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Four different unknown functions satisfying the triangle mean value property for harmonic polynomials

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Abstract. It is shown that if all quartets $f_0(x), f_1(x), f_2(x), f_3(x)$ are real-valued continuous functions of a complex variable x in the whole complex plane and satisfy the functional equation $f_0(x+y)+f_1(x+\theta y)+f_2(x+\theta^2 y) = 3f_3(x)$ for all complex variables x and y, where θ is a primitive cube root of unity, then all f_j have the triangle mean value property for harmonic polynomials.

1. Introduction. The problem to be considered in this note is the determination of all quartets $f_0(x)$, $f_1(x)$, $f_2(x)$, $f_3(x)$ of real-valued continuous functions of a complex variable x in the whole complex plane and satisfying

$$\text{(M)} \qquad \sum_{j=0}^{2} f_j(x+\theta^j y) = 3f_3(x),$$

for all complex variables x and y, where θ is a primitive cube root of unity.

If all f_j, $j = 0, 1, 2, 3$, are equal to one and the same function f, then f is said to have the *triangle mean value property*. That is, for every x_0, the value $f(x_0)$ is the arithmetic mean of the value of f at the vertices of any equilateral triangle whose centre is at x_0.

It is known [5] that if f satisfies the triangle mean value property, then f is a harmonic polynomial of degree $\leqslant 2$. For equations similar to (M), but with all unknown functions equal, see [1]–[5] (among others).

The problem is solved by showing that each f_j, $j = 0, 1, 2, 3$, has the triangle mean value property. Hence the following theorem is a generalization of the triangle mean value property.

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THEOREM. *If all quartets $f_0(x)$, $f_1(x)$, $f_2(x)$, $f_3(x)$ are real-valued continuous functions of a complex variable x in the whole complex plane and satisfy* (M) *for all complex variables x and y, then all f_0, f_1, f_2, f_3 have the triangle mean value property, and hence are harmonic polynomials of degree* $\leqslant 2$.

2. Proof of the Theorem. Simple computations show that (M) implies

$$(\mathrm{M}^*) \qquad \sum_{j=0}^{2} f_j(x+\theta^j y) = \sum_{j=0}^{2} f_j(x), \qquad \sum_{j=0}^{2} f_j(x) = 3f_3(x),$$

for all x and y.

It is convenient to introduce translation operators T_j^y and T_j^0 defined by

$$T_j^y f(x) = f_j(x+y), \qquad T_j^0 f(x) = f_j(x)$$

for $j = 0, 1, 2$.

Consider an equilateral triangle ABC centred at x with the vertices $x+y$, $x+\theta y$, $x+\theta^2 y$, denoted briefly by $\triangle\, ABC$. Take the midpoints D, E and F of sides AB, BC and CA respectively to obtain four equilateral triangles $\triangle ADF$, $\triangle DBE$, $\triangle DEF$ and $\triangle FEC$. Then, from the first equation of (M^*) and the five equilateral triangles, since (M^*) is satisfied for all x and y, one obtains the following three equations:

$$\begin{aligned}
(T_0^y + T_1^{-\frac{1}{2}\theta^2 y} + T_2^{-\frac{1}{2}\theta y}) f(x) &= (T_0^{\frac{1}{2}y} + T_1^{\frac{1}{2}y} + T_2^{\frac{1}{2}y}) f(x), \\
(T_0^{\theta y} + T_1^{-\frac{1}{2}y} + T_2^{-\frac{1}{2}\theta^2 y}) f(x) &= (T_0^{\frac{1}{2}\theta y} + T_1^{\frac{1}{2}\theta y} + T_2^{\frac{1}{2}\theta y}) f(x), \\
(T_0^{\theta^2 y} + T_1^{-\frac{1}{2}\theta y} + T_2^{-\frac{1}{2}y}) f(x) &= (T_0^{\frac{1}{2}\theta^2 y} + T_1^{\frac{1}{2}\theta^2 y} + T_2^{\frac{1}{2}\theta^2 y}) f(x).
\end{aligned}$$

Add both sides of the above three equations to obtain

$$(1) \qquad \sum_{j=0}^{2} [(T_0^{\theta^j y} + T_1^{-\frac{1}{2}\theta^j y} + T_2^{-\frac{1}{2}\theta^j y}) f(x)] = \sum_{j=0}^{2} [(T_0^{\frac{1}{2}\theta^j y} + T_1^{\frac{1}{2}\theta^j y} + T_2^{\frac{1}{2}\theta^j y}) f(x)].$$

Similarly, one also obtains the equations

$$(2) \qquad \sum_{j=0}^{2} [(T_0^{-\frac{1}{2}\theta^j y} + T_1^{-\frac{1}{2}\theta^j y} + T_2^{-\frac{1}{2}\theta^j y}) f(x)] = 3(T_0^0 + T_1^0 + T_2^0) f(x),$$

$$(3) \qquad \sum_{j=0}^{2} [(T_0^{\frac{1}{2}\theta^j y} + T_1^{\frac{1}{2}\theta^j y} + T_2^{\frac{1}{2}\theta^j y}) f(x)] = 3(T_0^0 + T_1^0 + T_2^0) f(x).$$

By combining equations (1), (3), (2), one obtains

$$\sum_{j=0}^{2} [(T_0^{\theta^j y} + T_1^{-\frac{1}{2}\theta^j y} + T_2^{-\frac{1}{2}\theta^j y}) f(x)] = \sum_{j=0}^{2} [(T_0^{-\frac{1}{2}\theta^j y} + T_1^{-\frac{1}{2}\theta^j y} + T_2^{-\frac{1}{2}\theta^j y}) f(x)],$$

which implies

$$(T_0^y + T_0^{\theta y} + T_0^{\theta^2 y}) f(x) = (T_0^{-\frac{1}{2}y} + T_0^{-\frac{1}{2}\theta y} + T_0^{-\frac{1}{2}\theta^2 y}) f(x).$$

By iteration

$$(T_0^y + T_0^{\theta y} + T_0^{\theta^2 y}) f(x) = (T_0^{(-\frac{1}{2})^n y} + T_0^{(-\frac{1}{2})^n \theta y} + T_0^{(-\frac{1}{2})^n \theta^2 y}) f(x)$$

and by continuity it follows that

$$\lim_{n\to\infty} (T_0^{(-\frac{1}{2})^n y} + T_0^{(-\frac{1}{2})^n \theta y} + T_0^{(-\frac{1}{2})^n \theta^2 y}) f(x) = 3T_0^0 f(x).$$

Hence

$$(T_0^y + T_0^{\theta y} + T_0^{\theta^2 y}) f(x) = 3T_0^0 f(x),$$

which is the required equation

$$\sum_{j=0}^{2} f_0(x + \theta^j y) = 3f_0(x).$$

Similarly, the desired results follow for $j = 1, 2$. Hence, from the second equation of (M^*), the theorem is valid for all $j = 0, 1, 2, 3$.

References

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