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π-GEODESICS AND LINES OF SHADOW

BY

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Let V_n^Γ be an n-dimensional space with a linear connection Γ given with the aid of components Γ_{jk}^i in each local map U on a differential manifold V_n. Space V_n^Γ will be called *Γ-space*. The set of differentiable vector fields on V_n will be denoted by $\mathscr{T}(V_n)$, the set of differentiable real-valued functions by $\mathscr{F}(V_n)$, and the set of real numbers by R.

Let (U, φ) be a local map on V_n, $U \subset V_n$, $\varphi\colon U \to R\times R\times \ldots \times R$, $\varphi(X) = (x^1, \ldots, x^n)$, where $X \in V_n$ and $(x^1, \ldots, x^n)$ is a coordinate system on U.

We introduce the following notations:

(a) if $\{e_i\}$ is a natural basis of $\mathscr{T}(U)$ and $v \in \mathscr{T}(U)$, then $v = v^i e_i$ and $v_x = v_x^i e_i(X) \in T_x$, where T_x is the vector space tangent to V_n at the point $X \in V_n$;

(b) if $f \in \mathscr{F}(U)$ and $v \in \mathscr{T}(U)$, then $(d_v f)_x = (\partial_i f)_x v_x^i$, $d_v f\colon U \to \mathscr{F}(U)$ and $d_v f\colon X \mapsto (d_v f)_x$, $X \in U$, (if $f \in \mathscr{F}(V_n)$ and $v \in \mathscr{T}(V_n)$, then $d_v f\colon V_n \to \mathscr{F}(V_n)$ and $d_v f\colon X \mapsto (d_v f)_x$), where $(\partial_i f)_x$ is the partial derivative $\partial f/\partial x^i$ at $X \in U$;

(c) if $\nabla_i F$, $i = 1, \ldots, n$, is a covariant derivative of a tensor field F on V_n (or on U) and $v \in \mathscr{T}(V_n)$ (or $v \in \mathscr{T}(U)$), then $(\nabla_v F)_x = (\nabla_i F)_x v_x^i$ and $\nabla_v F\colon X \mapsto (\nabla_v F)_x$, where $X \in V_n$ (or $X \in U$).

Let us consider now a fixed differentiable tensor field π of the type (0,2) on V_n, $\pi\colon \mathscr{T}(V_n)\times\mathscr{T}(V_n) \to \mathscr{F}(V_n)$, and a vector field $w \in \mathscr{T}(V_n)$ with $w_x \neq 0$ for each $X \in V_n$. On a local map $U \subset V_n$ tensor π can be written in the form

$$\pi\colon (v_1, v_2) \mapsto \pi_{ij} v_1^i v_2^j, \qquad v_k = v_k^i e_i \in \mathscr{T}(U),\ \pi_{ij} \in \mathscr{F}(U).$$

Tensor π will be called *non-singular* if there exists an atlas $\mathscr{A}$ on V_n such that $\det(\pi_{ij}) \neq 0$ in each local chart of $\mathscr{A}$.

We will construct a covariant vector field

$$\pi^w\colon v_1 \to \pi(v, w), \qquad v, w \in \mathscr{T}(V_n). \tag{1}$$

Definition 1. A vector field $w \in \mathscr{T}(V_n)$, $w_x \neq 0$ for each $x \in V_n$, is called *π-geodesic* if

$$\nabla_w \pi^w = \lambda \pi^w, \qquad \lambda \in \mathscr{F}(V_n),\ \pi^w\colon v \mapsto \pi(v, w), \tag{2}$$

where $\pi\colon \mathcal{T}(V_n)\times\mathcal{T}(V_n) \to \mathcal{F}(V_n)$ is a non-singular tensor field of the type (0,2) on V_n.

Integral curves of π-geodesic vector field will be called *π-geodesics* (they are curves satisfying, in each local map of V_n, $w^i = dx^i/dt$, $w = w^i e_i$).

Let us write condition (2) in the coordinate system $(x^1, \ldots, x^n)$ of a local map U of V_n, where Γ^k_{ij} are components of a linear connection on V_n. Then

$$v = v^i e_i, \qquad w = w^i e_i,$$

$$\pi^w\colon v \mapsto \pi(v, w) = \pi_{ij} v^i w^j = \pi^w_i v^i, \qquad \pi^w_i = \pi_{ij} w^j,$$

$$\begin{aligned}\nabla_w \pi^w_i = \nabla_k \pi^w_i w^k &= (\partial_k \pi^w_i - \Gamma^s_{ki}\pi^w_s) w^k = (\partial_k \pi_{ij} w^j + \pi_{ij}\partial_k w^j - \Gamma^s_{ki}\pi_{sj} w^j) w^k \\ &= (\nabla_k \pi_{ij} w^j + \pi_{ij}\nabla_k w^j) w^k;\end{aligned}$$

$$\nabla_w \pi^w\colon v \mapsto \nabla_w \pi^w_i v^i.$$

Setting $w^i = dx^i/dt$, we obtain a differential equation of π-geodesics

$$\pi_{ij}\frac{d^2x^j}{dt^2} + (\nabla_k \pi_{is} + \pi_{ij}\Gamma^j_{ks})\frac{dx^k}{dt}\frac{dx^s}{dt} = \lambda\pi_{ij}\frac{dx^j}{dt}$$

or

$$\frac{d^2x^i}{dt^2} + (\nabla_k \pi_{ps}\pi^{pi} + \Gamma^i_{ks})\frac{dx^k}{dt}\frac{dx^s}{dt} = \lambda\frac{dx^i}{dt}, \qquad \pi_{ps}\pi^{pi} = \delta^i_s. \tag{3}$$

Functions

$$G^i_{ks} = \nabla_k \pi_{ps}\pi^{pi} + \Gamma^i_{ks} \tag{4}$$

are components of linear connection G on V_n, and (3) is a differential equation of geodesics in the ordinary sense in the space V^G_n with the linear connection G. Thus we obtain

THEOREM 1. *π-geodesics in a space V^Γ_n are geodesics in ordinary sense in the space V^G_n, where connection G is given by formula* (4).

From this theorem it follows immediately

THEOREM 2. *If g is a metric tensor of a Riemannian space V_n, then g-geodesics in V_n are geodesics in the ordinary sense.*

Let V^Γ_n and $\hat{V}^{\hat{\Gamma}}_n$ be two spaces with the linear connections Γ and $\hat{\Gamma}$. We consider π-geodesics in V^Γ_n and $\hat{\pi}$-geodesics in $\hat{V}^{\hat{\Gamma}}_n$. Theorem 1 allows us to treat a mapping $f\colon V_n \to \hat{V}_n$, which maps π-geodesics onto $\hat{\pi}$-geodesics, as a geodesic mapping of a G-space (V_n, G) onto a $\hat{G}$-space $(\hat{V}_n, \hat{G})$, where

$$G^i_{jk} = \nabla_j \pi_{pk}\pi^{pi} + \Gamma^i_{jk}, \qquad \hat{G}^i_{jk} = \hat{\nabla}_j \hat{\pi}_{pk}\hat{\pi}^{pi} + \hat{\Gamma}^i_{jk}, \tag{5}$$

($\hat{\nabla}$ denotes the covariant derivative with respect to $\hat{\Gamma}^i_{jk}$).

Assuming that mapping f is given by equal coordinates in local charts $U \subset V_n$ and $\hat{U} \subset \hat{V}_n$, we can write differential equations of geodesics in V_n^G and $\hat{V}_n^{\hat{G}}$ in the form

$$(6) \qquad \frac{d^2x^k}{dt^2} + G_{ij}^k \frac{dx^i}{dt}\frac{dx^j}{dt} = \lambda \frac{dx^k}{dt}, \qquad \frac{d^2x^k}{dt^2} + \hat{G}_i^k \frac{dx^i}{dt}\frac{dx^j}{dt} = \mu \frac{dx^k}{dt},$$

respectively, whence, by subtraction, we receive

$$(G_{ij}^k - \hat{G}_{ij}^k)\frac{dx^i}{dt}\frac{dx^j}{dt} = (\lambda - \mu)\frac{dx^k}{dt}.$$

Setting

$$(7) \qquad p_{rj}^k = G_{rj}^k - \hat{G}_{rj}^k = \nabla_r \pi_{ij} \pi^{ik} + \Gamma_{rj}^k - \hat{\nabla}_r \hat{\pi}_{ij} \hat{\pi}^{ik} - \hat{\Gamma}_{rj}^k, \qquad \lambda - \mu = \gamma,$$

we have

$$(8) \qquad p_{sj}^k \frac{dx^j}{dt}\frac{dx^s}{dt} = \gamma \frac{dx^k}{dt}.$$

Multiplying (8) by dx^q/dt and $p_{sj}^q\,(dx^j/dt)(dx^s/dt) = \gamma(dx^q/dt)$ by dx^k/dt and subtracting, we obtain

$$p_{sj}^k \frac{dx^s}{dt}\frac{dx^q}{dt}\frac{dx^j}{dt} = p_{sj}^q \frac{dx^s}{dt}\frac{dx^k}{dt}\frac{dx^j}{dt}$$

or

$$(8^*) \qquad p_{sj}^k \frac{dx^s}{dt}\frac{dx^j}{dt}\delta_r^q \frac{dx^r}{dt} - p_{sj}^q \frac{dx^s}{dt}\frac{dx^j}{dt}\delta_r^k \frac{dx^r}{dt} = 0,$$

whence

$$(9) \qquad p^k{}_{(sj}\delta_{r)}^q - p^q{}_{(sj}\delta_{r)}^k = 0,$$

where ( ) denotes the symmetrisation with respect to indexes in brackets. Contraction over r and q yields

$$(n+1)(p_{sj}^k + p_{js}^k) = (p_{sq}^q + p_{qs}^q)\delta_j^k + (p_{jq}^q + p_{qj}^q)\delta_s^k,$$

whence, setting

$$(10) \qquad p_s = \frac{1}{n+1} p_{(sq)}^q = \frac{1}{n+1}(G_{(sq)}^q - \hat{G}_{(sq)}^q),$$

we obtain

$$(11) \qquad p_{(sj)}^k = p_s \delta_j^k + p_j \delta_s^k.$$

From (11) it follows (9) and, subsequently, (8*). Setting $a_k dx^k/dt = 1$, $p_{sj}^k (dx^s/dt)(dx^j/dt) a_k = \gamma$, we infer (8) from (8*). Calculating $G_{ij}^k (dx^i/dt)(dx^j/dt)$

from formula (8) and inserting it into the first equation of (6), we obtain the second equation of (6), that is, the equation of geodesics in $\hat{G}$-space. Thus we have proved that Weyl's condition (11) is sufficient and necessary for the existence of a (local) geodesic map of G-space onto $\hat{G}$-space (by equal coordinates).

In particular, if

$$G^i_{jk} = \Gamma^i_{jk} = \hat{\Gamma}^i_{jk}, \qquad \hat{G}^i_{jk} = \nabla_j \pi_{pk}\pi^{pi} + \Gamma^i_{jk}, \qquad V_n = \hat{V}_n,$$

then, by (11),

$$p^i_{jk} = -\nabla_j \pi_{pk}\pi^{pi}, \qquad 2p_s = -\frac{1}{n+1}(\nabla_j \pi_{ps}\pi^{pj} + \nabla_s \pi_{pj}\pi^{pj}),$$

$$\nabla_{(j}\pi_{|p|k)}\pi^{pi} = \frac{1}{n+1}(\nabla_{(j}\pi_{|p|s)}\pi^{ps}\delta^i_k + \nabla_{(k}\pi_{|p|s)}\pi^{ps}\delta^i_j). \tag{12}$$

Thus we obtain

THEOREM 3. *Condition* (12) *is necessary and sufficient for π-geodesics in Γ-space be* (*locally*) *geodesics in the ordinary sens.*

If a tensor π satisfies condition (12) *and $f \in \mathscr{F}(V_n)$, then the tensor $\eta = f\pi$ also satisfies* (12).

Since in the indicated case condition (12) is equivalent to (11), the local mapping given by equal coordinates of V^Γ_n onto $V^{\hat{G}}_n$, where $\hat{G}^i_{jk} = \nabla_j \pi_{pk}\pi^{pi} + \Gamma^i_{jk}$, is geodesic, and thus if (12) is satisfied, geodesics in V^Γ_n and $V^{\hat{G}}_n$ are the same curves in V_n. The second part of Theorem 3 follows immediately from 12 by inserting $f\pi_{ij}$ instead of π_{ij}.

In the same way, if we set $V_n = \hat{V}_n$, $\Gamma^i_{jk} = \hat{\Gamma}^i_{jk}$, $G^i_{jk} = \nabla_j \pi_{pk}\pi^{pi} + \Gamma^i_{jk}$, $\hat{G}^i_{jk} = \nabla_j \hat{\pi}_{pk}\hat{\pi}^{pi} + \Gamma^i_{jk}$, one can show

THEOREM 4. *If*

$$\nabla_{(j}\pi_{|p|k)}\pi^{pi} - \nabla_{(j}\hat{\pi}_{|p|k)}\hat{\pi}^{pi} = \frac{1}{n+1}[(\nabla_{(q}\pi_{|p|k)}\pi^{pq} - \nabla_{(q}\hat{\pi}_{|p|k)}\hat{\pi}^{pq})\delta^i_j + (\nabla_{(q}\pi_{|p|j)}\pi^{pq} - \nabla_{(q}\hat{\pi}_{|p|j)}\hat{\pi}^{pq})\delta^i_k], \tag{13}$$

then π-geodesics and $\hat{\pi}$-geodesics in V^Γ_n are the same (*locally*) *curves in V_n.*

If tensors π and $\hat{\pi}$ satisfy (13), *then tensors $\eta = f\pi$ and $\hat{\eta} = g\hat{\pi}$, where $f, g \in \mathscr{F}(V_n)$, also satisfy* (13).

Now let us restrict our considerations to a Euclidean space E_n and an oriented surface S_{n-1} in it. Orthonormal coordinates in E_n will be denoted by x^a, and local coordinates in $U \subset S_{n-1}$ by u^i; thus S is locally given by the mapping $x\colon (u^1, \ldots, u^{n-1}) \mapsto x(u^1, \ldots, u^{n-1})$, $x = OX$, $x\colon R \times \ldots \times R \to \overline{U}$, $X \in U$, $\overline{U} = \{OX; X \in U\}$, where OX is a vector with the origin at the point $O \in E_n$.

The non-singular second fundamental tensor of the oriented surface S_{n-1} will be denoted by b,

$$b\colon \mathscr{T}(S_{n-1})\times\mathscr{T}(S_{n-1}) \to \mathscr{F}(S_{n-1}), \qquad b\colon (v, w) \to b_{ij}v^i w^j, \qquad v, w \in \mathscr{T}(U);$$

$$v = v^i x_i, \qquad w = w^i x_i, \qquad x_i = \partial x/\partial u^i, \qquad b_{ij} = x_{ij} n = -x_i n_j,$$

where n is the unit vector normal to S_{n-1}, $n_i = \partial n/\partial u^i$, $x_{ij} = \partial^2 x/(\partial u^i \partial u^j)$.

We write formula (12) for b. Since $\nabla_{(j} b_{kr)} = \nabla_j b_{kr}$, we have

$$\nabla_j b_{kr} b^{pr} = (\partial_j b_{kr} - \Gamma^s_{jk} b_{sr} - \Gamma^s_{jr} b_{ks}) b^{pr},$$

$$\begin{aligned}\partial_j b_{kr} &= -\partial_j (n_k x_r) = -n_{kj} x_r - n_k x_{jr}\\ &= -(\Sigma^s_{kj} n_s + B_{kj} n) x_r - n_k(\Gamma^s_{jr} x_s + b_{jr} n) = \Sigma^s_{kj} b_{sr} + \Gamma^s_{jr} b_{sk},\end{aligned}$$

where Γ^s_{jr} are components of the Levi-Cività connection on S_{n-1}, and Σ^s_{jr} on the unit sphere $\overset{\circ}{S}_{n-1}$ being the image by normals to S_{n-1}. Now, we have

$$\nabla_j b_{kr} b^{pr} = \Sigma^p_{jk} - \Gamma^p_{jk} \tag{14}$$

and (12) can be written

$$\begin{gathered}\Sigma^s_{jk} - \Gamma^s_{jk} = p_j \delta^s_k + p_k \delta^s_j,\\ p_j = \frac{1}{n}\nabla_j b_{sr} b^{sr} = \frac{1}{n}(\Sigma^s_{sj} - \Gamma^s_{sj}) = \frac{1}{2n}\partial_j \log\frac{\sigma}{g},\end{gathered} \tag{15}$$

where g is the determinant of the first, and σ of the third fundamental tensor of S_{n-1}. But (15) is Weyl's condition for a geodesic (local) mapping of S_{n-1} onto $\overset{\circ}{S}_{n-1}$. Thus, we obtain

THEOREM 5. *b-geodesics on a surface S_{n-1} in a Euclidean space E_n are (locally) geodesics in the ordinary sense if and only if the (local) mapping of S_{n-1} onto unit sphere $\overset{\circ}{S}_{n-1}$ by normals is geodesic.*

If S_{n-1} and $\hat{S}_{n-1}$ are two surfaces in E_n, then from (7) and (14) we obtain $p^k_{rj} = \Sigma^k_{rj} - \hat{\Sigma}^k_{rj}$, and from Voss-Weyl's theorem (cf. [3], p. 149) and (11) we have

$$\Sigma^k_{rj} - \hat{\Sigma}^k_{rj} = p_r \delta^k_j + p_j \delta^k_r, \qquad p_j = \frac{1}{2n}\partial_j \log\frac{\sigma}{\hat{\sigma}}, \tag{16}$$

where σ and $\hat{\sigma}$ are determinants of the third fundamental tensors of S_{n-1} and $\hat{S}_{n-1}$, respectively. Formula (16) can be interpreted as follows:

THEOREM 6. *The (local) mapping $f\colon S_{n-1} \to \hat{S}_{n-1}$ by equal coordinates on surfaces S_{n-1} and $\hat{S}_{n-1}$ in E_n is b-geodesic if and only if the mapping by equal coordinates of their spherical images by normals is geodesic.*

A mapping $f\colon S_{n-1} \to \hat{S}_{n-1}$ is called *b-geodesic* if it maps b-geodesics on S_{n-1} onto b-geodesics on $\hat{S}_{n-1}$.

We consider now an oriented surface S_2 in E_3.

Definition 2. A vector field w on S_2 is called *shadow* if there exists on S_2 a vector field $v \neq 0$, $v \neq w$, such that $d_w v = 0$ and $\nabla_w v = 0$, where $d_w v = \partial_r v w^r$, $\nabla_w v = \nabla_w v^i x_i = \nabla_r v^i w^r x_i$, $w = w^i x_i = w^a e_a$, $(i = 1, 2;$ $a = 1, 2, 3)$.

Integral curves of a shadow vector field w will be called *lines of shadow on S_2*.

In a local map $U \subset S$ we have along a line of shadow L,

$$t \to u^i(t), \quad v = v^i x_i = v^a e_a, \quad w = w^i x_i, \quad w^i = du^i/dt,$$

$$i = 1, 2;\ a = 1, 2, 3;\ x_i = x_i^a e_a,$$

$$\begin{aligned} d_w v^a &= d_w(v^i x_i^a) = d_w v^i x_i^a + v^i d_w x_i^a = d_w v^i x_i^a + v^i x_{ij}^a w^j \\ &= d_w v^i x_i^a + v^i (\Gamma_{ij}^r x_r^a + b_{ij} n^a) w^j \\ &= -\Gamma_{rk}^i x_i^a v^k w^r + v^k \Gamma_{kj}^r w^j x_i^a + v^k b_{ij} w^j n^a = 0, \end{aligned}$$

because $\nabla_w v^i = d_w v^i + \Gamma_{jk}^i w^j v^k = 0$. Thus we have

$$v^k b_{kj} w^j = 0 \quad \text{or} \quad v^k b_k^w,$$

where $b_k^w = b_{kj} w^j$ along L.

Derivating covariantly the last identity, we obtain

$$v^k b_k^w = 0, \quad v^k \nabla_w b_k^w = 0, \quad k = 1, 2,$$

or

$$\nabla_w b_k^w = \lambda b_k^w, \quad w \in \mathcal{T}(S_2), \quad \lambda \in \mathcal{F}(S_2). \tag{17}$$

Comparing (17) with (2), we have for a surface S_2 with the Gaussian curvature $K \neq 0$ the following result:

THEOREM 7. *Lines of shadow on a surface S_2 with $K \neq 0$ in E_3 are (locally) b-geodesics on S_2 and inversely.*

Let S_2 and $\hat{S}_2$ be two surfaces in E_3 with the first fundamental tensors g and $\hat{g}$, second fundamental tensors b and $\hat{b}$, and third fundamental tensors ν and $\hat{\nu}$, respectively,

$$\nu\colon (v, w) \to \nu_{ij} v^i w^j, \quad \nu_{ij} = n_i n_j, \quad \hat{\nu}_{ij} = \hat{n}_i \hat{n}_j,$$

$$\det \nu_{ij} = \sigma, \quad \det \hat{\nu}_{ij} = \sigma, \quad \det g_{ij} = \gamma, \quad \det \hat{g}_{ij} = \hat{\gamma}.$$

By virtue of [1], p. 206, we have $\sigma = K^2 \gamma$, $\hat{\sigma} = \hat{K}^2 \hat{\gamma}$.

For a b-geodesic mapping $f\colon S_2 \to \hat{S}_2$ by equal coordinates, formula (16) holds, where

$$p_j = \frac{1}{6}\,\partial_j \log \frac{K^2\gamma}{\hat{K}^2\hat{\gamma}}.$$

Now assume that a b-geodesic mapping $f\colon S_2 \to \hat{S}_2$ is isometric. Then from (16) we have $\Sigma_{rj}^k = \hat{\Sigma}_{rj}^k$, and from Theorem 2 of [2], p. 10, we obtain $\nu_{ij} = c\hat{\nu}_{ij}$, $c = \text{const}$, $c > 0$.

For isometric surfaces we have $\det \nu_{ij} = \det \hat{\nu}_{ij}$. Then $c = 1$, and since $\nu_{ij} = 2Hb_{ij} - Kg_{ij}$, where H is the mean curvature of S_2, $Hb_{ij} = \hat{H}\hat{b}_{ij}$. Multiplying last identity by g^{ij} and summing with respect to i, j, we obtain $H^2 = \hat{H}^2$ and $b_{ij} = \pm \hat{b}_{ij}$.

Thus

THEOREM 8. *If two surfaces S_2 and $\hat{S}_2$ in E_3 are isometric and isometry preserves b-geodesics, then S_2 and $\hat{S}_2$ differ (locally) by a Euclidean motion (i.e., they are congruent).*

We have decided to publish these simple results to call attention to a possibility of applications of π-geodesics to a study of surface in Euclidean or Riemannian spaces. If a tensor π is defined in a natural way, then π-geodesics are curves having geometrical interpretations and may serve in explaining some further properties of surfaces or spaces with linear connections. One may wonder why these so naturally defined curves are not used in handbooks for students, although the theory of π-geodesics is a part of the theory of geodesics in spaces with linear connections. We have given an example of b-geodesics, but we cannot give a geometrical interpretation of other natural π-geodesics, for instance, if π is Ricci tensor of a Riemannian space or third fundamental tensor of a surface in E_3.

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