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The radius of p-valent starlikeness for certain classes of analytic functions*

by DONALD J. WRIGHT (Kentucky)

1. Introduction. Let $E = \{z: |z| < 1\}$. Suppose p is a positive integer and let $S^*(p, a)$ denote the class of functions $f(z) = z^p + \sum_{p+1}^{\infty} a_k z^k$ which are regular in E and satisfy $\operatorname{Re}\{zf'(z)/f(z)\} > a$, $z \in E$, $0 \leqslant a < p$. The members of $S^*(p, a)$ are p-valent and starlike in E [2]. Let $n > p$ be a positive integer and suppose $g_n(z) = \sum_{n}^{\infty} b_k z^k$, $b_n \neq 0$, is regular in E. We consider the class of functions $h_n(z) = f(z) + g_n(z)$, where $f(z) \in S^*(p, a)$ and $g_n(z)$ satisfies $\operatorname{Re}\{g_n(z)/f(z)\} > -1$, $z \in E$. In the first part of this paper we determine the radius of p-valent starlikeness for this class and also for the subclass consisting of those functions $h_n(z) = f(z) + g_n(z)$ for which $|g_n(z)| \leqslant |f(z)|$, $z \in E$.

Let $CS^*(p, a)$ denote the class of functions $h(z) = z^p + \sum_{p+1}^{\infty} c_k z^k$ which are regular in E and satisfy $\operatorname{Re}\{h(z)/f(z)\} > 0$, $z \in E$, for some $f(z) \in S^*(p, a)$. When $p = 1$, $a = 0$, this definition gives the class of close-to-star functions introduced by Reade [6]. If $h(z) \in CS^*(p, a)$, then $h(z) = f(z) + [h(z) - f(z)] = f(z) + g_n(z)$, where $f(z) \in S^*(p, a)$ and $g_n(z) = \sum_{n}^{\infty} b_k z^k$ $(n > p)$ is regular and satisfies $\operatorname{Re}\{g_n(z)/f(z)\} > -1$, $z \in E$. Similarly, if $|h(z)/f(z) - 1| < 1$, $z \in E$, then $h(z) = f(z) + g_n(z)$, where $|g_n(z)| \leqslant |f(z)|$, $z \in E$. Thus, the results mentioned above yield the radius of p-valent starlikeness for the class $CS^*(p, a)$ and that of the subclass of $CS^*(p, a)$ consisting of those functions $h(z)$ which satisfy $|h(z)/f(z) - 1| < 1$, $z \in E$, for some $f(z) \in S^*(p, a)$.

Suppose $0 < \beta \leqslant 1$. In the last section we give the radius of p-valent starlikeness for the two subclasses of $CS^*(p, a)$ consisting of the functions $h(z)$ which satisfy respectively $\operatorname{Re}\{h(z)/f(z)\}^{1/\beta} > 0$, and $|\{h(z)/f(z)\}^{1/\beta} - 1| < 1$, $z \in E$, for some $f(z) \in S^*(p, a)$.

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The results in the paper are extensions of some similar work done by MacGregor [3] and [4].

2. Preliminaries. We shall make frequent use of the fact that for a function $h(z) = z^p + \sum_{p+1}^{\infty} c_k z^k$ which is regular in $|z| < r$, the condition $\mathrm{Re}\{zh'(z)/h(z)\} > 0$, $|z| < r$, is necessary and sufficient for $h(z)$ to be p-valent and starlike for $|z| < r$ [2].

The following lemma is well-known for the case $p = 1, a = 0$ ([5], p. 173, problem 11). The general result is easily obtained from this special case.

LEMMA 1. *If* $P(z) = p + \sum_{1}^{\infty} p_k z^k$ *is regular in* E *and satisfies* $\mathrm{Re}\{P(z)\} > a$, $0 \leqslant a < p$, *then*

$$\mathrm{Re}\{P(z)\} \geqslant \frac{p-(p-2a)|z|}{1+|z|}, \quad z \in E. \tag{1}$$

We shall also need the following extension of Schwartz's lemma ([1], p. 290).

LEMMA 2. *If* $\varphi(z) = d_0 + \sum_{m}^{\infty} d_k z^k$, $m \geqslant 1$, *is regular and bounded by* 1 *in* E, *then*

$$|\varphi'(z)| \leqslant \frac{m|z|^{m-1}\left(1-|\varphi(z)|^2\right)}{1-|z|^{2m}}, \quad z \in E. \tag{2}$$

If $d_0 = 0$, *then*

$$|\varphi(z)| \leqslant |z|^m, \quad z \in E. \tag{3}$$

3. Main results.

THEOREM 1. *If* $f(z) \in S^*(p, a)$ *and* $\mathrm{Re}\{g_n(z)/f(z)\} > -1$, $z \in E$, *then* $h_n(z) = f(z) + g_n(z)$ *is p-valent and starlike for* $|z| < r(p, a, n)$, *where* $r(p, a, n)$ *is the smallest positive root of*

$$\lambda(p, a, n; x) = p-(p-2a)x-2(n-p)x^{n-p}-2(n-p)x^{n-p+1}- \\ -px^{2(n-p)}+(p-2a)x^{2(n-p)+1} = 0.$$

Proof. The function $k(z) = -2z/(1+z)$ maps E onto the half plane $\mathrm{Re}\{w\} > -1$, and by hypothesis, $g_n(z)/f(z)$ is subordinate to $k(z)$. Thus, there is a function $\varphi(z)$ which is regular and bounded by 1 in E such that

$$\frac{g_n(z)}{f(z)} = \frac{-2\varphi(z)}{1+\varphi(z)}.$$

Furthermore, $\varphi(z)$ has a zero of order $n-p$ at $z = 0$. It follows that

$$h_n(z) = f(z)\left\{\frac{1-\varphi(z)}{1+\varphi(z)}\right\},$$

and a computation yields

$$\frac{zh_n'(z)}{h_n(z)} = \frac{zf'(z)}{f(z)} - \frac{2z\varphi'(z)}{1-\varphi^2(z)}.$$

The functions $zf'(z)/f(z)$ satisfies the hypotheses of Lemma 1, so from (1) we obtain

$$\operatorname{Re}\left\{\frac{zh_n'(z)}{h_n(z)}\right\} \geqslant \frac{p-(p-2a)|z|}{1+|z|} - \frac{2|z|\,|\varphi'(z)|}{|1-\varphi^2(z)|}.$$

Applying (2) with $m = n-p$ yields

$$\frac{|z|\,|\varphi'(z)|}{|1-\varphi^2(z)|} \leqslant \frac{(n-p)|z|^{n-p}\left(1-|\varphi(z)|^2\right)}{\left(1-|z|^{2(n-p)}\right)\left(1-|\varphi(z)|^2\right)} = \frac{(n-p)|z|^{n-p}}{1-|z|^{2(n-p)}},$$

and thus

$$\operatorname{Re}\left\{\frac{zh_n'(z)}{h_n(z)}\right\} \geqslant \frac{p-(p-2a)|z|}{1+|z|} - \frac{2(n-p)|z|^{n-p}}{1-|z|^{2(n-p)}}$$

$$= \frac{\lambda(p,a,n;|z|)}{(1+|z|)(1-|z|^{2(n-p)})}.$$

The last expression is positive for $|z| < r(p,a,n)$, and so $h_n(z)$ is p-valent and starlike for $|z| < r(p,a,n)$.

If $f(z) = z^p/(1+z)^{2(p-a)}$ and $g_n(z) = -2z^{n-p}f(z)/(1+z^{n-p})$, then $\operatorname{Re}\{zh_n'(z)/h_n(z)\} = 0$ for $z = r(p,a,n)$. Thus, for this choice of $f(z)$ and $g_n(z)$ the function $h_n(z)$ is not p-valent and starlike in $|z| < r$ for any $r > r(p,a,n)$.

COROLLARY. $\operatorname{Re}\{g_n(z)/z^p\} > -1$, $z \in E$, *then* $h_n(z) = z^p + g_n(z)$ *is p-valent and starlike for*

$$|z| < \left\{\frac{p-n+\sqrt{(n-p)^2+p^2}}{p}\right\}^{1/(n-p)}.$$

Proof. Letting $f(z) = z^p$ in Theorem 1 yields

$$\operatorname{Re}\left\{\frac{zh_n'(z)}{h_n(z)}\right\} \geqslant p - \frac{2(n-p)|z|^{n-p}}{1-|z|^{2(n-p)}} = \frac{p-2(n-p)|z|^{n-p}-p|z|^{2(n-p)}}{1-|z|^{2(n-p)}},$$

so $\operatorname{Re}\{zh_n'(z)/h_n(z)\} > 0$ for

$$|z| < \left\{\frac{n-p-\sqrt{(n-p)^2+p^2}}{-p}\right\}^{1/(n-p)}.$$

The radius is exact for the choice $g_n(z) = -2z^n/(1+z^{n-p})$.

THEOREM 2. *If* $f(z)\in S^*(p, a)$ *and* $|g_n(z)| \leqslant |f(z)|$, $z\in E$, *then* $h_n(z) = f(z)+g_n(z)$ *is p-valent and starlike for* $|z| < R(p, a, n)$, *where* $R(p, a, n)$ *is the smallest positive root of*

$$\mu(p, a, n; x) = p-(p-2a)x-nx^{n-p}-(n+2a-2p)x^{n-p+1} = 0.$$

Proof. Let $\varphi(z) = g_n(z)/f(z) = \sum_{n-p}^{\infty} d_k z^k$. Then $\varphi(z)$ is regular and bounded by 1 in E, and

$$h_n(z) = f(z)\{1+\varphi(z)\}.$$

A computation yields

$$\frac{zh_n'(z)}{h_n(z)} = \frac{zf'(z)}{f(z)}+\frac{z\varphi'(z)}{1+\varphi(z)},$$

and so

$$\mathrm{Re}\left\{\frac{zh_n'(z)}{h_n(z)}\right\} \geqslant \frac{p-(p-2a)|z|}{1+|z|}-\frac{|z|\,|\varphi'(z)|}{|1+\varphi(z)|}.$$

Applying (2) and (3) with $m = n-p$ we get

$$\frac{|z|\,|\varphi'(z)|}{|1+\varphi(z)|} \leqslant \frac{(n-p)|z|^{n-p}\big(1-|\varphi(z)|^2\big)}{\big(1-|z|^{2(n-p)}\big)\big(1-|\varphi(z)|\big)} \leqslant \frac{(n-p)|z|^{n-p}(1+|z|^{n-p})}{1-|z|^{2(n-p)}}.$$

Thus,

$$\mathrm{Re}\left\{\frac{zh_n'(z)}{h_n(z)}\right\} \geqslant \frac{p-(p-2a)|z|}{1+|z|}-\frac{(n-p)|z|^{n-p}}{1-|z|^{n-p}} = \frac{\mu(p, a, n; |z|)}{(1+|z|)(1-|z|^{n-p})},$$

and the last expression is positive for $|z| < R(p, a, n)$.

To see that the result is sharp let $f(z) = z^p/(1+z)^{2(p-a)}$ and $g_n(z) = -z^{n-p}f(z)$, in which case, $\mathrm{Re}\{zh_n'(z)/h_n(z)\} = 0$ for $z = R(p, a, n)$.

COROLLARY. *If* $|g_n(z)| \leqslant |z|^p$, $z\in E$, *then* $h_n(z) = z^p+g_n(z)$ *is p-valent and starlike for* $|z| < (p/n)^{1/(n-p)}$.

Proof. Letting $f(z) = z^p$ in Theorem 2 yields

$$\mathrm{Re}\left\{\frac{zh_n'(z)}{h_n(z)}\right\} \geqslant p-\frac{(n-p)|z|^{n-p}}{1-|z|^{n-p}} = \frac{p-n|z|^{n-p}}{1-|z|^{n-p}},$$

and the result follows. The radius $(p/n)^{1/(n-p)}$ is exact for the choice $g_n(z) = -z^n$.

4. Throughout this section $h(z)$ denotes a function of the form $h(z) = z^p+\sum_{p+1}^{\infty} c_k z^k$ which is regular in E and vanishes only at $z = 0$. We assume $0 < \beta \leqslant 1$.

THEOREM 3. *If* $f(z)\in S^*(p, a)$ *and* $\operatorname{Re}\{h(z)/f(z)\}^{1/\beta} > 0$, $z\in E$, *then* $h(z)$ *is p-valent and starlike for*

$$|z| < \sigma(p, a, \beta) = \frac{(p+\beta-a)-\sqrt{(p+\beta-a)^2-p(p-2a)}}{p-2a},$$

where the expression above is defined by its limit when $a = p/2$.

Proof. With the appropriate choice of the branch, $\{h(z)/f(z)\}^{1/\beta}$ takes the value 1 at $z = 0$ and is subordinate to $(1-z)/(1+z)$. Thus

$$h(z) = f(z)\left\{\frac{1-\varphi(z)}{1+\varphi(z)}\right\}^{\beta},$$

where $\varphi(z)$ is regular and bounded by 1 in E, $\varphi(0) = 0$. A computation yields

$$\frac{zh_n'(z)}{h_n(z)} = \frac{zf'(z)}{f(z)} - \frac{2\beta z\varphi'(z)}{1-\varphi^2(z)},$$

and from (2) with $m = 1$ we get

$$\operatorname{Re}\left\{\frac{zh_n'(z)}{h_n(z)}\right\} \geqslant \frac{p-(p-2a)|z|}{1+|z|} - \frac{2\beta|z|\,|\varphi'(z)|}{1-|\varphi(z)|^2}$$

$$\geqslant \frac{p-(p-2a)|z|}{1+|z|} - \frac{2\beta|z|}{1-|z|^2}$$

$$= \frac{p-2(p+\beta-a)|z|+(p-2a)|z|^2}{1-|z|^2}.$$

The last expression is positive for $|z| < \sigma(p, a, \beta)$, and so $h_n(z)$ is p-valent and starlike for $|z| < \sigma(p, a, \beta)$.

The radius $\sigma(p, a, \beta)$ is exact for the choice $f(z) = z^p/(1+z)^{2(p-a)}$ and $h(z) = f(z)\{(1-z)/(1+z)\}^{\beta}$.

COROLLARY. *If* $\operatorname{Re}\{h(z)/z^p\}^{1/\beta} > 0$, $z\in E$, *then* $h(z)$ *is p-valent and starlike for*

$$|z| < \frac{-\beta+\sqrt{\beta^2+p^2}}{p}.$$

Proof. If $f(z) = z^p$ in Theorem 3, then

$$\operatorname{Re}\left\{\frac{zh'(z)}{h(z)}\right\} \geqslant p - \frac{2\beta|z|}{1-|z|^2} = \frac{p-2\beta|z|-p|z|^2}{1-|z|^2},$$

and the result follows.

The radius is exact for the choice $h(z) = z^p\{(1-z)/(1+z)\}^{\beta}$.

THEOREM 4. *If* $f(z)\in S^*(p,a)$ *and* $|\{h(z)/f(z)\}^{1/\beta}-1|<1$, $z\in E$, *then* $h(z)$ *is p-valent and starlike for*

$$|z|<\Sigma(p,a,\beta)=\frac{(2p+\beta-2a)-\sqrt{(2p+\beta-2a)^2-4p(p-2a-\beta)}}{2(p-2a-\beta)},$$

where the expression above is defined by its limit when $a=(p-\beta)/2$.

Proof. With the appropriate choice of the branch, $\{h(z)/f(z)\}^{1/\beta}$ takes the value 1 at $z=0$ and is subordinate to $1+z$. Thus,

$$h(z)=f(z)\{1+\varphi(z)\}^{\beta},$$

where $\varphi(z)$ is regular and bounded by 1 in E, $\varphi(0)=0$. It follows that

$$\frac{zh'(z)}{h(z)}=\frac{zf'(z)}{f(z)}+\frac{\beta z\varphi'(z)}{1+\varphi(z)},$$

and from (2) and (3) with $m=1$ we get

$$\begin{aligned}\operatorname{Re}\left\{\frac{zh'(z)}{h(z)}\right\} &\geqslant \frac{p-(1-2a)|z|}{1+|z|}-\frac{\beta|z|\,|\varphi'(z)|}{1-|\varphi(z)|}\\ &\geqslant \frac{p-(p-2a)|z|}{1+|z|}-\frac{\beta|z|}{1-|z|}\\ &=\frac{p-(2p+\beta-2a)|z|+(p-2a-\beta)|z|^2}{1-|z|^2}.\end{aligned}$$

The last expressions is positive for $|z|<\Sigma(p,a,\beta)$.

The radius $\Sigma(p,a,\beta)$ is exact for the choice $f(z)=z^p/(1+z)^{2(p-a)}$ and $h(z)=f(z)(1-z)^{\beta}$.

COROLLARY. *If* $|\{h(z)/z^p\}^{1/\beta}-1|<1$, $z\in E$, *then* $h(z)$ *is p-valent and starlike for* $|z|<p/(p+\beta)$.

Proof. Letting $f(z)=z^p$ in Theorem 4 yields

$$\operatorname{Re}\left\{\frac{zh'(z)}{h(z)}\right\}\geqslant p-\frac{\beta|z|}{1-|z|}=\frac{p-(p+\beta)|z|}{1-|z|},$$

and so $\operatorname{Re}\{zh'(z)/h(z)\}>0$ for $|z|<p/(p+\beta)$.

The radius $p/(p+\beta)$ is exact for the choice $h(z)=z^p(1-z)^{\beta}$.

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UNIVERSITY OF KENTUCKY
LEXINGTON, Kentucky

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