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ON A LATTICE CHARACTERIZATION OF HILBERT SPACES

BY

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Let V be a vector space (in general, infinite-dimensional) over D, where D is one of the following division rings: R (real numbers), C (complex numbers), or Q (quaternions). Let L be a given lattice of subspaces (linear manifolds) of V. In this paper we consider the problem under what conditions there exists a D-valued inner product $(.,.)$ on $V\times V$ such that under $(.,.)$ V becomes a Hilbert space over D and L coincides with the lattice of all closed subspaces of the Hilbert space V. It is clear that a necessary condition is that L contain all finite-dimensional subspaces of V and admit an orthocomplementation, i. e. a map $^{\perp}$: $L\to L$ such that $a^{\perp}\vee a = 1$ (the greatest element of L), $a\leqslant b$ implies $b^{\perp}\leqslant a^{\perp}$, and $(a^{\perp})^{\perp} = a$. Birkhoff and von Neumann showed in [2] that for a finite-dimensional V this condition is essentially sufficient as well. Namely, they showed that if V is a vector space of dimension $n\geqslant 3$ over $D=R$ or Q, and ξ is an orthocomplementation of the lattice of all subspaces of V, then there exists an inner product $(.,.)$ converting V into a Hilbert space such that ξ coincides with the orthocomplementation induced by $(.,.)$, and $(.,.)$ is uniquely determined by ξ up to a multiplicative positive real number. If $D=C$, then in order to obtain a Hilbert space in the above way we must additionally assume that the orthocomplementation ξ is regular in the sense that the anti-automorphism of C associated with ξ (and uniquely determined by it) is the complex conjugation. A proof and discussion of the Birkhoff and von Neumann theorem can also be found in [8] (Theorem 4.7). A generalization of this theorem to the infinite-dimensional case was given by Kakutani and Mackey [3]. They showed that if V is an infinite-dimensional Banach space over D and L is the lattice of all closed subspaces of V, then for any orthocomplementation $M\to M^{\perp}$ of L there exists a D-valued inner product $(.,.)$ on $V\times V$ such that (i) under $(.,.)$ V becomes a Hilbert space over D; (ii) the topology of V induced by the norm associated with $(.,.)$ coincides with its original topology; (iii) the map $M\to M^{\perp}$ coincides with the orthocomplementation induced by $(.,.)$ (for the proof of this theorem see also [8], Theorem 7.1).

In the present paper we show that a theorem similar to the above one holds even without the assumption that V is a Banach space. The proof of this theorem will be based upon a lemma used in the proof of Kakutani and Mackey's theorem given by V. S. Varadarajan and a theorem proved by I. Amemiya and H. Araki (both to be formulated in the sequel, the latter in connection with [7]). First let us state the lemma.

LEMMA 1 (Kakutani-Mackey [3], Varadarajan [8], Lemma 7.2). *Let V be a vector space of infinite dimension over D. Let L be a lattice of subspaces of V such that*

(i) *L contains all finite-dimensional subspaces of V;*

(ii) *if $M, N \in L$ and at least one of them is finite-dimensional, then $M \vee N = M + N$.*

Suppose that $M \to M^{\perp}$ is an orthocomplementation in L. Then, for any non-zero vector $w_0 \in W$, there exist an involutive anti-automorphism θ of D and a θ-symmetric θ-bilinear form $(.,.)$ on $V \times V$ such that

(i) $(w_0, w_0) = 1$, *and*

(ii) $(x, y) = 0$ *if and only if* $x \in (D \cdot y)^{\perp}$.

θ and $(.,.)$ are uniquely determined. The form $(.,.)$ is definite, i. e. $(x, x) = 0$ if and only if $x = 0$.

Observe that since $M+N$ is the least subspace of V containing both M and N, condition (ii) can be replaced by the following: $M+N \in L$ whenever $M, N \in L$ and at least one of them is finite-dimensional.

As shown in the proof of Lemma 1 given in [8], in case $D = R$, θ must be the identity and, in case $D = Q$, θ must be the canonical conjugation. Hence, in both cases, $(.,.)$ is a positive definite inner product. To conclude that $(.,.)$ is a positive definite inner product also in the case where $D = C$, we must know that θ is the complex conjugation, which is true if and only if θ is continuous. Consequently, in case $D = C$ we additionally assume that the given orthocomplementation is regular in the sense that the associated anti-automorphism, uniquely determined by the orthocomplementation, is continuous, and hence is the complex conjugation. It is interesting that, as shown in the proof of Kakutani and Mackey's theorem, in the case where V is a complex Banach space and L the lattice of all closed subspaces of V any orthocomplementation is regular.

Before we state our main theorem let us recall that a lattice L with orthocomplementation $^{\perp}$ is said to be *orthomodular* if $a \leqslant b$ implies that there is a $c \in L$ such that $c \perp a$ (i. e. $c \leqslant a^{\perp}$) and $a \vee c = b$ (see [5], p. 132).

THEOREM 1. *Let V be an infinite-dimensional vector space over D, where D is one of the following division rings: R (real numbers), C (complex numbers), or Q (quaternions). Let L be a lattice of subspaces of V such that*

(i) *L contains all finite-dimensional subspaces of V;*

(ii) *if* $M, N \in L$ *and at least one of them is finite-dimensional, then* $M+N \in L$; *and*

(iii) L *is closed under set intersection, i. e. for any family* $\{M_a\} \subset L$, *the set intersection* $\bigcap M_a$ *also belongs to* L.

Suppose $^\perp$ *is an orthocomplementation in* L (*regular in case* $D = C$) *with respect to which* L *is an orthomodular lattice. Then there exists a* D-*valued inner product* $(.,.)$ *on* $V \times V$ *such that under* $(.,.)$ V *is a Hilbert space over* D *and* L *coincides with the lattice of all closed subspaces of the Hilbert space* V. *The inner product* $(.,.)$ *is determined uniquely up to a multiplicative real number.*

Note that condition (iii) implies that L is a complete lattice ($\bigcap M_a$ is the meet $\bigwedge M_a$ in L, and since de Morgan's laws hold in an orthocomplemented lattice, $\bigvee M_a = (\bigcap M_a^\perp)^\perp$ exists in L).

Proof. Taking into account that in case $D = C$ the orthocomplementation is regular, in view of the discussion preceding the theorem, we infer from Lemma 1 that there exists a D-valued inner product $(.,.)$ on $V \times V$ (determined up to a multiplicative real number) such that $(x, y) = 0$ if and only if $x \in (D\cdot y)^\perp$. Hence V becomes an inner product space.

We show that L coincides with the lattice of all $(.,.)$-closed subspaces of V.

Let us recall that a subspace $N \subset V$ is said to be $(.,.)$-*closed* if $N'' = (N')' = N$, where $N' = \{x \in V\colon (x, y) = 0 \text{ for all } y \in N\}$.

Let $M \in L$ and let $\{D\cdot y_a\}$ be the set of all one-dimensional subspaces of V contained in M. We have $M = \bigvee(D\cdot y_a)$, where $\bigvee$ denotes the lattice join in L (M is the set union of all one-dimensional subspaces contained in M). By de Morgan's law we infer that

$$M^\perp = \bigwedge(D\cdot y_a)^\perp = \bigcap(D\cdot y_a)^\perp$$

(set intersection). Hence $x \in M^\perp$ if and only if $x \in (D\cdot y_a)^\perp$ for all $y_a \in M$, i. e. if and only if $(x, y_a) = 0$ for all $y_a \in M$. Hence $M'' = (M^\perp)^\perp = M$, i. e. every member of L is $(.,.)$-closed.

Conversely, let $N \subset V$ be $(.,.)$-closed, i. e. $N'' = N$. We first show that $N' \in L$. Let $\{D\cdot y_a\}$ be the set of all one-dimensional subspaces of V contained in N. By (i) we have $D\cdot y_a \in L$ for all a. By the definition of N' we have $x \in N'$ if and only if $(x, y_a) = 0$ for all $y_a \in N$, i. e., by Lemma 1, $x \in N'$ if and only if $x \in (D\cdot y_a)^\perp$ for all a, that is, $x \in N'$ if and only if $x \in \bigcap(D\cdot y_a)^\perp$. Since (iii) holds, we obtain $\bigcap(D\cdot y_a)^\perp \in L$, i. e. $N' \in L$. Since for $M \in L$ we have $M' = M^\perp \in L$, we conclude that $N'' = N \in L$. Hence we have shown that L coincides with the lattice of all $(.,.)$-closed subspaces of V, and the orthocomplementation induced by $(.,.)$ coincides with the original orthocomplementation $M \to M^\perp$ given in L.

We shall now apply a theorem proved by Amemiya and Araki [1] stating that if V is an inner product space (with the inner product $(.,.)$), then the lattice of all $(.,.)$-closed subspaces of V is orthomodular if and only if V is complete with respect to the topology induced by the norm associated with the inner product, that is, if and only if V is a Hilbert space with respect to $(.,.)$ (proof of this theorem see [5], Theorem 34.9). In this case it is well known that the lattice of all $(.,.)$-closed subspaces of the Hilbert space V coincides with the lattice of all closed subspaces of V. Since the lattice L of all $(.,.)$-closed subspaces of V is orthomodular by assumption, we conclude, by Amemiya and Araki's theorem, that V is a Hilbert space with respect to the inner product $(.,.)$ and L coincides with the lattice of all closed subspaces of V. This completes the proof of Theorem 1.

It is clear that the converse of Theorem 1 also holds; that is, the lattice of all closed subspaces of every Hilbert space satisfies conditions (i)-(iii) of Theorem 1 and admits a regular orthocomplementation (namely, the one induced by the inner product).

We can restate Theorem 1 in another form without postulating orthocomplementation and orthomodularity explicitly, but using the notion of a full set of probability measures.

Let L be a σ-complete lattice. We say that a set F of mappings from L into the closed interval $[0,1]$ is a *full set* of probability measures on L provided that

1° $f(a) \leqslant f(b)$ for all $f \in F$ implies $a \leqslant b$;

2° for each $a \in L$ there is a $b \in L$ such that $f(a)+f(b) = 1$ for all $f \in F$;

3° for every sequence $a_1, a_2, \ldots$ (finite or infinite) satisfying $f(a_i)+f(a_j) \leqslant 1$ for $i \neq j$ and all $f \in F$, we have

$$f(a_1 \vee a_2 \vee \ldots) = f(a_1)+f(a_2)+\ldots \quad \text{for all } f \in F.$$

It is easy to show that if a σ-complete lattice admits a full set of probability measures, then it admits an orthocomplementation with respect to which it is an orthomodular lattice. In fact, observe first that 1° implies that $a = b$ if and only if $f(a) = f(b)$ for all $f \in F$. Consequently, we can define a map $a \to a^\perp$ of L into L by asserting that $b = a^\perp$ if and only if $f(a)+f(b) = 1$ for all $f \in F$. This map constitutes an orthocomplementation on L. Namely, $f(a \vee a^\perp) = f(b \vee b^\perp) = 1$ for all $a, b \in L$, so that $a \vee a^\perp$ is the greatest element of L. Since $a^{\perp\perp} = a$ and $a \leqslant b$ if and only if $b^\perp \leqslant a^\perp$, de Morgan's laws hold in L and $a \wedge a^\perp$ is the least element of L. To show that L is orthomodular, let $a \leqslant b$. Then, by (iii),

$$\begin{aligned} f\big(a \vee (a \vee b^\perp)^\perp\big) &= f(a)+f\big((a \vee b^\perp)^\perp\big) \\ &= f(a)+1-\big(f(a)+1-f(b)\big) = f(b) \quad \text{for all } f \in F, \end{aligned}$$

hence

$$b = a \vee (a \vee b^{\perp})^{\perp},$$

which shows that L is orthomodular (see [5], Theorem 29.13).

Let us note that the notion of a full set of probability measures can be defined and discussed in a more general setting of partially ordered sets (see, e. g., [6]).

A full set of probability measures is said to be *regular* if the orthocomplementation induced by it is regular.

We have the following theorem:

THEOREM 2. *Let V be an infinite-dimensional vector space over D and let L be a lattice of subspaces of V satisfying conditions* (i)-(iii) *of Theorem* 1. *Then L is the lattice of closed subspaces of a Hilbert space based on V if and only if L admits a full set of probability measures* (*regular in case $D = C$*).

Proof. In view of the discussion preceding the theorem, if L admits a full set of probability measures, then L admits an orthocomplementation with respect to which it is orthomodular, so that from Theorem 1 it follows that L is the lattice of all closed subspaces of a Hilbert space. Conversely, assume that L is the lattice of all closed subspaces of the Hilbert space V with an inner product $(.,.)$. We are going to show that L admits a full set of probability measures. Let S be the unit sphere of V, and for each $M \in L$, let P_M be the orthogonal projection onto M. For each $u \in S$, we define a function f_u from L into $[0,1]$ by setting $f_u(M) = (P_M u, u)$ for all $M \in L$. It is well known that $F = \{f_u \colon u \in S\}$ is a full set of probability measures on L ($f(M)+f(N) = 1$ for all $f \in F$ is equivalent to $N = M^{\perp}$, and $f(M) + f(N) \leqslant 1$ is equivalent to $M \perp N$). Thus the proof of Theorem 2 is complete.

Theorem 2 shows that, roughly speaking, the lattice of all closed subspaces of a Hilbert space can be characterized among other lattices of subspaces of a vector space by the fact that it admits a full set of probability measures. Conditions (i)-(iii) are clearly not essential since any lattice of subspaces can be extended to a lattice satisfying these conditions, whereas the property of admitting a full set of probability measures cannot, in general, be achieved by extending the underlying lattice (since the lattice of all subspaces of an infinite-dimensional vector space does not have this property).

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