where Γ_Φ is the graph of a multifunction Φ: $X \to n(Y)$.

DEFINITION 4. A multifunction Φ: $X \to n(Y)$ is *closed* if for any $x_0 \in X$, $y_0 \in Y$ with $y_0 \notin \Phi(x_0)$ there exist two open neighbourhoods U and V of the points x_0 and y_0, respectively, such that $\Phi(x) \cap V = \emptyset$ for every $x \in U$.

The proof of the following theorem is almost the same as the proof of Lemma 7 in [2].

THEOREM 2. *Let* X *be a topological space and* Y *be a Hausdorff topological space. If a function* H: $X \times Y \to Y$ *is continuous,* f: $X \to X$ *is a continuous open mapping,* Φ_0: $X \to c(Y)$ *is a u.s.c. solution of inequality* (2) *and* $\Phi \subset \Phi_0$ *is a solution of* (1), *then* cl Φ *is a u.s.c. solution of* (1) *with compact values.*

LEMMA 2. *Let X be a topological space and Y be a Banach space and let $\Phi\colon X \to c(Y)$ be u.s.c. Then*

$$\operatorname{conv} \Phi(x) := \operatorname{conv}[\Phi(x)], \qquad x \in X,$$

is a u.s.c. multifunction and has compact values. Here $\operatorname{conv}[\Phi(x)]$ *denotes the smallest closed convex set containing $\Phi(x)$.*

Proof. Let $x_0 \in X$ and let $\operatorname{conv} \Phi(x_0) \subset G$, where G is an open set. There exists a convex open set S such that

$$\operatorname{conv} \Phi(x_0) \subset S \subset G.$$

Indeed, let us define $S := \{y \in G\colon \operatorname{dist}(y, \operatorname{conv} \Phi(x_0)) < \frac{1}{2}r\}$, where $r := \operatorname{dist}(\operatorname{conv} \Phi(x_0), G^c)$. The set S is open. We shall prove that S is convex. Let $y, z \in S$. It follows by the compactness of the set $\operatorname{conv} \Phi(x_0)$ that there exist points u, v belonging to $\operatorname{conv} \Phi(x_0)$ such that

$$\|y-u\| < \tfrac{1}{2}r, \qquad \|z-v\| < \tfrac{1}{2}r.$$

Hence, we have for $0 \leqslant \lambda \leqslant 1$

$$\|(1-\lambda)y+\lambda z-(1-\lambda)u-\lambda v\| \leqslant (1-\lambda)\|y-u\|+\lambda\|z-v\| < \tfrac{1}{2}r.$$

Consequently, the set S is convex and $\Phi(x_0) \subset S$. There exists an open neighbourhood W of x_0 such that $\Phi(x) \subset S$ for every $x \in W$. The set $S_1 := \{y \in G\colon \operatorname{dist}(y, \operatorname{conv} \Phi(x_0)) \leqslant \frac{2}{3}r\}$ is convex and closed. Hence $\Phi(x) \subset S \subset S_1$. Thus $\operatorname{conv} \Phi(x) \subset S_1 \subset G$ for every $x \in W$. This completes the proof.

LEMMA 3. *Suppose that X is a non-empty set and Y is a Banach space, $f\colon X \to X$, $H\colon X \times Y \to Y$ is linear with respect to the second variable and for every $x \in X$ the function $H(x, \cdot)$ is continuous. If $\Phi\colon X \to c(Y)$ is a solution of* (1), *then* $\operatorname{conv} \Phi\colon X \to cc(Y)$ *is also a solution of* (1).

Proof. From the inclusion

$$\Phi(x) = H(x, \Phi[f(x)]) \subset H(x, \operatorname{conv} \Phi[f(x)])$$

it follows that

$$\operatorname{conv} \Phi(x) \subset H(x, \operatorname{conv} \Phi[f(x)]),$$

since the set $H(x, \operatorname{conv} \Phi[f(x)])$, $x \in X$, is convex and compact. Now, we take $y \in H(x, \operatorname{conv} \Phi[f(x)])$. There exists $z \in \operatorname{conv} \Phi[f(x)]$ such that $y = H(x, z)$. Thus there exist numbres $\lambda_i \geqslant 0$ and points $z_i \in \Phi[f(x)]$, $i = 1, 2, \ldots, n$, for which $\sum_{i=1}^{n} \lambda_i = 1$ and $z = \sum_{i=1}^{n} \lambda_i z_i$. Since $H(x, z_i) \in H(x, \Phi[f(x)]) = \Phi(x)$, we have

$$y = H(x, z) = \sum_{i=1}^{n} \lambda_i H(x, z_i) \in \operatorname{conv} \Phi(x).$$

The proof is finished.

DEFINITION 5. A multifunction Φ: $X \to n(Y)$ is *lower semi-continuous* (l.s.c.) at $x_0 \in X$ iff for every open set $G \subset Y$ such that $G \cap \Phi(x_0) \neq \emptyset$ there exists an open neighbourhood U of x_0 such that $G \cap \Phi(x) \neq \emptyset$ for every $x \in U$.

DEFINITION 6. A multifunction Φ is *continuous* at x_0 iff it is lower and upper semi-continuous at x_0.

LEMMA 4. *If Φ: $X \to c(\mathbf{R})$ is continuous at x_0, then $\varphi(x) = \min \Phi(x)$ is continuous at x_0.*

Proof. Let us fix $\delta > 0$. By the definition of φ it follows that $\Phi(x_0) \subset (\varphi(x_0) - \delta, +\infty)$. Since Φ is u.s.c. at x_0, there exists a neighbourhood U of x_0 such that $\Phi(x) \subset (\varphi(x_0) - \delta, +\infty)$ for every $x \in U$. Thus $\varphi(x_0) - \delta < \varphi(x)$ for $x \in U$. We also have $(-\infty, \varphi(x_0) + \delta) \cap \Phi(x_0) \neq \emptyset$. By the lower semi-continuity of Φ at x_0 there exists an open neighbourhood of x_0 such that

$$(-\infty, \varphi(x_0) + \delta) \cap \Phi(x) \neq \emptyset$$

for $x \in V$. Then $\varphi(x) < \varphi(x_0) + \delta$ for $x \in V$ and the continuity of φ at x_0 is proved.

THEOREM 3. *Assume that X is a topological space, Y is a uniformly convex Banach space, f: $X \to X$ is a continuous open function, H: $X \times Y \to Y$ is continuous, linear with respect to the second variable and fulfils the condition*

$$(5) \qquad \|y_1\| \leqslant \|y_2\| \Rightarrow \|H(x, y_1)\| \leqslant \|H(x, y_2)\|.$$

If Φ_0: $X \to cc(Y)$ is a u.s.c. solution of inequality (2), then a minimal u.s.c. solution Φ: $X \to cc(Y)$ of (1) such that $\Phi \subset \Phi_0$ is almost everywhere single-valued in the sense of category.

Proof. It follows by Theorem 1 that there exists a minimal u.s.c. solution Φ: $X \to cc(Y)$ with $\Phi \subset \Phi_0$. Let put $S(x) = \{\|y\|: y \in \Phi(x)\}$. The multifunction S has compact values and is u.s.c. because it is the composition of a u.s.c mapping and a continuous mapping. Define a function s: $X \to \mathbf{R}$ by the formula

$$s(x) := \min S(x), \qquad x \in X.$$

The multifunction S is a.e. continuous in X in the sense of category (cf. [1]). In virtue of Lemma 4 the function s is a.e. continuous in X. We shall prove that the equality $\mathrm{cl}\,\varphi(x) = \varphi(x)$ holds at points of continuity of s, where φ is defined by

$$\varphi(x) := \{y \in \Phi(x): \|y\| = s(x)\}.$$

Since Y is uniformly convex, the function φ is single-valued. Let $y_0 \in \mathrm{cl}\,\varphi(x_0)$ and let x_0 be a point of continuity of s. Suppose that $2\varepsilon := |\|y_0\| - \|\varphi(x_0)\|| > 0$. There exists an open neighbourhood U of x_0 such that

$|s(x)-s(x_0)| < \varepsilon$ holds for $x \in U$. We put $V := \{y: \|y-y_0\| < \varepsilon\}$. We can find $x \in U$ and $y \in V$ such that $y = \varphi(x)$, because $(U \times V) \cap \Gamma_\varphi \neq \emptyset$. Since $\|\varphi(x)\| = s(x)$, we have $\big|\|\varphi(x)\| - \|\varphi(x_0)\|\big| < \varepsilon$ and $\|\varphi(x)-y_0\| < \varepsilon$; hence $2\varepsilon = \big|\|y_0\| - \|\varphi(x_0)\|\big| < 2\varepsilon$. This contradiction shows that $\|y_0\| = \|\varphi(x_0)\|$. The point y_0 belongs to $\Phi(x_0)$ because Φ is a closed multifunction and $y_0 \in \mathrm{cl}\,\varphi(x_0)$. It is the unique element of minimal norm in $\Phi(x_0)$; thus $y_0 = \varphi(x_0)$.

Now we prove that Φ fulfils equation (1). For every $x \in X$, the points $\varphi(x)$ and $H(x, \varphi[f(x)])$ belong to $\Phi(x)$. The norm of $\varphi(x)$ is the minimal norm of points in $\Phi(x)$; hence

$$\|\varphi(x)\| \leqslant \big\|H(x, \varphi[f(x)])\big\|.$$

On the other hand, $\varphi(x) \in H(x, \Phi[f(x)])$. Thus there exists $z \in \Phi[f(x)]$ such that $\varphi(x) = H(x, z)$. By $\varphi[f(x)] \in \Phi[f(x)]$ we get $\|\varphi[f(x)]\| \leqslant \|z\|$. Using (5) we obtain

$$\big\|H(x, \varphi[f(x)])\big\| \leqslant \|H(x, z)\| = \|\varphi(x)\|.$$

It has been proved that $\varphi(x)$, $H(x, \varphi[f(x)])$ belong to $\Phi(x)$ and their norms are equal; thus

$$\varphi(x) = H(x, \varphi[f(x)]).$$

From Theorem 2 it follows that $\mathrm{cl}\,\varphi$ is a u.s.c. solution of (1) with compact values. By Lemmas 2 and 3, $\mathrm{conv\,cl}\,\varphi$ is a u.s.c. solution of (1) with values in $cc(Y)$ and $\mathrm{conv\,cl}\,\varphi \subset \Phi_0$. Φ is a minimal u.s.c. solutions of (1) with values in $cc(Y)$, contained in Φ_0, and $\mathrm{conv\,cl}\,\varphi \subset \Phi$. Therefore $\mathrm{conv\,cl}\,\varphi = \varphi$ and $\mathrm{conv\,cl}\,\varphi$ is a.e. single-valued. Thus the theorem is proved.

Condition (5) is fulfilled, for example, by isometric mappings.

An application. Let X be a partially ordered space by "$\leqslant$" and let $L(x) := \{y \in X: y \leqslant x\}$. The topology determined by the system of neighbourhoods $\{L(x)\}$, $x \in X$, is called the *topology* generated by the partial order.

DEFINITION 7. A multifunction $\Phi: X \to n(Y)$ is *increasing* iff

$$x_1 \leqslant x_2 \Rightarrow \Phi(x_1) \subset \Phi(x_2).$$

LEMMA 5. *If $\Phi: X \to n(Y)$ is closed, then Φ is increasing. If Φ is increasing, then Φ is u.s.c.*

Proof. Suppose that there exist $x', x \in X$, $x' \leqslant x$, such that $\Phi(x') \not\subset \Phi(x)$. Choose $y \in \Phi(x')$ and suppose that $y \notin \Phi(x)$. Since Φ is a closed multifunction, we have $y \notin \Phi(u)$ for every $u \in L(x)$, which contradicts $y \in \Phi(x')$. Thus Φ is increasing.

Let Φ be increasing and let $\Phi(x) \subset G$, where G is an open set in Y. If $u \in L(x)$, then $\Phi(u) \subset \Phi(x) \subset G$, which shows that Φ is u.s.c. at x.

Let X be a space of all bounded sequences in $\boldsymbol{R}^k$, where k is a positive integer. Let us define an order in X:

$$x \leqslant y \Leftrightarrow x \quad \text{is a subsequence of a sequence } y.$$

We shall treat X as a topological space with the topology generated by the order just defined. We define a function f in X putting

$$f(x) := (x_2, x_3, \ldots) \quad \text{for} \quad x = (x_1, x_2, \ldots) \in X.$$

We observe that the function f is continuous and open. It is enough to prove that

$$f[L(x)] = L[f(x)].$$

If $y \in f[L(x)]$, then $y = f(u)$ for $u \in L(x)$. Hence $u \leqslant x$ and $f(u) \leqslant f(x)$; thus $y \in L[f(x)]$. If $y \in L[f(x)]$, then $y \leqslant f(x)$ and $y = (y_1, y_2, \ldots)$. The sequence y is a subsequence of x and $y_1 = x_{k_1}$ for some $k_1 \in N$, $k_1 \geqslant 2$. Let us put $u := (x_1, y_1, y_2, \ldots)$. Then u is a subsequence of x and $f(u) = y$; hence $y \in f[L(x)]$.

Let us define $\Phi_0(x) := \mathrm{cl}\{x_1, x_2, \ldots\}$. Since $\{x_2, x_3, \ldots\} \subset \{x_1, x_2, \ldots\}$, we have $\Phi_0[f(x)] \subset \Phi_0(x)$. Let us notice that Φ_0: $X \to c(\boldsymbol{R}^k)$ and Φ_0 is u.s.c. because it is increasing.

From Theorem 1.3. in [2] it follows that the equation

$$\Phi[f(x)] = \Phi(x) \tag{6}$$

has property (P) in the family of all u.s.c. functions Φ: $X \to c(Y)$. Thus the formulas

$$\Phi(x) = \bigcap_{n=0}^{\infty} \Phi_n(x), \quad \Phi_{n+1}(x) = \Phi_n[f(x)], \quad n = 0, 1, 2, \ldots \tag{7}$$

define a u.s.c. solution of (6). By Lemmas 2 and 3, conv $\bar{\Phi}$ is a u.s.c. solution of (6) with compact values.

Now we prove that $\bar{\Phi}(x)$ is the set of cluster points of x. Let us take $y \in \bar{\Phi}(x)$. Then $y \in \Phi_0(x) = \mathrm{cl}\{x_1, x_2, \ldots\}$. We can find $n_1 \in N$ such that $|x_{n_1} - y| < 1$. But $y \in \Phi_{n_1}(x) = \Phi_0[f^{n_1}(x)] = \mathrm{cl}\{x_{n_1+1}, \ldots\}$. Consequently, there exists $n_2 \in N$, $n_2 > n_1$ such that $|y - x_{n_2}| < \frac{1}{2}$. By induction we can find an increasing sequence (x_{n_k}) such that

$$|y - x_{n_k}| < 1/k, \quad k = 1, 2, \ldots$$

Thus y is a cluster point of x. Conversely, if y is a cluster point of x, then $y \in \mathrm{cl}\{x_{n+1}, \ldots\} = \Phi_n(x)$, $n = 0, 1, 2, \ldots$, whence $y \in \bar{\Phi}(x)$.

THEOREM 4. *The set of all divergent bounded sequences in $\boldsymbol{R}^k$ is of the first category in the set X in the topology generated by the partial order in X.*

Proof. Let Φ be given by (7). By Lemmas 2 and 3, conv $\bar{\Phi}$: $X \to cc(Y)$ is a u.s.c. solution of (6). It follows from Theorem 2 that equation (6) has a minimal solution $\underline{\Phi}$: $X \to cc(\boldsymbol{R}^k)$ which is u.s.c. and a.e. single-valued and $\underline{\Phi} \subset$ conv $\bar{\Phi}$. It is sufficient to show that $\underline{\Phi} =$ conv $\bar{\Phi}$. Suppose the contrary, i.e., that there exists

$$y \in \mathrm{conv}\, \bar{\Phi}(x) \tag{8}$$

such that $y \notin \underline{\Phi}(x)$. We can find an open neighbourhood V of y with

(9) $$V \cap \underline{\Phi}(x) = \emptyset.$$

By (8) we have $V \cap \hat{\bar{\Phi}}(x) \neq \emptyset$, where $\hat{\bar{\Phi}}(x)$ is the convex hull of $\bar{\Phi}(x)$. There exist $n \in N$, $\lambda_i \geqslant 0$, $y_i \in \bar{\Phi}(x)$ $(i = 1, 2, \ldots, n)$ such that $\sum_{i=1}^{n} \lambda_i = 1$, $\sum_{i=1}^{n} \lambda_i y_i \in V$. Suppose that $y_i \in \underline{\Phi}(x)$ for $i = 1, 2, \ldots, n$; then $\sum_{i=1}^{n} \lambda_i y_i \in \underline{\Phi}(x)$, which is impossible by (9). Thus there exists an index $i \in \{1, \ldots, n\}$ for which

(10) $$y_i \notin \underline{\Phi}(x).$$

But y_i is a cluster point of x, i.e., there exists a subsequence $x' = (x_{n_k})$ of x such that $y_i = \lim_{k \to \infty} x_{n_k}$. Consequently,

$$\underline{\Phi}(x') \subset \operatorname{conv} \bar{\Phi}(x') = \{y_i\}.$$

This implies the equality

$$\{y_i\} = \Phi(x').$$

Since, by Lemma 5, $\underline{\Phi}$ is increasing, we obtain

$$y_i \in \underline{\Phi}(x') \subset \underline{\Phi}(x),$$

which contradicts (10).

The set X defined above is not of the first category. Indeed, let Y be the set of all constant sequences. Suppose that Y is a first category set, i.e., $Y = \bigcup_{n=1}^{\infty} F_n$, where $\operatorname{Int} \operatorname{cl} F_n = \emptyset$. Take an $x \in Y$; then $x \in F_n$ for some $n \in \boldsymbol{N}$. Since the set $\{x\}$ is open, $x \in \{x\} = \operatorname{Int}\{x\} \subset \operatorname{Int} \operatorname{cl} F_n = \emptyset$. This contradiction shows that Y is not a first category set.

References

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