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Convexity of a class of functions related to classes of starlike functions and functions with boundary rotation

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Abstract. Let $N_k^\lambda(\beta, b, c)$ denote the class of functions $H_f = z(f')^b(f/z)^c$, where $f(z) = z + a_2z^2 + \ldots$ is analytic on $|z| < 1$,

$$\int_0^{2\pi} \left|\operatorname{Re} dJ_{f(z)} - \frac{\beta}{1-\beta}\right| d\theta \leqslant k\pi, \quad z = re^{i\theta},\ 0 \leqslant r < 1,$$

$$J_{f(z)} = bz\frac{f''(z)}{f'(z)} + cz\frac{f'(z)}{f(z)} + (1-c), \quad d = e^{i\lambda}\sec\lambda(1-\beta)^{-1};$$

b and c are complex numbers, $-\pi/2 < \lambda < \pi/2$, $0 \leqslant \beta < 1$ and $k \geqslant 2$ an integer. In this paper, we obtain a disc of convexity for the class $N_k^\lambda(\beta, b, c)$ and thereby unify and, at the same time, generalize results concerning discs of convexity of generalized Robertson functions, generalized Moulis functions, generalized λ-spirallike functions and functions in other related classes.

1. Introduction. Let N denote the set of all regular functions f on the unit disc $|z| < 1$ such that $f(0) = 0$, $f'(0) = 1$ and for such a function f, let

$$J_{f(z)} = J_{f(z)}(b, c) = bz\frac{f''(z)}{f'(z)} + cz\frac{f'(z)}{f(z)} + (1-c), \tag{1.1}$$

where b and c are complex numbers. Let then $N_k^\lambda(\beta, b, c)$ denote the class of functions

$$H_f = z(f')^b(f/z)^c, \tag{1.2}$$

where

$$\int_0^{2\pi} \left|\operatorname{Re} dJ_{f(z)} - \frac{\beta}{1-\beta}\right| d\theta \leqslant k\pi, \quad z = re^{i\theta},\ 0 \leqslant r < 1, \tag{1.3}$$

$$d = e^{i\lambda}\sec\lambda(1-\beta)^{-1}, \quad 0 \leqslant \beta < 1, \tag{1.4}$$

$-\frac{1}{2}\pi < \lambda < \frac{1}{2}\pi$, $k \geqslant 2$ an integer.

We make the following additional notations:

(1.5) $$\hat{J}_f = \hat{J}_f(b, c) = \frac{J_f - 1}{z} = b\frac{f''}{f'} + c\left\{\frac{f'}{f} - \frac{1}{z}\right\},$$

(1.6) $$N_{f(z)}(v) = N_{f(z)}(b, c, v) = \hat{J}'_{f(z)} - v\hat{J}^2_{f(z)} \quad (v \text{ complex}),$$

so that

(1.7) $$N_{f(z)}(1, 0, \tfrac{1}{2}) = \{f, z\},$$

the Schwarzian of f. In what follows we also use the following polynomials:

(1.8) $$Q_\delta(r) = a_0 - a_1 r + a_2 r^2$$

with

(1.9) $$a_0 = \operatorname{Re}\delta > 0, \quad a_1 = k|\delta d^{-1}|, \quad a_2 = \operatorname{Re}(2\delta d^{-1} - \delta),$$

(1.10) $$T_u(r) = |d/u|\{2Q_1(r) + Q_d(r) - 2\operatorname{Re}(d-1)(1-r^2)\}Q_{\delta'}(r)Q_u(r) - \\ - 2|d/u|(1-r^2)^2 Q_u(r) - 12\tilde{J}(0, k, 1)Q_{\delta'}(r)r^2,$$

where $u = 1$ or d and $\delta' = 2/(2-d)$

(1.11) $$\tilde{J}(0, k, 1) = \begin{cases}(k^2-4)/12, & k \geqslant 4,\\ (k-1)/3, & 2 \leqslant k \leqslant 4.\end{cases}$$

Further, let

(1.12) $$R_\delta = \begin{cases}\dfrac{a_1 - \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}, & a_2 \neq 0,\\ a_0/a_1, & a_2 = 0,\end{cases}$$

the radius of positivity of $Q_\delta(r)$.

In Section 3 we prove the following main results.

THEOREM 1.1. *Let H be in $N_k^\lambda(\beta, b, c)$. Let*

(1.13) $$0 \leqslant \beta < \frac{3 - \sqrt{4\sec^2\lambda - 3}}{4} \leqslant \tfrac{1}{2}, \quad \cos\lambda > 1/\sqrt{3},$$

and let R_0 be the least positive root of

(1.14) $$T_u(r) = 0$$

with

$$u = 1 \quad \textit{if } R_1 \textit{ or } R_{\delta'} \textit{ is } \min\{R_1, R_d, R_{\delta'}\},$$
$$u = d \quad \textit{if } R_d \textit{ is } \min\{R_1, R_d, R_{\delta'}\}.$$

Then

(1.15) $$\mathrm{Re}(1+zH''/H')>0 \quad in\ 0\leqslant r=|z|<R_0.$$

COROLLARY 1.1. *If f in N satisfies* (1.3) *with $b=1$, $c=0$, that is, if f is in the Moulis class $V_k^{\lambda}(\beta)$* [4], *then zf' is convex in the disc $|z|<R_0$, where R_0 is as in the theorem.*

In particular, the corollary is true for

(i) *Robertson functions (put $\beta=0$),*

(ii) *functions of bounded boundary rotation ($\lambda=\beta=0$),*

(iii) *convex functions of order β ($\lambda=0$, $k=2$).*

COROLLARY 1.2. *If f in N satisfies* (1.3) *with $b=0$, $c=1$, that is, if f is in the generalized Pinchuk class $U_k^{\lambda}(\beta)$* ([7], [6], [2]), *then f is convex in the disc $|z|<R_0$, where R_0 is as in the theorem.*

In particular, the corollary holds for functions in (i) the class $U_k^0(0)$ [7], (ii) the class $U_k^0(\beta)$ [6] when $k\geqslant 4$. Further, in the latter case, R_0 can be obtained as the least positive root of

(1.16) $$T_0(r)\equiv 1-3k(1-\beta)r+\{6-8\beta+k^2(1-\beta)^2\}r^2-$$
$$-k(1-\beta)(3-4\beta)r^3+(1-2\beta)^2r^4=0, \qquad k\geqslant 4.$$

Thus a result of Padmanabhan and Parvatham [6] is contained in our Theorem.

2. Some lemmas. We first note that the class V_k of Paatero [5] consists of those functions f in N satisfying (1.3) with $\lambda=\beta=0$, $b=1$, $c=0$.

LEMMA 2.1. *Let f and g in N be related by*

(2.1) $$g'(z)=[H_f/z]^d.$$

Then f satisfies (1.3) *if and only if g is in V_k.*

Proof. Taking logarithmic derivatives in (2.1), we have

$$zg''/g'=d(J_f-1),$$

and hence

$$\mathrm{Re}\left(1+\frac{zg''}{g'}\right)=\mathrm{Re}\,dJ_f-\frac{\beta}{1-\beta}.$$

Using Pinchuk [7] criteria for g to be in V_k, we get the required result.

LEMMA 2.2. *Let f in N satisfy* (1.3). *Let F be another function in N defined by*

(2.2) $$F'(z)=\frac{1}{(1+\bar{a}z)^2}\left[\frac{H_{f(\xi)}}{\xi}\,\frac{a}{H_{f(a)}}\right]^d, \qquad \xi=\frac{z+a}{1+\bar{a}z},\ |a|<1.$$

Then F is in V_k.

Proof. Given f in N satisfying (1.3), we define g in N by (2.1). Then, by the above lemma, g is in V_k. We now define F by

$$F(0) = 0, \qquad F(z) = \frac{g(\xi)-g(a)}{(1-|a|^2)g'(a)}.$$

Then, by variational principle of Robertson [8], F is in V_k. Substituting it for g, we have (2.2).

LEMMA 2.3. *If f in N satisfies (1.3) and δ any nonzero complex number with* $\operatorname{Re}\delta > 0$, *then*

$$\text{(2.3)} \qquad \operatorname{Re}\delta J_f(b, c) \geqslant \frac{Q_\delta(r)}{1-r^2} > 0, \qquad 0 \leqslant r = |z| < R_\delta \leqslant 1,$$

where $Q_\delta(r)$ and R_δ are as in (1.8) and (1.12) respectively.

Proof. By Lemma 2.2 we can choose g in the Paatero class V_k such that

$$g'(z) = \text{const}\cdot(1+\bar{a}z)^{-2}\left[\frac{H_{f(\xi)}}{\xi}\right]^d, \qquad \xi = \frac{z+a}{1+\bar{a}z}, \ |a|<1.$$

Hence

$$\frac{g''(z)}{g'(z)} = -\frac{2\bar{a}}{1+\bar{a}z}+\frac{1-|a|^2}{(1+\bar{a}z)^2}d\hat{J}_{f(\xi)}.$$

With $z = 0$, this gives, on using the Pick [1] estimate: $|g''(0)| \leqslant k$,

$$\left|d(1-|a|^2)\hat{J}_{f(a)}-2\bar{a}\right| \leqslant k.$$

Changing a to z and multiplying throughout by $|zd^{-1}|$, we get

$$|(1-r^2)\hat{J}_{f(z)}-2r^2d^{-1}| \leqslant k|d^{-1}|r.$$

Hence

$$\operatorname{Re}(\delta J_f) = \operatorname{Re}(\delta+\delta z\hat{J}_f) \geqslant \operatorname{Re}\delta+\operatorname{Re}\frac{2r^2\delta d^{-1}}{1-r^2}-\frac{k|d^{-1}\delta|r}{1-r^2} \geqslant \frac{Q_\delta(r)}{1-r^2}$$

where $Q_\delta(r)$ is as in (1.8).

That $Q_\delta(r) > 0$ when $r = |z| < R_\delta$, where R_δ is as in (1.12), follows from the standard result for positivity of a quadratic form. Finally, it is easy to check that $R_\delta \leqslant 1$ is equivalent to $1+a_2 \leqslant a_1$ which is true.

LEMMA 2.4. *If f in N satisfies (1.3), then*

$$|N_{f(z)}(b, c, d/2)| \leqslant \frac{6\tilde{J}(0, k, 1)}{|d|(1-r^2)^2},$$

where $\tilde{J}$ is the Moulis function [3] (in a slightly different notation) given by (1.11).

Proof. By Lemma 2.1, we can take a g in V_k satisfying (2.1). Differentiating (2.1) logarithmically, we have $g''/g' = d\hat{J}_f$, and hence $dN_{f(z)}(b, c, d/2) = \{g, z\}$, the Schwarzian of g. The theorem is now at once proved by using Theorem 8 of Moulis [3] with $J = \tilde{J}$, $\alpha = 0$ and $f = g$ there.

3. Proof of the main results.

Proof of Theorem 1.1. By the definition of $N_k^\lambda(\beta, b, c)$, there exists an f in N satisfying (1.3) such that (1.2) holds for $H = H_f$. Differentiating (1.2) logarithmically, we have

$$z\frac{H'}{H} = J_f. \tag{3.1}$$

This similarly yields

$$1+\frac{zH''}{H'} = J_f+\frac{zJ_f'}{J_f}. \tag{3.2}$$

From Lemma 2.4 we have, with $\delta' = 2/(2-d)$,

$$\frac{6\tilde{J}(0, k, 1)}{|d|(1-r^2)^2} \geqslant \left|\hat{J}_f'-\frac{d}{2}\hat{J}_f^2\right| = \left|\frac{J_f}{z^2}\right|\left|\frac{zJ_f'}{J_f}-\left(\tfrac{1}{2}dJ_f-\frac{1}{\delta' J_f}+1-d\right)\right|. \tag{3.3}$$

Now, since

$$\operatorname{Re} d = \frac{1}{1-\beta} > 0 \quad \text{and} \quad \operatorname{Re}\delta' = \frac{2(1-\beta)(1-2\beta)}{(1-2\beta)^2+\tan^2\lambda} > 0, \tag{3.4}$$

applying Lemma 2.3 in (3.3) repeatedly (with $\delta = 1$, d and δ') and using (3.2), we have

$$\begin{aligned}\operatorname{Re}\left(1+z\frac{H''}{H'}\right) &\geqslant \frac{Q_1(r)}{(1-r^2)}+\operatorname{Re}\frac{dJ_f}{2}-\frac{1}{\operatorname{Re}(\delta' J_f)}-\frac{6r^2\tilde{J}(0, k, 1)}{(1-r^2)^2|dJ_f|}+\operatorname{Re}(1-d)\\ &\geqslant \frac{T_1(r)}{2|d|(1-r^2)Q_{\delta'}(r)Q_1(r)} \equiv Z_1(r) \quad \text{in } 0 \leqslant r < R_1,\end{aligned} \tag{3.5}$$

where $T_1(r)$ is as in (1.10) with $u = 1$.

Here, we have assumed $R_1 = \min\{R_1, R_d, R_{\delta'}\}$, so that $Q_{\delta'}(r)$, $Q_1(r)$ are strictly positive (by Lemma 2.3) and $Z_1(r)$ is continuous. The other two cases are treated in the end. Now, from the above definitions of Z_1 and T_1 we have, on simplification,

$$\operatorname{Sgn} T_1(0) = \operatorname{Sgn}(\beta-\beta_1)(\beta-\beta_2), \tag{3.6}$$

where

$$\beta_1, \beta_2 = \frac{3 \pm \sqrt{4\sec^2_\lambda - 3}}{4},$$

respectively. Conditions (1.13) in (3.6) and (3.5) give

$$T_1(0) > 0 \quad \text{and} \quad Z_1(0) > 0. \tag{3.7}$$

For the latter we have also used $Q_{\delta'}(0) = \operatorname{Re}\delta' > 0$ from (3.4).

Now, (1.10) gives

$$T_1(R_1) = -12\tilde{J}(0, k, 1) R_1^2 Q_{\delta'}(R_1) \leqslant 0 \tag{3.8}$$

since, by Lemma 2.3, $Q_{\delta'}(R_1) \geqslant 0$ $(R_1 \leqslant R_{\delta'})$.

That there exists a positive root, and hence the least positive R_0 $(\leqslant R_1)$ of $T_1(r) = 0$ follows from (3.7) and (3.8). Thus $T_1(r) > 0$ in $0 \leqslant r < R_0$. Using this in (3.5) gives (1.15), proving the theorem in the case $R_1 = \min\{R_1, R_d, R_{\delta'}\}$. In the case $R_{\delta'} = \min\{R_1, R_d, R_{\delta'}\}$, the above arguments can easily be modified to show that R_0, the least positive root of $T_1(r) = 0$, exists with $R_0 \leqslant R_{\delta'}$ and (1.15) holds for this R_0. Lastly, in the case $R_d = \min\{R_1, R_d, R_{\delta'}\}$ we can retrace steps from (3.5) onwards and show that (1.15) holds when R_0 is the least positive root of $T_d(r) = 0$, where $T_d(r)$ is as in (1.10) with $u = d$. This completes the proof of Theorem 1.1.

Proof of Corollary 1.1. If f in N satisfies (1.3) with $b = 1$, $c = 0$, then (1.2) gives $H_f = zf'$ and for the proof of the corollary it is enough to put $H = H_f$ in Theorem 1.1.

Proof of Corollary 1.2. If f in N satisfies (1.3) with $b = 0$, $c = 1$, then (1.2) gives $H_f = f$ and for the proof of the first part of the corollary it is enough to put $H = H_f = f$ in Theorem 1.1. For the particular case, we have that if $\lambda = 0 = b$ and $c = 1$, then

$$d = \frac{1}{1-\beta} \quad \text{and} \quad \delta' = \frac{2}{2-d} = \frac{2(1-\beta)}{1-2\beta}$$

are real and positive. Thus, substituting in (1.8) and (1.12), we get

$$\frac{Q_d}{d} = \frac{Q_{\delta'}}{\delta'} = Q_1 \quad \text{and} \quad R_d = R_{\delta'} = R_1. \tag{3.9}$$

Hence the three cases of Theorem 1.1 merge. Substituting (3.9) in (1.10), we have

$$T_1(r) = T_d(r) = A T_0(r) Q_1(r), \tag{3.10}$$

where $A = 4/(1-2\beta) > 0$,

$$A T_0(r) = d\delta' \{(2+d) Q_1(r) - 2(1-r^2)(d-1)\} Q_1(r) - \\ -2d(1-r^2)^2 - 12\delta' \tilde{J}(0, k, 1) r^2 \tag{3.11}$$

which (but for the factor A) reduces to the expression in (1.16) on simplification. Now, from (3.7), (3.10) and (3.11) it is easy to see that $T_0(r) = 0$ has a positive root and the least positive root R_0 is such that $R_0 < R_1$ and that this is also the least positive root of $T_1(r) = 0$. If f is in N satisfying (1.3) with $b = 0$, $c = 1$ (1.2) gives $H_f = f$ and convexity of f in $0 \leqslant r < R_0$ now follows from (1.15) on taking $H = H_f = f$.

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