ON THE SPACE OF VECTOR-VALUED FUNCTIONS
INTEGRABLE WITH RESPECT TO THE WHITE NOISE

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1. Introduction and notation. In this paper $E$ denotes a Banach space with a norm $\| \cdot \|$, and $E'$ stands for the dual space of $E$. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A random vector $X: \Omega \rightarrow E$ denotes a strongly measurable function.

$L^p(\Omega, P; E)$, $0 \leq p \leq \infty$, denotes the Fréchet space (Banach space if $1 \leq p \leq \infty$) of random vectors $X: \Omega \rightarrow E$ for which

$$\|X\|_p \overset{df}{=} E \frac{\|X\|}{1 + \|X\|} \quad \text{if } p = 0,$$

$$\|X\|_p \overset{df}{=} (E \|X\|^p)^{1/p} < \infty \quad \text{if } 0 < p < \infty, \quad r = \max\{1, p\},$$

and

$$\|X\|_\infty \overset{df}{=} \esssup_{\Omega} \|X\| < \infty \quad \text{if } p = \infty.$$

A random vector is called symmetric if $P(X \in A) = P(-X \in A)$ for every $A \in \mathcal{B}_E$, where $\mathcal{B}_E$ is the Borel $\sigma$-algebra on $E$.

A probability measure on $(E, \mathcal{B}_E)$ defined by

$$\mu(A) = P(X \in A) \quad \text{for every } A \in \mathcal{B}_E$$

is called the distribution law of $X$. The characteristic functional of a measure $\nu$ on $(E, \mathcal{B}_E)$ is defined by

$$\hat{\nu}(x') = \int_E \exp[i \langle x', x \rangle] \nu(dx) \quad \text{for every } x' \in E'.$$

A random vector $X$ is gaussian if, for each $x' \in E'$, $\langle x', X \rangle$ is a gaussian random variable. $X$ is pregaussian if there exists a gaussian measure $\gamma$ on $(E, \mathcal{B}_E)$ such that

$$\hat{\gamma}(x') = \exp \left[ -\frac{1}{2} B \langle x', X \rangle^2 \right].$$
Let \((S, \Sigma, \mu)\) be a measure space with \(\mu(S) = 1\). A mapping
\[
W: \Sigma \rightarrow L^0(\Omega, \mathcal{F}, P)
\]
is called a \textit{gaussian random measure} on \((S, \Sigma, \mu)\) if
(a) for every sequence \(A_1, A_2, \ldots\) of disjoint sets from \(\Sigma\) we have
\[
W(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} W(A_n),
\]
where the series converges with probability 1;
(b) for every sequence \(A_1, \ldots, A_n\) of disjoint sets from \(\Sigma\) the random variables \(W(A_1), \ldots, W(A_n)\) are independent;
(c) for every \(A \in \Sigma\), \(W(A)\) has a normal distribution with mean 0 and variance \(\mu(A)\).

Let \(f: S \rightarrow E\) be a simple function,
\[
f = \sum_{i=1}^{n} x_i 1_{A_i},
\]
where \(A_i \in \Sigma\) are disjoint, \(x_i \in E\), \(i = 1, \ldots, n\). For each \(B \in \Sigma\) we set
\[
\int_B f dW = \sum_{i=1}^{n} x_i W(A_i \cap B).
\]
\(\int_S (\cdot) dW\) is a linear operator on the vector space of \(E\)-valued simple functions on \(S\) with values in the space of gaussian random vectors in \(L^2(\Omega, P; E)\). If \(E\) is of type 2 (see Section 5 for the definition of Banach spaces of type 2), then, as Hoffmann-Jørgensen and Pisier [5] have shown, there exists a unique extension of this operator on \(L^2(S, \mu; E)\).

Using the idea of Urbanik and Woyczyński [13] we define a random integral of vector-valued functions with values in any Banach space.

The purpose of this paper is to study the class of random integrable functions with respect to a gaussian random measure. Section 2 of this paper contains the basic properties of random integrable functions. In Section 3 we give some counterexamples which show the difference between the random integral for Banach space valued functions and the random integral for Hilbert space valued functions. Section 4 contains a characterization of random integrable functions. In Section 5 we study properties of the random integral which depends on geometry of a Banach space. In Section 6 we investigate some properties of the space of functions which are integrable with respect to a gaussian random measure.

2. \textbf{A gaussian random integral of vector-valued functions.} Let \((S, \Sigma, \mu)\) be a measure space \(\mu(S) = 1\) and let \(W\) be a gaussian random
measure on \((S, \Sigma, \mu)\). Let \(E\) be a Banach space and let \(f: S \to E\) be a simple function, i.e.

\[ f = \sum_{i=1}^{n} x_i 1_{A_i}, \]

where \(A_i \in \Sigma\) are pairwise disjoint and \(x_i \in E\). For every \(B \in \Sigma\) we set

\[ \int_B f dW \stackrel{\text{at}}{=} \sum_{i=1}^{n} x_i W(A_i \cap B). \]

**Definition 2.1.** A strongly measurable function \(f: S \to E\) is said to be **integrable with respect to a gaussian random measure** \(W\) if there exists a sequence of simple functions \(f_n: S \to E\) such that

1. \(f_n \to f\) in \(\mu\),
2. \(\int_B f_n dW\) converges in \(P\) for every \(B \in \Sigma\).

Then for \(B \in \Sigma\) we set

\[ \int_B f dW \stackrel{\text{at}}{=} \text{P-lim} \int_B f_n dW. \]

This integral is uniquely determined. Definition 2.1 is the extension of the definition of Urbanik and Woyczyński (cf. [13]) of random integral in the case of Banach valued functions.

Let \(\mathcal{L}(S, W; E) \subset L^0(S, \mu; E)\) denote the set of all integrable functions with respect to the gaussian random measure \(W\). The set \(\mathcal{L}(S, W; E)\) is a vector space. Moreover, \(\mathcal{L}(S, W; E)\) is a Fréchet space with \(F\)-norm

\[ |||f|||_o = ||f||_o + \left\| \int_S f dW \right\|_o, \]

and the set of simple functions is dense in \(\mathcal{L}(S, W; E)\).

The following properties are immediate consequences of Definition 2.1 and we omit their proofs.

**Proposition 2.1.** (1) For every \(f, g \in \mathcal{L}(S, W; E)\) and \(B \in \Sigma\) we have

\[ \int_B (f + g) dW = \int_B f dW + \int_B g dW \quad \text{P-a.e.} \]

(2) Let \(E, F\) be Banach spaces and let \(A: E \to F\) be a continuous linear operator. If \(f \in \mathcal{L}(S, W; E)\), then \(Af \in \mathcal{L}(S, W; F)\) and

\[ A \int_B f dW = \int_B Af dW \quad \text{P-a.e.} \]

(2') In particular, if \(x' \in E'\), then for every \(f \in \mathcal{L}(S, W; E)\) we have

\[ \langle x', f \rangle \in \mathcal{L}(S, W; \mathbb{R}) \quad \text{and} \quad \int_B \langle x', f \rangle dW = \int_B \langle x', f \rangle dW \quad \text{P-a.e.} \]
(3) If \( f \in \mathcal{L}(S, W; E) \), then for every \( B \in \Sigma \) we have \( f1_B \in \mathcal{L}(S, W; E) \) and

\[
\int_B f \, dW = \int_S f1_B \, dW.
\]

**Proposition 2.2.** If \( (f_n) \subset L^0(S, \mu; E) \) is a sequence of simple functions such that \( f_n \to f \) in \( \mu \) and \( \int_S f_n \, dW \) converges in \( P \), then \( f \in \mathcal{L}(S, W; E) \).

**Proof.** Let \( B \in \Sigma \). The random vectors

\[
\int_B (f_n - f_m) \, dW \quad \text{and} \quad \int_{S \setminus B} (f_n - f_m) \, dW
\]

are independent and symmetric. By the inequality

\[
P\left( \left\| \int_B (f_n - f_m) \, dW \right\| > \varepsilon \right) \leq 2P\left( \left\| \int_S (f_n - f_m) \, dW \right\| > \varepsilon \right)
\]

for every \( \varepsilon > 0 \), we infer that \( \int_B f_n \, dW \) converges in \( P \).

In the sequel we shall use some other \( F \)-norms in the space \( \mathcal{L}(S, W; E) \), equivalent to the original one. First we prove the following lemma:

**Lemma 2.1.** Let \( X_n \) be symmetric gaussian random vectors such that \( X_n \to X \) in \( P \). Then \( X \) is a symmetric gaussian random vector and, for every \( p, 0 < p < \infty \),

\[
E \|X_n - X\|^p \to 0 \quad \text{as} \quad n \to \infty.
\]

**Proof.** Let \( Y \) be a symmetric gaussian random vector. Combining the results of [3], [4] and [7], we infer that for every \( p, q \in (0, \infty) \) there exists a constant \( C_{p,q} \) (dependent only on \( E \)) such that

\[
(\mathbb{E} \|Y\|^p)^{1/p} \leq C_{p,q} (\mathbb{E} \|Y\|^q)^{1/q}.
\]

Inequality (2) from 1.5 in [8] applied to \( \|Y\|^2 \) gives

\[
P\left( \|Y\|^2 > t \mathbb{E} \|Y\|^2 \right) \geq (1 - t)^2 \frac{(\mathbb{E} \|Y\|^2)^2}{\mathbb{E} \|Y\|^4} \geq (1 - t)^2 C_{4,2}^{-2}
\]

for every \( t \in (0, 1) \). Putting \( Y = X_n \) and \( t = 1/2 \) we get

\[
P\left( \|X_n\|^2 > \frac{1}{2} \mathbb{E} \|X_n\|^2 \right) \geq 4^{-1} C_{4,2}^{-2} = \text{const} \quad \text{for every} \quad n \in \mathbb{N}.
\]

By the assumptions of the lemma and inequality (2) we have

\[
\sup_n \mathbb{E} \|X_n\|^2 < \infty,
\]

and, by (1),

\[
\sup_n \mathbb{E} \|X_n\|^p < \infty \quad \text{for each} \quad p \geq 1.
\]
By the Fatou lemma, $E \|X\|^p < \infty$ for each $p > 0$. Let $p \in (0, \infty)$ be fixed. Using the Hölder inequality we infer that for each $\varepsilon > 0$

$$E \|X_n - X\|^p \leq \int_{\|X_n - X\|^p > \varepsilon} \|X_n - X\|^p \, dP + \int_{\|X_n - X\|^p < \varepsilon} \|X_n - X\|^p \, dP$$

$$\leq C^{p/(p+1)} \left[ E \left\{ \|X_n - X\|^p > \varepsilon \right\} \right]^{p/(p+1)} + \varepsilon$$

where

$$C = \sup_n E \|X_n\|^{p+1}.$$ 

Therefore, $E \|X_n - X\|^p \to 0$ as $n \to \infty$. The fact that $X$ is gaussian is trivial.

By Lemma 2.1 and Definition 2.1 we obtain immediately

**Corollary 2.1.** Let for every $f \in \mathcal{L}(S, W; E)$

$$\|f\|_p \overset{\text{def}}{=} \|f\|_0 + \left\| \int_S f \, dW \right\|_p, \quad 0 < p < \infty.$$ 

Then for every $p \in (0, \infty)$ the $F$-norm $\|\cdot\|_p$ is equivalent to $\|\cdot\|_0$.

**Remark 2.1.** In the definition of $\|\cdot\|_p$ the first component $\|\cdot\|_0$ cannot be omitted in general (see Example 3.3 in Section 3).

3. **Some counterexamples.** The examples given in the sequel show that the basic properties of the space $\mathcal{L}(S, W; E)$ and of the random integral, which are evidently fulfilled in Hilbert spaces, are not usually fulfilled in arbitrary Banach spaces.

**Proposition 3.1.** Let

$$f = \sum_{n=1}^{\infty} x_n 1_{A_n},$$

where $x_n \in E$, $A_n \in \Sigma$ are disjoint, $n = 1, 2, \ldots$ and $\bigcup_{n=1}^{\infty} A_n = S$. Then $f \in \mathcal{L}(S, W; E)$ if and only if $\sum_{n=1}^{\infty} x_n W(A_n)$ converges a.s. Moreover,

$$\int_S f \, dW = \sum_{n=1}^{\infty} x_n W(A_n).$$

The proposition is an immediate consequence of Proposition 2.2 and the theorem of Ito and Nisio [6].

Let $S = [0, 1]$, $\Sigma = \mathcal{B}_{[0,1]}$, $\mu(\text{dt}) = \text{dt}$ and let $W$ be the random measure generated by the Brownian motion $\omega$ on $[0, 1]$, i.e. $W([s, t]) = \omega_t - \omega_s$ for $s, t \in [0, 1]$.

**Example 3.1.** There exist a Banach space $E$ and $f : [0, 1] \to E$ strongly measurable such that

$$\sup_{t \in [0, 1]} \|f(t)\| = 1 \quad \text{and} \quad f \notin \mathcal{L}([0, 1], W; E).$$
Let $E = l^p$, $1 \leq p < 2$, and let, for $1 < r < 2p^{-1}$,

$$c = \sum_{n=1}^{\infty} n^{-r}.$$ 

Let $e_1, e_2, \ldots$ be the standard Schauder basis in $l^p$. Let

$$t_0 = 0, \quad t_n = e^{-1} \sum_{i=1}^{n} i^{-r} \text{ for } n \geq 1.$$ 

We set

$$f(t) = e_n \text{ for } t \in (t_{n-1}, t_n], \quad f(0) = e_1.$$ 

Then $\|f(t)\| = 1$ for each $t \in [0, 1]$.

We assume that $f \in \mathcal{L}(S, W; l^p)$. Then, by Proposition 3.1,

$$\int_0^1 f dW = \sum_{n=1}^{\infty} e_n (w_{n} - w_{n-1}) \text{ a.s.}$$ 

Moreover,

$$E \left\| \int_0^1 f dW \right\|^p = \sum_{n=1}^{\infty} E \left| w_{n} - w_{n-1} \right|^p = \sum_{n=1}^{\infty} E \left| \frac{X}{(cn^r)^{1/2}} \right|^p$$

$$= E |X|^p e^{-p/2} \sum_{n=1}^{\infty} \frac{1}{n^{rp/2}} = \infty,$$

where $X$ is a gaussian random variable, $X \sim \mathcal{N}(0, 1)$. This gives a contradiction.

**Example 3.2.** There exist a Banach space $E$ and a sequence $(f_n)$ \( \in \mathcal{L}([0, 1], W; E) \) such that

$$\sup_{t \in [0, 1]} \|f_n(t)\| \to 0 \quad \text{as } n \to \infty$$

and \( \int_0^1 f_n dW \) diverges in $P$.

Let $E = l^p$, $1 \leq p < 2$, and let $e_1, e_2, \ldots$ be the standard Schauder basis in $l^p$. We set

$$f_n(t) = n^{-r/p} \sum_{k=1}^{n} e_k I_{((k-1)/n, k/n]}(t) \quad \text{for } t \in (0, 1],$$

$$f_n(0) = n^{-r/p} e_1,$$

where $0 < r < 1 - p/2$.

We obtain

$$\left\| \int_0^1 f_n dW \right\|^p = E \left( \sum_{k=1}^{n} n^{-r} |w_{k/n} - w_{(k-1)/n}|^p \right) = E |X|^p n^{1-p/2-r} \to \infty \quad \text{as } n \to \infty.$$
(X \sim \mathcal{N}(0, 1)) and
\[ \sup_t \|f_n(t)\| = n^{-r/p} \rightarrow 0. \]

**Example 3.3.** There exist a Banach space E and a sequence \( (f_n) \subset \mathcal{L}([0, 1], W; E) \) such that
\[ \int_0^1 f_n \, dW \rightarrow 0 \quad \text{in } P \]
and \( (f_n) \) diverges in \( \mu \).

Let \( E = \ell^p, \ p > 2, \) and let \( e_1, e_2, \ldots \) be the standard Schauder basis in \( \ell^p \). We set
\[ f_n(t) = n^{-r/p} \sum_{k=1}^n e_k 1_{(k-1)/n, k/n]}(t) \quad \text{for } t \in (0, 1], \]
\[ f_n(0) = n^{-r/p} e_1, \]
where \( 0 < r < p/2 - 1. \)

We have
\[ E \| \int_0^1 f_n \, dW \|^p = n^{1-p/2+r} E |X|^p \rightarrow 0 \quad \text{as } n \rightarrow \infty. \]

(X \sim \mathcal{N}(0, 1)) and
\[ \|f_n(t)\| = n^{-r/p} \rightarrow \infty \quad \text{for each } t \in [0, 1]. \]

**Example 3.4.** There exist a Banach space \( E \) a function \( f \)
in \( \mathcal{L}([0, 1], W; E) \) such that for each \( r > 0 \)
\[ \int_0^1 \|f(t)\|^r \, dt = \infty. \]

Let \( E = \ell^p, \ p > 2, \) and let \( e_1, e_2, \ldots \) be the standard Schauder basis in \( \ell^p \). Let
\[ c_n = (\log^{1/2} n) \log \log n \quad \text{if } n \geq 3 \text{ and } c_1 = c_2 = 1. \]

We have
\[ c = \sum_{n=1}^\infty (nc_n^2)^{-1} < \infty. \]

Let
\[ t_0 = 0, \quad t_n = \frac{1}{c} \sum_{k=1}^n (kc_k^2)^{-1} \quad \text{for } n \geq 1. \]

We set
\[ f(t) = \sum_{n=1}^\infty c_n e_n 1_{(t_{n-1}, t_n]}(t). \]
By Proposition 3.1 we have
\[ E \left\| \int_0^1 f \, dW \right\|^p = \sum_{n=1}^{\infty} c_n^p E|w_n - w_{n-1}|^p = E|X|^p \sum_{n=1}^{\infty} c_n^p (t_n - t_{n-1})^{p/2} < \infty, \]
where \( X \sim \mathcal{N}(0, 1) \), but
\[ \int_0^1 \|f(t)\|^r \, dt = \sum_{n=1}^{\infty} c_n^r (t_n - t_{n-1}) = \sum_{n=1}^{\infty} (nc_n^{2-r})^{-1} = \infty \]
for each \( r > 0 \).

4. Characterization of elements in \( L(S, W; E) \). In this section the characterization of elements in \( L(S, W; E) \) is given. As some applications of this characterization we study the definition of random integral in the sense of Pettis and we give a description of \( L(S, W; E) \) for \( 1 \leq p < \infty \).

Suppose that \( f \in L(S, W; E) \); then for each \( x' \in E' \)
\[ \langle x', f \rangle \in L(S, W; R) = L^2(S, \mu; R) \]
and, by Corollary 5.31 in [12], \( f \) is integrable in the sense of Pettis.

**Lemma 4.1.** If \( f: S \to E \) is strongly measurable and integrable in the sense of Pettis, then there exists a sequence of finite \( \sigma \)-algebras \( \Sigma_1 \subset \Sigma_2 \subset \ldots \subset \Sigma \) such that
\[ E_n(f | \Sigma_n) \to f \] strongly \( \mu \)-a.s.,
where \( E_n(f | \Sigma_n) \) denotes the weak conditional expectation.

**Proof.** Notice first that if \( \Sigma' \) is a finite sub-\( \sigma \)-algebra of \( \Sigma \), then it is generated by atoms \( A_1, \ldots, A_n \) and
\[ E_n(f | \Sigma') = \sum_{i=1}^{n} \left[ \mu(A_i) \right]^{-1} \int_{A_i} f \, d\mu \mathbb{1}_{A_i}, \]
where \( \int f \, d\mu \) denotes the Pettis integral and we take \( \left[ \mu(A_i) \right]^{-1} = 0 \) if \( \mu(A_i) = 0 \).

Now, since \( f \) is strongly measurable, for each \( n \in \mathbb{N} \) there exists a disjoint covering of \( S \) by sets \( A_i^n, \ldots, A_{k_n}^n, A_{k_n + 1}^n \in \Sigma \) such that
\[ \mu\left( \bigcup_{i=1}^{k_n} A_i^n \right) > 1 - 2^{-n} \]
and
\[ \text{diam} \{f(A_i^n)\} = \sup \{ ||f(t) - f(s)|| : t, s \in A_i^n \} \leq 2^{-n} \]
for \( i = 1, \ldots, k_n \).

Write
\[ \Sigma_1 = \sigma(A_1^1, \ldots, A_{k_1}^1, A_{k_1 + 1}^1) \]
and
\[ \Sigma_n = \sigma(A^n_1, \ldots, A^n_n, A : A \in \Sigma_{n-1}) \quad \text{for } n \geq 2. \]

Then \( \Sigma_1 \subset \Sigma_2 \subset \ldots \subset \Sigma \) and \( \Sigma_n \) is finite for each \( n \).

Let \( n \in \mathbb{N} \) be fixed and let \( B_1, \ldots, B_n \in \Sigma_n \) be generators of \( \Sigma_n \), i.e. \( B_i \) are pairwise disjoint, \( \bigcup_{i=1}^{r_n} B_i = \mathcal{S} \) and each set in \( \Sigma_n \) is the union of some number of sets \( B_i \). Let
\[ I = \{ 1 \leq i \leq r_n : B_i \subset \bigcup_{j=1}^{k_n} A^n_j \}. \]

We have
\[ [\mu(B_i)]^{-1} \int_{B_i} f d\mu \in \text{conv} \{ f(B_i) \} \]
(see, e.g., [11], Theorem 3.1) and for each \( i \in I \) there exists a \( j, 1 \leq j \leq k_n \), such that \( B_i \subset A^n_j \), which gives
\[ \text{diam} \left( \text{conv} \{ f(B_i) \} \right) \leq \text{diam} \{ f(A^n_j) \} \leq 2^{-n}. \]

Thus for each \( i \in I \) and \( s \in B_i \) we have
\[ \left\| f(s) - [\mu(B_i)]^{-1} \int_{B_i} f d\mu \right\| \leq 2^{-n} \]
and
\[ \mu \left( \bigcup_{i \in I} B_i \right) \geq 1 - 2^{-n}. \]

We obtain
\[ \mu \{ \| f - E_\mu(f|\Sigma_n) \| \leq 2^{-n} \} \geq 1 - 2^{-n}, \]
and this completes the proof.

**Theorem 4.1.** A strongly measurable function \( f : \mathcal{S} \to \mathcal{E} \) is integrable with respect to a gaussian random measure \( W \) associated with \( \mu \) if and only if

(i) \( \int_{\mathcal{S}} \langle x', f \rangle^2 d\mu < \infty \) for each \( x' \in \mathcal{E}' \),

(ii) \( \varphi(x') = \exp \left[ -\frac{1}{2} \int_{\mathcal{S}} \langle x', f \rangle^2 d\mu \right] \) is the characteristic functional of some measure on \( (\mathcal{E}, \mathcal{B}_\mathcal{E}) \).

**Proof.** It follows non-trivially from (i) and (ii) that \( f \) is integrable with respect to \( W \), i.e. there exist simple functions \( f_n \) such that \( f_n \to f \) in \( \mu \) and \( \int_{\mathcal{S}} f_n dW \) converges in \( P \) (Proposition 2.2). By (i), \( f \) is Pettis integrable (see [12], Corollary 5.31) and, by Lemma 4.1, there exists a sequence of finite \( \sigma \)-algebras \( \Sigma_1 \subset \Sigma_2 \subset \ldots \subset \Sigma \) such that

\[ E_\mu(f|\Sigma_n) \to f \text{ strongly } \mu \text{-a.s.} \]

Put \( f_n = E_\mu(f|\Sigma_n) \). The functions \( \{f_n\} \) are simple and we have to prove that \( \int_{\mathcal{S}} f_n dW \) converges in \( P \). First we prove that \( \int_{\mathcal{S}} f_n dW \) are partial sums of some series of independent gaussian random vectors.
Let 
\[ X_1 = \int_s f_1 dW, \quad X_n = \int_s f_n dW - \int_s f_{n-1} dW \text{ for } n \geq 2. \]

Since $W$ is Gaussian by the linearity of random integral, it is sufficient to prove that 
\[ \mathbb{E} \langle x', X_n \rangle \langle y', X_m \rangle = 0 \quad \text{for each } x', y' \in E' \text{ and } n \neq m. \]

We have 
\[
\mathbb{E} \langle x', X_n \rangle \langle y', X_m \rangle = \mathbb{E} \int_s \langle x', f_n - f_{n-1} \rangle dW \int_s \langle y', f_m - f_{m-1} \rangle dW \\
= \int_s \langle x', f_n - f_{n-1} \rangle \langle y', f_m - f_{m-1} \rangle d\mu = 0,
\]
since $\{f_n, \Sigma_n\}$ is a martingale in $E$.

We infer that 
\[
\int_s f_n dW = \sum_{i=1}^n X_i
\]
and $\{X_i\}_{i \geq 1}$ are independent symmetric Gaussian random vectors.

Let $x' \in E'$. We have 
\[
\mathbb{E} \exp \left[ i \left\langle x', \sum_{i=1}^n X_i \right\rangle \right] = \mathbb{E} \exp \left[ i \left\langle x', \int_s f_n dW \right\rangle \right] \\
= \exp \left[ -\frac{1}{2} \int_s \langle x', f_n \rangle^2 d\mu \right] \mathbb{E} \exp \left[ -\frac{1}{2} \int_s \langle x', f \rangle^2 d\mu \right] \quad \text{as } n \to \infty.
\]

By (ii) and by the theorem of Ito and Nisio [6], the sums 
\[
\sum_{i=1}^n X_i = \int_s f_n dW
\]
converge a.s. This completes the proof.

**Corollary 4.1.** Let the function $f: S \to E$ be strongly measurable. Then $f \in \mathcal{L}(S, W; E)$ if and only if $f$ is pregaussian (as a random element on the probability space $(S, \Sigma, \mu)$).

The random integral with vector-valued functions may be defined in the sense of Pettis (see also [15]). Namely, a strongly measurable function $f: S \to E$ is weakly integrable with respect to a Gaussian random measure $W$ if for each $x' \in E'$ the integral $\int \langle x', f \rangle dW$ exists and for each $B \in \Sigma$ there exists a random vector $X_B$ such that for each $x' \in E'$
\[
\langle x', X_B \rangle = \int_{\tilde{B}} \langle x', f \rangle dW \quad \text{a.s.}
\]
In view of Theorem 4.1 we obtain

**Corollary 4.2.** A function \( f: S \to E \) is weakly integrable with respect to a Gaussian random measure \( W \) if and only if \( f \in L(S, W; E) \).

**Proof.** Indeed, let \( f \) be weakly integrable. Then

\[
\int_S \langle x', f \rangle^2 \, d\mu < \infty
\]

and

\[
\varphi(x') = \exp\left[-\frac{1}{2} \int_S \langle x', f \rangle^2 \, d\mu\right]
\]

is the characteristic functional of the random vector \( X_S \). Conditions (i) and (ii) of Theorem 4.1 are fulfilled.

**Corollary 4.3.** Let \( f: S \to \mathcal{V} \), where \( f = (f_n)_{n \geq 1} \), \( 1 \leq p < \infty \), be measurable. Then \( f \) is integrable with respect to a Gaussian random measure \( W \) if and only if

\[
\sum_{n=1}^{\infty} \left( \int_S f_n^2 \, d\mu \right)^{p/2} < \infty.
\]

The corollary is a consequence of Theorem 4.1 and Vakhania’s characterization of covariance operators of Gaussian measures in \( \mathcal{L}^p \) (see [14]).

5. Random integral in the spaces of type and cotype 2. Let \( E \) be a Banach space. We say that \( E \) is of type \( p \), \( p \in (1, 2] \) (cotype \( q \), \( q \in [2, \infty) \)) if for a Rademacher sequence \( (r_n) \) and for every \( (x_n) \in E \) the following implication holds (see [4] and [10]):

if \( \sum_{n=1}^{\infty} \|x_n\|^p < \infty \), then \( \sum_{n=1}^{\infty} r_n x_n \) converges a.e.

(if \( \sum_{n=1}^{\infty} r_n x_n \) converges a.e., then \( \sum_{n=1}^{\infty} \|x_n\|^q < \infty \)).

For example, the spaces \( L^p \) and \( l^p \) are of type \( 2 \) and cotype \( p \) if \( 2 \leq p < \infty \) and of type \( p \) and cotype \( 2 \) if \( 1 \leq p < 2 \).

**Proposition 5.1** (cf. [4] and [10]). The following statements are equivalent:

(a) \( E \) is of type \( p \) (cotype \( q \)).

(b) There exists a constant \( C_1 \) (\( C_2 \)) depending only on \( E \) such that

\[
E \left\| \sum_{i=1}^{n} \eta_i x_i \right\|^p \leq C_1 \sum_{i=1}^{n} \|x_i\|^p
\]

\[
\left( \sum_{i=1}^{n} \|x_i\|^q \right) \leq C_2 E \left\| \sum_{i=1}^{n} \eta_i x_i \right\|^{q}
\]

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for every $n \geq 1$ and $x_1, \ldots, x_n \in E$, where $(\eta_n)$ is a sequence of independent normally distributed random variables with mean 0 and variance 1.

(c) If $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$, then $\sum_{n=1}^{\infty} \eta_n x_n$ converges a.s.

(if $\sum_{n=1}^{\infty} \eta_n x_n$ converges a.s., then $\sum_{n=1}^{\infty} \|x_n\|^q < \infty$),

where $\eta_n$ are as in (b).

The gaussian random integral, as a linear operator on $L^2(S, \mu; E)$ ($E$ of type 2), was constructed by Hoffmann-Jørgensen and Pisier [5]. The next proposition follows immediately from Proposition 5.1 (see also [5]).

**Proposition 5.2.** Let $E$ be a Banach space of type 2. Then

$L^2(S, \mu; E) \subset \mathcal{L}(S, W; E)$.

Moreover, the identity embedding

$I: L^2(S, \mu; E) \to \mathcal{L}(S, W; E)$

is continuous and there exists a constant $C$ such that for each $f \in L^2(S, \mu; E)$

$$E \left\| \int_S f \, dW \right\|^2 \leq C \int_S \|f\|^2 \, d\mu.$$  

**Proposition 5.3.** Let $E$ be a Banach space of cotype 2. Then

$\mathcal{L}(S, W; E) \subset L^2(S, \mu; E)$.

Moreover, the identity embedding

$I: \mathcal{L}(S, W; E) \to L^2(S, \mu; E)$

is continuous and there exists a constant $C$ such that for each $f \in \mathcal{L}(S, W; E)$

$$\int_S \|f\|^2 \, d\mu \leq C E \left\| \int_S f \, dW \right\|^2. \quad (3)$$

**Proof.** Let $E$ be of cotype 2. By Proposition 5.1 we obtain (3) for each simple function $f$. If $f \in \mathcal{L}(S, W; E)$, then there exists a sequence $(f_n)$ of simple functions such that $f_n \to f$ in $\mu$ and $\int_S f_n \, dW$ converges in $P$. Therefore, by Lemma 2.1,

$$\int_S (f_n - f_m) \, dW \to 0 \quad \text{in} \quad L^2(\Omega, P; E) \quad \text{as} \quad n, m \to \infty,$$

and so $f_n \to f$ in $L^2(S, \mu; E)$.

This shows that inequality (3) holds for each $f \in \mathcal{L}(S, W; E)$ and that the identity embedding of $\mathcal{L}(S, W; E)$ in $L^2(S, \mu; E)$ is continuous.
Corollary 5.1. If $E$ is of cotype 2, then $\mathcal{L}(S, W; E)$ is the Banach space with the norm

$$
|||f|||_p = \left( E \left\| \int_S f dW \right\|^p \right)^{1/p}, \quad p \geq 1
$$

($|||\cdot|||_p$ are equivalent as norms on $\mathcal{L}(S, W; E)$ for each $p \geq 1$).

For example, we consider the space $\mathcal{L}(S, W; l^p)$ for $1 \leq p \leq 2$. Let $F$ be a Banach space and $p \geq 1$. By $l^p(F)$ we denote the Banach space of all sequences $(x_n) \in F$ for which

$$
\|(x_n)\| = \left( \sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p} < \infty.
$$

Corollary 5.2. Let $1 \leq p \leq 2$. The space $\mathcal{L}(S, W; l^p)$ is isomorphic and isometric with $l^p(L^2(S, \mu; R))$.

Proof. If $f = (f_n) \in \mathcal{L}(S, W; l^p)$, then, in view of Corollary 4.3, $f \in l^p(L^2(S, \mu; R))$.

Conversely, if $f \in l^p(L^2(S, \mu; R))$, then by the inequality

$$
\int_S |f_n|^p d\mu \leq \left( \int_S |f|^2 d\mu \right)^{p/2} \quad \text{for } 1 \leq p \leq 2
$$

we have $f \in L^p(S, \mu; l^p)$ and, by Corollary 4.3, $f \in \mathcal{L}(S, W; l^p)$. Since $l^p$ is of cotype 2 ($1 \leq p \leq 2$), by Corollary 5.1 the space $\mathcal{L}(S, W; l^p)$ is a Banach space with the norm

$$
|||f|||_p = \left( E \left\| \int_S f dW \right\|^p \right)^{1/p}.
$$

We have

$$
|||f|||_p^p = E \left\| \int_S f dW \right\|^p = \sum_n E \left\| \int_S f_n dW \right\|^p = c_p \sum_n \left[ E \left( \int_S f_n dW \right)^2 \right]^{p/2}
$$

$$
= c_p \sum_n \left( \int_S |f_n|^2 d\mu \right)^{p/2} = c_p \|f\|_{L^p(L^2(S, \mu; R))}^p,
$$

where

$$
c_p = (2\pi)^{-1/2} \left( \int_{-\infty}^{\infty} |x|^p \exp \left[ -\frac{x^2}{2} \right] dx \right).
$$

We infer that the operator $I(f) = c_p^{1/p} f$ forms an isometry between $\mathcal{L}(S, W; l^p)$ and $l^p(L^2(S, \mu; R))$.

6. Some questions concerning the space $\mathcal{L}(S, W; E)$. Examples 3.1 and 3.4 show that bounded functions are not always integrable with respect to a gaussian random measure and that $f \in \mathcal{L}(S, W; E)$ does not always imply that $f \in L^r(S, \mu; E)$ for some $r > 0$. Corollary 5.1 shows that if $E$ is of cotype 2, then $\mathcal{L}(S, W; E)$ is the Banach space.
In this section we answer the following questions:
For which Banach spaces $E$ are the following conditions satisfied:
(A) $L^\infty(S, \mu; E) \subseteq \mathcal{L}(S, W; E)$,
(B) $\mathcal{L}(S, W; E) \subseteq \bigcup_{r > 0} L^r(S, \mu; E)$,
(C) $\mathcal{L}(S, W; E)$ admits a Banach norm equivalent to $|||\cdot|||_0$ !
In this section we assume that $\mu$ is atomless.
The following proposition answers the question (A).

**Proposition 6.1.** The following conditions are equivalent:
(a) $L^\infty(S, \mu; E) \subseteq \mathcal{L}(S, W; E)$;
(b) $E$ is of type 2.

**Proof.** Let $L^\infty(S, \mu; E) \subseteq \mathcal{L}(S, W; E)$. We have to show that $E$
is of type 2. Suppose that this is not true. Then there exists a sequence $(x_n) \subseteq E$,
$$\sum_{n=1}^\infty ||x_n||^2 < \infty,$$
such that the series $\sum_{n=1}^\infty x_n \eta_n$ diverges a.e., where $\eta_n$ are independent
random variables, $\eta_n \sim N(0, 1), n = 1, 2, \ldots$ (Proposition 5.1).
Let
$$c = \sum_{n} ||x_n||^2.$$ 
Since $\mu$ is atomless, there exists a partition $A_1, A_2, \ldots$ of $S$ such that
$A_n$ are disjoint, $A_n \in \Sigma$ and $\mu(A_n) = c^{-1} ||x_n||^2, n = 1, 2, \ldots$
We set
$$f = \sum_{n=1}^\infty x_n [\mu(A_n)]^{-1/2} 1_{A_n}.$$ 
We have $||f(t)|| = c^{1/2}$ for each $t \in S$, and so $f \in L^\infty(S, \mu; E)$.
On the other hand, the series
$$\sum_{n} x_n [\mu(A_n)]^{-1/2} W(A_n)$$
diverges a.e. and, consequently, $f \notin \mathcal{L}(S, W; E)$, which gives a contradiction (Proposition 3.1).
The inverse implication of this proposition follows from Proposition 5.2.
The following proposition answers the question (B).

**Proposition 6.2.** The following conditions are equivalent:
(a) $\mathcal{L}(S, W; E) \subseteq \bigcup_{p > 0} L^p(S, \mu; E)$;
(b) $E$ is of cotype 2.
In order to prove this proposition we need the following lemma:

**Lemma 6.1.** If \((a_n)\) is a sequence of real numbers such that

\[
0 \leq a_n \leq M \quad \text{and} \quad \sum_{n=1}^{\infty} a_n = \infty,
\]

then there exists a sequence \(b_n \not\to 0\) such that

\[
\sum_{n=1}^{\infty} a_n b_n < \infty
\]

and, for each \(\varepsilon > 0\),

\[
\sum_{n=1}^{\infty} a_n b_n^{1-\varepsilon} = \infty.
\]

**Proof.** Let

\[
n_1 = \max \left\{ n : M \leq \sum_{i=1}^{n} a_i < 2M \right\}, \quad n_k = \max \left\{ n : M \leq \sum_{i=n_{k-1}+1}^{n} a_i < 2M \right\}.
\]

We have \(0 = n_0 < n_1 < n_2 < \ldots\)

Let \(b_j = (k \log^2 k)^{-1}\) for \(n_{k-1} < j \leq n_k, \ k = 1, 2, \ldots\) Then

\[
\sum_{n=1}^{\infty} a_n b_n = \sum_{k=1}^{\infty} (k \log^2 k)^{-1} \sum_{j=n_{k-1}+1}^{n_k} a_j \leq 2M \sum_{k=1}^{\infty} (k \log^2 k)^{-1} < \infty
\]

and

\[
\sum_{n=1}^{\infty} a_n b_n^{1-\varepsilon} = \sum_{k=1}^{\infty} (k \log^2 k)^{1-\varepsilon} \sum_{j=n_{k-1}+1}^{n_k} a_j
\]

\[
\geq M \sum_{k=1}^{\infty} (k \log^2 k)^{1-\varepsilon} = \infty \quad \text{for} \ \varepsilon > 0.
\]

**Proof of Proposition 6.2.** Suppose that \(E\) is not of cotype 2. Then there exists a sequence \((x_n) \subset E\) such that \(\sum x_n \eta_n\) converges a.e., where \(\eta_n\) are independent, \(\eta_n \sim N(0, 1)\), and \(\sum_{n} \|x_n\|^2 = \infty\) (Proposition 5.1).

The sequence \((\|x_n\|)\) is bounded. Indeed, the convergence of the series \(\sum x_n \eta_n\) implies that, for each \(x' \in E'\),

\[
E \langle x', \sum_{n} x_n \eta_n \rangle^2 = \sum_{n} \langle x', x_n \rangle^2 < \infty,
\]

so \((x_n)\) is weakly bounded and, therefore, by the Banach-Steinhaus theorem \((x_n)\) is strongly bounded.

By Lemma 6.1 there exists a sequence \(b_n \not\to 0\) such that

\[
\sum_{n} b_n \|x_n\|^2 < \infty
\]
and, for each \( \varepsilon > 0 \),
\[
\sum_n b_n^{1-\varepsilon} \|x_n\|^2 = \infty.
\]

Let
\[
b = \sum_n b_n \|x_n\|^2
\]

and let \((A_n) \subset \Sigma\) be a partition of \( S \) such that \( \mu(A_n) = b^{-1} b_n \|x_n\|^2 \).

Setting
\[
f(t) = \sum_{n=1} x_n [\mu(A_n)]^{-1/2} 1_{A_n}(t),
\]
we infer that
\[
\int_S f dW = \sum_n x_n [\mu(A_n)]^{-1/2} W(A_n) \text{ converges a.e.}
\]

Thus \( f \in \mathcal{L}(S, W; E) \).

Let \( r > 0 \). Then
\[
\int_S \|f\|^r d\mu = \sum_n \|x_n\|^r [\mu(A_n)]^{-r/2} \mu(A_n) = b^{r/2-1} \sum_n b_n^{1-r/2} \|x_n\|^2 = \infty.
\]

The inverse implication of this proposition follows from Proposition 5.3.

The following proposition answers the question \((O)\).

PROPOSITION 6.3. The following conditions are equivalent:
(a) \( \mathcal{L}(S, W; E) \) admits a Banach norm equivalent to \( \|\cdot\|_0 \);
(b) \( E \) is of cotype 2.

In the proof of this proposition we use the following lemma:

LEMMA 6.2. Suppose that \( E \) is not of cotype 2. Then there exists a sequence of simple functions \((f_n)\) such that \( \int_S f_n dW \) are bounded in \( P \) and
\[
\inf_{s \in S} \|f_n(s)\| \to \infty \quad \text{as } n \to \infty.
\]

Proof. Since \( E \) is not of cotype 2, then there exists a sequence \((x_n) \subset E\) such that \( \sum_n \eta_n x_n \) converges a.s. and
\[
\sum_n \|x_n\|^2 = \infty
\]
\((\eta_n \sim \mathcal{N}(0, 1) \text{ independent}).\) Put
\[
a_n = \sum_{i=1}^n \|x_i\|^2.
\]

Then \( a_n \to \infty \).

Let \( n \in \mathbb{N} \) be fixed and let \( A_1, \ldots, A_n \) be a partition of \( S \) such that
\[
\mu(A_i) = a_n^{-1} \|x_i\|^2, \ i = 1, \ldots, n.
\]
We write

\[ f_n = a_n^{1/2} \sum_{i=1}^{n} ||x_i||^{-1} x_i I_{A_i}. \]

We have

\[ \int_S f_n dW = a_n^{1/2} \sum_{i=1}^{n} ||x_i||^{-1} x_i W(A_i) = \sum_{i=1}^{n} x_i \eta_i \] in law,

and for each \( s \in S \)

\[ ||f_n(s)|| = a_n^{1/2} \to \infty \quad \text{as} \quad n \to \infty. \]

**Proof of Proposition 6.3.** Suppose that there exists a Banach norm \( \cdot \) on \( L(S, W; E) \) equivalent to \( |||\cdot|||_1 \) (which is equivalent to \( |||\cdot|||_0 \)). Thus there exist \( r_1 \) and \( r_2, \ 0 < r_1 < \frac{1}{2}, \ r_2 > 0 \), such that \( S_0 \supset S_1 \supset S_2 \), where

\[ S_0 = \{ f : |f| < 1 \}, \]
\[ S_1 = \{ f : |||f|||_1 < r_1 \}, \]
\[ S_2 = \{ f : |f| < r_2 \}. \]

Suppose, to the contrary, that \( E \) is not of cotype 2. Then by Lemma 6.2 there exist simple functions \( f_n \) such that

\[ b_n^2 = \inf_{s \in S} ||f_n(s)|| \to \infty \]

and \( \int_S f_n dW \) are bounded in \( P \).

Put \( g_n = b_n^{-1} f_n, \ n = 1, 2, \ldots \)

We have

\[ \inf_{s \in S} ||g_n(s)|| = b_n \to \infty \quad \text{and} \quad \int_S g_n dW \to 0 \text{ in } P. \]

Let \( A_1, \ldots, A_k \) be a partition of \( S \) such that \( \mu(A_i) < \frac{1}{2} r_1 \).

We write \( h_i^n = g_n I_{A_i}, \ i = 1, \ldots, k. \) Then

\[ ||h_i^n||_0 < \mu(A_i) < \frac{1}{2} r_1. \]

Let \( N \) be a positive integer such that for each \( n \geq N \)

\[ E \left\| \int_S g_n dW \right\| < \frac{1}{2} r_1. \]

Take fixed \( n \geq N. \) We have

\[ E \left\| \int_S h_i^n dW \right\| = E \left\| \int_{A_i} g_n dW \right\| \leq E \left\| \int_S g_n dW \right\| < \frac{1}{2} r_1, \]

whence

\[ |||h_i^n|||_1 = ||h_i^n||_0 + E \left\| \int_S h_i^n dW \right\| < r_1, \]
so $h^n_i \in S_1$ for each $i = 1, \ldots, k$ and $n \geq N$. Consequently,

$$k^{-1}g_n = k^{-1} \sum_{i=1}^{k} h^n_i \in S_0 \quad \text{and} \quad k^{-1}r_2g_n \in S_2,$$

whence $k^{-1}r_2g_n \in S_1$. We obtain a contradiction, since

$$\inf_{s \in S} ||k^{-1}r_2g_n(s)|| \to \infty \quad \text{as} \quad n \to \infty.$$

**Corollary 6.1.** $L(S, W; E) = L^2(S, \mu; E)$ if and only if $E$ is isomorphic to a Hilbert space.

This follows from the result of Kwapien [9], which states that $E$ is isomorphic to a Hilbert space if and only if $E$ is of type 2 and of cotype 2, and from Propositions 6.1 and 6.2.

**Remark 6.1.** Chobanian and Tarieladze (Theorem 4.1 in [1]) have shown that if there exists a $p > 0$ such that each pregaussian measure on a Banach space $E$ has the $p$-th strong order, then $E$ is of cotype 2. From Theorem 4.1 and Proposition 6.2 we obtain:

*If each pregaussian measure $\mu$ on $E$ has some $p$-th strong order, then $E$ is of cotype 2.*

This strengthens the above-mentioned result of Chobanian and Tarieladze [1].

**REFERENCES**


