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On the functional equation $F(A\cdot B)=F(A)\cdot F(B)$

by M. Kucharzewski (Katowice) and M. Kuczma (Kraków)

Dedicated to our beloved master and teacher
Professor Stanisław Gołąb
on the occasion of his 60-th birthday

Introduction

The functional equation

$$F(A\cdot B)=F(A)\cdot F(B), \tag{1}$$

where A and B denote $n\times n$ matrices, F is an $m\times m$ matrix and the dot denotes the multiplication of matrices, plays an important role in the theory of geometric objects and in the theory of invariants. This equation has been studied by many authors. For some values of n and m it has been solved, under strong assumptions of regularity of the function F, by O. Perron [12], P. Reisch [13] and I. Schur [14], [15], [16].

Without any regularity supposition about the function F equation (1) has been solved for $n=2$, $m=1$ by S. Gołąb [3]. The result of S. Gołąb has been generalized to the case of $m=1$ and an arbitrary n by the first of the authors of the present paper [7] (cf. also [5], [10], [17]).

In the general case (m, n arbitrary) equation (1) has been solved by S. Kurepa [11]. However, though the author makes no assumptions concerning the regularity of the function F, he imposes on F conditions of another kind (invariance under some operations), as a result of which he does not obtain all solutions of equation (1). Thus the problem of solving equation (1) in the general case, without any supposition whatever about the required function F, remains open.

In the present paper we consider equation (1) for $m=n=2$. (¹) Below we shall find all solutions of equation (1) in this case, making no suppositions about the required function F. We shall assume that relation (1) holds only for non-singular matrices A, B.

In the sequel we shall not repeat that we consider equation (1) for $m=n=2$. We shall always use capital letters (Latin as well as Greek)

(¹) We assume that the elements of all the occurring matrices are real.

to denote 2×2 matrices, while scalar values will be denoted by small letters. The only exception will be the letter Δ, which will be used to denote the determinant of matrices (i.e. a scalar). We shall denote by the dot the multiplication of matrices, while to denote the multiplication of scalars we shall not use any sign.

We shall adopt the following short notation:

$$
\begin{gathered}
E \overset{\text{df}}{=} \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad \Theta \overset{\text{df}}{=} \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}, \quad \Xi \overset{\text{df}}{=} \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}, \\
J \overset{\text{df}}{=} \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}, \quad K \overset{\text{df}}{=} \begin{Vmatrix} -1 & 0 \\ 0 & -1 \end{Vmatrix}, \quad M \overset{\text{df}}{=} \begin{Vmatrix} -1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad N \overset{\text{df}}{=} \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix},
\end{gathered} \tag{2}
$$

and

$$
\Phi(x) = \begin{Vmatrix} \varphi(x) & 0 \\ 0 & \varphi(x) \end{Vmatrix}, \tag{3}
$$

where $\varphi(x)$ is an arbitrary function satisfying the functional equation

$$
\varphi(xy) = \varphi(x)\varphi(y)\,. \tag{4}
$$

(Such functions will be called *multiplicative*).

I. Preliminaries

In the sequel we shall frequently make use of the following theorem of S. Gołąb [3]:

If a function $f(A)$ satisfies for all non-singular matrices A, B the functional equation

$$
f(A\cdot B) = f(A)f(B)\,, \tag{5}
$$

then

$$
f(A) = \varphi(\Delta)\,,
$$

where Δ denotes the determinant of the matrix A and $\varphi(x)$ is a multiplicative function (i.e. satisfying equation (4)*).*

We shall also need the following lemmas (see e.g. [2], p. 210 and following):

LEMMA I. *Equation* ([2])

$$
X\cdot X = X \tag{6}
$$

has the solutions

$$
X = E\,, \quad X = \Theta\,, \quad X = C\cdot\Xi\cdot C^{-1}\,,
$$

where C is an arbitrary non-singular matrix.

([2]) Matrices fulfilling (6) are called idempotent (cf. [2]).

LEMMA II. *The equation*

$$X\cdot X=E \tag{7}$$

has the solutions

$$X=E\,,\qquad X=K\,,\qquad X=C\cdot M\cdot C^{-1}\,, \tag{8}$$

where C is an arbitrary non-singular matrix.

DEFINITION. Two matrices A and B will be called *similar* (cf. [2], p. 64) if there exists a non-singular matrix C such that $A=C\cdot B\cdot C^{-1}$.

One can easily verify that if a function $F(A)$ satisfies equation (1) and C is a non-singular matrix, then $C\cdot F(A)\cdot C^{-1}$ also satisfies equation (1) (in other words, a function similar to a solution of (1) is also a solution of (1)). This fact will turn out to be very useful in our further proceedings.

The reader will easily verify the following

LEMMA III. *If $a_{11}\neq 0$, then*

$$A=\begin{Vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{Vmatrix}=\begin{Vmatrix} 1 & 0\\ \dfrac{a_{21}}{a_{11}} & 1\end{Vmatrix}\cdot\begin{Vmatrix} a_{11} & 0\\ 0 & 1\end{Vmatrix}\cdot\begin{Vmatrix} 1 & 0\\ 0 & \dfrac{\Delta}{a_{11}}\end{Vmatrix}\cdot\begin{Vmatrix} 1 & \dfrac{a_{12}}{a_{11}}\\ 0 & 1\end{Vmatrix},$$

where Δ denotes the determinant of the matrix A.

II. Singular solutions of equation (1)

Although for applications only non-singular solutions of equation (1) are important, for the completeness of the results we shall determine also singular solutions.

If the matrix $F(E)$ is singular, then from the relation

$$F(A)=F(E)\cdot F(A) \tag{9}$$

(easily resulting from (1)) it follows that $F(A)$ is singular for all matrices A. On the other hand, if $F(E)$ is non-singular, then $F(A)$ is non-singular for all non-singular matrices A. Indeed, if there existed a non-singular matrix A_0 such that $F(A_0)$ would be singular, then $F(E)=F(A_0)\cdot F(A_0^{-1})$ would also have to be singular, which is a contradiction.

The matrix $F(E)$ is evidently idempotent. Among the solutions of equation (6) only $X=E$ is non-singular. Matrices Θ, Ξ and those similar to the latter are singular and thus represent the values that may be assumed by singular solutions of equation (1) for $A=E$.

If $F(E)=\Theta$, then it follows from relation (9) that

$$F(A)=\Theta \tag{10}$$

for all matrices A. Thus let us suppose that $F(E) = C \cdot \Xi \cdot C^{-1}$ and let us put $F^*(A) = C^{-1} \cdot F(A) \cdot C$. The function $F^*(A)$ evidently satisfies (like $F(A)$) equation (1) and moreover

$$F^*(E) = \Xi \,. \tag{11}$$

Since $F^*(A)$ satisfies (1), we have according to (11)

$$F^*(A) = F^*(A) \cdot \Xi \,, \qquad F^*(A) = \Xi \cdot F^*(A) \,. \tag{12}$$

Let

$$F^*(A) = \begin{Vmatrix} f_{11}(A) & f_{12}(A) \\ f_{21}(A) & f_{22}(A) \end{Vmatrix}.$$

Thus we have from (12) $f_{12}(A) = f_{21}(A) = f_{22}(A) = 0$ and consequently

$$F^*(A) = \begin{Vmatrix} f_{11}(A) & 0 \\ 0 & 0 \end{Vmatrix}.$$

Inserting again $F^*(A)$ in (1) we obtain the relation

$$f_{11}(A \cdot B) = f_{11}(A) f_{11}(B)$$

and thus by the theorem of S. Gołąb we have $f_{11}(A) = \varphi(\Delta)$, where $\Delta = \det A$ and $\varphi(x)$ is a multiplicative function. Of course

$$F(A) = C \cdot F^*(A) \cdot C^{-1} = C \cdot \begin{Vmatrix} \varphi(\Delta) & 0 \\ 0 & 0 \end{Vmatrix} \cdot C^{-1} \,,$$

which may also be written in the form

$$F(A) = \Phi(\Delta) \cdot C \cdot \Xi \cdot C^{-1} \,. \tag{13}$$

Thus finally we have obtained formulae (10) and (13) as the general singular solution of equation (1).

III. The functions $G(x)$ and $H(x)$

§ 1. Now we are going to determine non-singular solutions of equation (1). Thus in the sequel we shall assume that

$$F(E) = E \tag{14}$$

and consequently the matrix $F(A)$ is non-singular for all non-singular matrices A. Equation (1) is supposed to be satisfied for all non-singular matrices A, B.

We put

$$G(x) \stackrel{\mathrm{df}}{=} F\left(\begin{Vmatrix} x & 0 \\ 0 & 1 \end{Vmatrix}\right), \quad x \neq 0 \,, \tag{15}$$

$$H(x) \stackrel{\mathrm{df}}{=} F\left(\begin{Vmatrix} 1 & 0 \\ x & 1 \end{Vmatrix}\right). \tag{16}$$

It follows from (1) and (14) that the function $G(x)$ satisfies the equation

$$G(xy) = G(x) \cdot G(y) \tag{17}$$

and the condition

$$G(1) = E\,, \tag{18}$$

and that the function $H(x)$ satisfies the equation

$$H(x+y) = H(x) \cdot H(y) \tag{19}$$

and the condition

$$H(0) = E\,. \tag{20}$$

Moreover from the evident matrix equalities

$$\begin{Vmatrix} x & 0 \\ 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 1 & 0 \\ xy & 1 \end{Vmatrix} = \begin{Vmatrix} x & 0 \\ xy & 1 \end{Vmatrix} = \begin{Vmatrix} 1 & 0 \\ y & 1 \end{Vmatrix} \cdot \begin{Vmatrix} x & 0 \\ 0 & 1 \end{Vmatrix}$$

follows the relation

$$G(x) \cdot H(xy) = H(y) \cdot G(x)\,, \qquad x \neq 0\,. \tag{21}$$

The continuous solutions of equation (17) have been determined by *A.* Balogh [1]. In quite a similar manner one can find all solutions of equation (27), without any suppositions about the function $G(x)$ (cf. [9]). The continuous solutions of equation (19) (which can easily be reduced to equation (17)) have been given by G. Hajós [4] and F. Karteszi and F. Zigány [6].

§ 2. LEMMA IV. *If a function* $F(A)$ *satisfies (for non-singular* A *and* B*) equation* (1) *and fulfils condition* (14) *and the relations*

$$F\left(\begin{Vmatrix} 1 & 0 \\ 0 & x \end{Vmatrix}\right) = F\left(\begin{Vmatrix} x & 0 \\ 0 & 1 \end{Vmatrix}\right), \quad F\left(\begin{Vmatrix} 1 & x \\ 0 & 1 \end{Vmatrix}\right) = F\left(\begin{Vmatrix} 1 & 0 \\ x & 1 \end{Vmatrix}\right), \tag{22}$$

then for $A = \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}$ *such that* $a_{11} \neq 0$

$$F(A) = G(\Delta)\,, \tag{23}$$

where Δ — *as usualy* — *denotes the determinant of the matrix* A.

Proof. We have on account of lemma III and relations (1), (15), (16) and (22)

$$F(A) = H\left(\frac{a_{21}}{a_{11}}\right) \cdot G(a_{11}) \cdot G\left(\frac{\Delta}{a_{11}}\right) \cdot H\left(\frac{a_{12}}{a_{11}}\right),$$

whence by (17)

$$F(A) = H\left(\frac{a_{21}}{a_{11}}\right) \cdot G(\Delta) \cdot H\left(\frac{a_{12}}{a_{11}}\right). \tag{24}$$

Now let us put in equation (1)

$$A = \begin{Vmatrix} 1/a & 0 \\ 0 & a \end{Vmatrix}, \quad B = \begin{Vmatrix} 1 & 0 \\ 1 & 1 \end{Vmatrix} \tag{25}$$

(a arbitrary, different from zero). We have $A \cdot B = \begin{Vmatrix} 1/a & 0 \\ a & a \end{Vmatrix}$.

It follows from (24) that for A, B defined by (25) we have

$$F(A \cdot B) = H(a^2) \cdot G(1) \cdot H(0),$$

$$F(A) = H(0) \cdot G(1) \cdot H(0), \quad F(B) = H(1) \cdot G(1) \cdot H(0),$$

whence by (18) and (20)

$$H(a^2) = H(1) \overset{\text{df}}{=} H_0. \tag{26}$$

It follows from equation (19) and condition (20) that

$$H(-x) = H^{-1}(x). \tag{27}$$

Moreover, we have by (19) (for $x > 0$ and $y > 0$) and (26)

$$H_0 = H_0 \cdot H_0.$$

Thus according to lemma I H_0, being non-singular, equals E, whence on account of (26), (27) and (20) we obtain

$$H(x) = E \quad \text{for all } x. \tag{28}$$

Formula (23) results immediately from (24) and (28).

IV. Non-singular solutions of equation (1)

§ 1. Since $J \cdot J = E$, according to (14) $F(J)$ fulfils equation (7). Thus by lemma II we may assume that one of the following possibilities

$$F(J) = E, \tag{29}$$

$$F(J) = K, \tag{30}$$

$$F(J) = M \tag{31}$$

occurs, for otherwise we could find a function similar to $F(A)$ satisfying equation (1) and assuming for $X = J$ one of the values E, K, M.

Now we shall prove

LEMMA V. *If a function $F(A)$ satisfies (for non-singular A and B) equation* (1) *and fulfils condition* (14) *and one of relations* (29) *and* (30), *then for $A = \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}$ such that $a_{11} \neq 0$ relation* (23) *holds.*

Proof. It is enough to show that relation (29) or (30) implies relations (22). But it follows immediately from the relations

$$J\cdot\begin{Vmatrix}1&0\\0&x\end{Vmatrix}\cdot J=\begin{Vmatrix}x&0\\0&1\end{Vmatrix}, \tag{32}$$

$$J\cdot\begin{Vmatrix}1&x\\0&1\end{Vmatrix}\cdot J=\begin{Vmatrix}1&0\\x&1\end{Vmatrix} \tag{33}$$

and from the fact that both the matrices E and K commute with every matrix.

The most difficult case $F(J)=M$ has remained to be considered. This will be treated in the next sections.

§ 2. Now we shall discuss case (31). We start with determining the value of $G(-1)$. It evidently fulfils equation (7) and thus, by lemma II, is one of matrices (8). We shall show that neither

$$G(-1)=E\,, \tag{34}$$

nor

$$G(-1)=K \tag{35}$$

can occur. For an indirect proof let us assume for the moment that relation (34) holds. Matrices J and M are similar, and thus also $F(J)$ and $F(M)=G(-1)$ are similar. But this is a contradiction, since M and E are not similar. Similarly, the assumption that (35) holds leads to the nonsensical conclusion that the matrices M and K are similar.

Consequently $G(-1)$ is similar to M. Since the trace and the determinant of a matrix are invariants of similarity, we have

$$G(-1)=\begin{Vmatrix}a&b\\c&-a\end{Vmatrix},\qquad a^2+bc=1\,. \tag{36}$$

From the equality

$$M\cdot J\cdot M\cdot J=J\cdot M\cdot J\cdot M$$

it follows by (31) that

$$G(-1)\cdot M\cdot G(-1)\cdot M=M\cdot G(-1)\cdot M\cdot G(-1)\,.$$

Inserting (36) in the above relation and comparing the corresponding terms, we obtain $ab=ac=0$, whence in view of (36) it follows that $G(-1)$ must have one of the forms

$$G(-1)=\begin{Vmatrix}0&b\\\frac{1}{b}&0\end{Vmatrix}\quad(b\neq 0)\,,\qquad G(-1)=M\,,\qquad G(-1)=N\,. \tag{37}$$

§ 3. Now we shall determine $G(x)$ (still under the assumption that (31) holds). The matrix $G(-1)$ is similar to M:

$$G(-1)=Y\cdot M\cdot Y^{-1}\,. \tag{38}$$

In view of (37) Y may be chosen as one of the matrices

(39) $$Y = \begin{Vmatrix} b & b \\ -1 & 1 \end{Vmatrix}, \quad Y = E, \quad Y = J.$$

We shall make use of this fact later.

We put

(40) $$G^*(x) \stackrel{\mathrm{df}}{=} Y^{-1} \cdot G(x) \cdot Y.$$

$G^*(x)$ satisfies equation (17) and thus

(41) $$G^*(x) \cdot G^*(-1) = G^*(-1) \cdot G^*(x).$$

By (38) we have $G^*(-1) = M$. Let us write

$$G^*(x) = \begin{Vmatrix} g_{11}(x) & g_{12}(x) \\ g_{21}(x) & g_{22}(x) \end{Vmatrix}.$$

From (41) it follows that $g_{12}(x) = g_{21}(x) = 0$. Writing $\psi(x) \stackrel{\mathrm{df}}{=} g_{11}(x)$, $\varphi(x) \stackrel{\mathrm{df}}{=} g_{22}(x)$ we obtain

(42) $$G^*(x) = \begin{Vmatrix} \psi(x) & 0 \\ 0 & \varphi(x) \end{Vmatrix}$$

and hence according to (40)

(43) $$G(x) = Y \cdot \begin{Vmatrix} \psi(x) & 0 \\ 0 & \varphi(x) \end{Vmatrix} \cdot Y^{-1}.$$

Since $G^*(x)$ satisfies (17), the functions $\psi(x)$ and $\varphi(x)$ are multiplicative.

§ 4. Now we shall determine $H(x)$. At first we shall show that

(44) $$\det H(x) \equiv 1.$$

The function $f(A) \stackrel{\mathrm{df}}{=} \det F(A)$ satisfies equation (5) and consequently, on account of S. Gołąb's theorem

$$f(A) = \chi[\det A],$$

where $\chi(x)$ is a multiplicative function. Hence

$$\det H(x) = f\left(\begin{Vmatrix} 1 & 0 \\ x & 1 \end{Vmatrix}\right) = \chi(1).$$

Putting $x = y = 1$ in equation (4) (with χ in the place of φ) we get $\chi^2(1) = \chi(1)$, whence either $\chi(1) = 0$ or $\chi(1) = 1$. But since $H(x)$ is non-singular, necessarily $\chi(1) = 1$, whence (44) follows immediately.

Let us put $x = -1$ in relation (21). We obtain

$$G(-1) \cdot H(-y) = H(y) \cdot G(-1),$$

whence, since $H(-y) = H^{-1}(y)$,

$$G(-1) = H(y) \cdot G(-1) \cdot H(y).$$

By (38) we get hence

$$Y\cdot M\cdot Y^{-1} = H(y)\cdot Y\cdot M\cdot Y^{-1}\cdot H(y)\,. \tag{45}$$

Writing

$$H^*(x) \overset{\text{df}}{=} Y^{-1}\cdot H(x)\cdot Y \tag{46}$$

we have by (45)

$$H^*(y)\cdot M = M\cdot\{H^*(y)\}^{-1}\,. \tag{47}$$

Let

$$H^*(y) = \begin{Vmatrix} h_{11}(y) & h_{12}(y) \\ h_{21}(y) & h_{22}(y) \end{Vmatrix}.$$

Since by (46) and (44) $\det H^*(y) = \det H(y) \equiv 1$,

$$\{H^*(y)\}^{-1} = \begin{Vmatrix} h_{22}(y) & -h_{12}(y) \\ -h_{21}(y) & h_{11}(y) \end{Vmatrix}.$$

Thus we obtain from (47) $h_{11}(y) = h_{22}(y)$. Further, we have on account of (21), (43), (46)

$$Y\cdot\begin{Vmatrix} \psi(x) & 0 \\ 0 & \varphi(x) \end{Vmatrix}\cdot Y^{-1}\cdot Y\cdot\begin{Vmatrix} h_{11}(xy) & h_{12}(xy) \\ h_{21}(xy) & h_{11}(xy) \end{Vmatrix}\cdot Y^{-1}$$
$$= Y\cdot\begin{Vmatrix} h_{11}(y) & h_{12}(y) \\ h_{21}(y) & h_{11}(y) \end{Vmatrix}\cdot Y^{-1}\cdot Y\cdot\begin{Vmatrix} \psi(x) & 0 \\ 0 & \varphi(x) \end{Vmatrix}\cdot Y^{-1}.$$

Hence

$$\begin{Vmatrix} \psi(x)h_{11}(xy) & \psi(x)h_{12}(xy) \\ \varphi(x)h_{21}(xy) & \varphi(x)h_{11}(xy) \end{Vmatrix} = \begin{Vmatrix} \psi(x)h_{11}(y) & \varphi(x)h_{12}(y) \\ \psi(x)h_{21}(y) & \varphi(x)h_{11}(y) \end{Vmatrix}$$

and

$$\begin{aligned} \psi(x)h_{11}(xy) &= \psi(x)h_{11}(y)\,, \\ \psi(x)h_{12}(xy) &= \varphi(x)h_{12}(y)\,, \\ \varphi(x)h_{21}(xy) &= \psi(x)h_{21}(y)\,. \end{aligned} \tag{48}$$

Putting $y = 1$ in relations (48) and writing shortly

$$d_0 \overset{\text{df}}{=} h_{11}(1)\,, \qquad d_1 \overset{\text{df}}{=} h_{12}(1)\,, \qquad d_2 \overset{\text{df}}{=} h_{21}(1)$$

and

$$\mu(x) \overset{\text{df}}{=} \frac{\psi(x)}{\varphi(x)}\,, \qquad x \neq 0\,,$$

we obtain (for $x \neq 0$)

$$h_{11}(x) = d_0\,, \qquad h_{12}(x) = \frac{d_1}{\mu(x)}\,, \qquad h_{21}(x) = d_2\mu(x)\,.$$

The function $\mu(x)$ is evidently multiplicative, i.e. satisfies the equation

$$\mu(xy) = \mu(x)\mu(y)\,. \tag{49}$$

Now we are passing to our proper task, to the finding of the unknown function $F(A)$.

§ 5. We must distinguish four cases.

1. $d_1 d_2 \neq 0$. The function $H^*(x)$ satisfies — like $H(x)$ — equation (19). Thus since

$$H^*(x)\cdot H^*(y) = \begin{Vmatrix} d_0^2 + d_1 d_2 \dfrac{\mu(y)}{\mu(x)} & d_0 d_1 \left[\dfrac{1}{\mu(x)} + \dfrac{1}{\mu(y)}\right] \\ d_0 d_2[\mu(x)+\mu(y)] & d_0^2 + d_1 d_2 \dfrac{\mu(x)}{\mu(y)} \end{Vmatrix}$$

and

$$H^*(x+y) = \begin{Vmatrix} d_0 & \dfrac{d_1}{\mu(x+y)} \\ d_2\mu(x+y) & d_0 \end{Vmatrix},$$

we have in particular

$$d_0^2 + d_1 d_2 \frac{\mu(y)}{\mu(x)} = d_0 \tag{50}$$

and

$$d_0 d_2[\mu(x)+\mu(y)] = d_2\mu(x+y)\,. \tag{51}$$

Moreover

$$\det H^*(x) = d_0^2 - d_1 d_2 = 1\,. \tag{52}$$

Putting in (50) $x = 1$ we obtain

$$\mu(y) = \frac{d_0 - d_0^2}{d_1 d_2} = \text{const}\,,$$

whence, in view of the fact that the function $\mu(x)$ satisfies equation (49), it follows that

$$\mu(x) \equiv 1\,. \tag{53}$$

Hence we have, according to (51),

$$d_0 = \tfrac{1}{2}\,. \tag{54}$$

From relations (50), (53) and (54) we obtain $d_1 d_2 = \frac{1}{4}$, while (52) and (54) imply $d_1 d_2 = -\frac{3}{4}$. Thus we have got a contradiction, which proves that the case $d_1 d_2 \neq 0$ cannot occur.

2. $d_1 = 0$, $d_2 \neq 0$. Then the function $H^*(x)$ takes the form

$$H^*(x) = \begin{Vmatrix} d_0 & 0 \\ d_2\mu(x) & d_0 \end{Vmatrix}, \qquad d_2 \neq 0\,,$$

where, since $\det H^*(x) = 1$,

$$d_0^2 = 1\,.$$

We obtain from equation (19)

$$\begin{Vmatrix} d_0^2 & 0 \\ d_0 d_2[\mu(x)+\mu(y)] & d_0^2 \end{Vmatrix} = \begin{Vmatrix} d_0 & 0 \\ d_2\mu(x+y) & d_0 \end{Vmatrix},$$

whence

$$d_0 = d_0^2 \tag{55}$$

and

$$d_2\mu(x+y) = d_0 d_2[\mu(x)+\mu(y)]\,. \tag{56}$$

It follows from (55) that $d_0 = 1$. Consequently we have by (56)

$$\mu(x+y) = \mu(x)+\mu(y)\,. \tag{57}$$

From (49) and (57) it follows (cf. e.g. [8]) that $\mu(x) = x$. Thus we obtain

$$H^*(x) = \begin{Vmatrix} 1 & 0 \\ d_2 x & 1 \end{Vmatrix} \tag{58}$$

and relation (58), found under the assumption $x \neq 0$, is evidently valid also for $x = 0$.

The function $G^*(x)$ now has the form

$$G^*(x) = \begin{Vmatrix} x\varphi(x) & 0 \\ 0 & \varphi(x) \end{Vmatrix} = \Phi(x)\cdot \begin{Vmatrix} x & 0 \\ 0 & 1 \end{Vmatrix}, \tag{59}$$

where $\Phi(x)$ denotes matrix (3). Thus, on account of lemma III and relations (31), (32) and (33), we have for $A = \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}$ with $a_{11} \neq 0$:

$$F(A) = H\left(\frac{a_{21}}{a_{11}}\right)\cdot G(a_{11})\cdot M\cdot G\left(\frac{\Delta}{a_{11}}\right)\cdot H\left(\frac{a_{12}}{a_{12}}\right)\cdot M\,,$$

i.e., according to (40), (46), (58) and (59),

$$F(A) = \Phi(\Delta)\cdot Y\cdot \begin{Vmatrix} 1 & 0 \\ d_2\dfrac{a_{21}}{a_{11}} & 1 \end{Vmatrix}\cdot Y^{-1}\cdot Y\cdot \begin{Vmatrix} a_{11} & 0 \\ 0 & 1 \end{Vmatrix}\cdot Y^{-1}\cdot M\cdot Y \times \tag{60}$$

$$\times \begin{Vmatrix} \dfrac{\Delta}{a_{11}} & 0 \\ 0 & 1 \end{Vmatrix}\cdot Y^{-1}\cdot Y\cdot \begin{Vmatrix} 1 & 0 \\ d_2\dfrac{a_{12}}{a_{11}} & 1 \end{Vmatrix}\cdot Y^{-1}\cdot M\,.$$

Now we put

$$F^*(A) \overset{\text{df}}{=} Y^{-1}\cdot F(A)\cdot Y\,. \tag{61}$$

Then we have from relation (60)

$$F^*(A) = \Phi(\Delta)\cdot \begin{Vmatrix} a_{11} & 0 \\ d_2 a_{21} & 1 \end{Vmatrix}\cdot Y^{-1}\cdot M\cdot Y\cdot \begin{Vmatrix} \dfrac{\Delta}{a_{11}} & 0 \\ d_2\dfrac{a_{12}}{a_{11}} & 1 \end{Vmatrix}\cdot Y^{-1}\cdot M\cdot Y\,. \tag{62}$$

Y may be regarded as one of the matrices (39). Hence it follows that either $Y^{-1}\cdot M\cdot Y = K\cdot J$, or $Y^{-1}\cdot M\cdot Y = M$, or $Y^{-1}\cdot M\cdot Y = N$. Consequently we must distinguish some subcases.

α) $Y^{-1}\cdot M\cdot Y = K\cdot J$. Then we have from (62)

(63) $$F^*(A) = \Phi(\Delta)\cdot\begin{Vmatrix} a_{11} & d_2 a_{12} \\ d_2 a_{21} & \dfrac{\Delta + d_2^2 a_{12}a_{21}}{a_{11}} \end{Vmatrix}.$$

The function $F^*(A)$ satisfies equation (1); then in particular the relation

(64) $$F^*(A\cdot A) = F^*(A)\cdot F^*(A)$$

holds. Assuming in (64) $A = \begin{Vmatrix} 1 & 1 \\ 1 & 0 \end{Vmatrix}$ we have $A\cdot A = \begin{Vmatrix} 2 & 1 \\ 1 & 1 \end{Vmatrix}$ and according to (63)

$$F^*(A\cdot A) = \Phi(1)\cdot\begin{Vmatrix} 2 & d_2 \\ d_2 & \dfrac{d_2^2+1}{2} \end{Vmatrix},$$

$$F^*(A)\cdot F^*(A) = \Phi(1)\cdot\begin{Vmatrix} d_2^2+1 & d_2^3 \\ d_2^3 & d_2^2+(d_2^2-1)^2 \end{Vmatrix}.$$

Hence

$$d_2^2 = 1\,.$$

Thus we obtain from (63) two solutions: the first (for $d_2 = 1$)

(65) $$F^*(A) = \Phi(\Delta)\cdot A$$

and the second (for $d_2 = -1$)

(66) $$F^*(A) = \Phi(\Delta)\cdot\begin{Vmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{Vmatrix} = \Phi(\Delta)\cdot M\cdot A\cdot M\,.$$

Matrices (65) and (66) are similar. Taking into account all similar solutions we can, according to (61), write in the present subcase ($d_1 = 0$, $d_2 \neq 0$, $Y^{-1}\cdot M\cdot Y = K\cdot J$) the general solution of equation (1) in the form

(67) $$F(A) = \Phi(\Delta)\cdot C\cdot A\cdot C^{-1}\,,$$

where C is an arbitrary non-singular matrix.

β) $Y^{-1}\cdot M\cdot Y = M$ or $Y^{-1}\cdot M\cdot Y = N$. In both these cases we have from (62)

(68) $$F^*(A) = \Phi(\Delta)\cdot\begin{Vmatrix} \Delta & 0 \\ d_2\left[\dfrac{a_{21}}{a_{11}}\Delta - \dfrac{a_{12}}{a_{11}}\right] & 1 \end{Vmatrix}.$$

Let us choose $A = \begin{Vmatrix} 2 & 0 \\ 1 & 2 \end{Vmatrix}$, $B = \begin{Vmatrix} 2 & 0 \\ 0 & 2 \end{Vmatrix}$. Obviously

$$A\cdot B=B\cdot A=\begin{Vmatrix}4&0\\2&4\end{Vmatrix}\quad\text{and}\quad F^*(A)\cdot F^*(B)=F^*(B)\cdot F^*(A)\,.$$

Hence, in view of (68)

$$\Phi(4)\cdot\begin{Vmatrix}4&0\\2d_2&1\end{Vmatrix}\cdot\Phi(4)\cdot\begin{Vmatrix}4&0\\0&1\end{Vmatrix}=\Phi(4)\cdot\begin{Vmatrix}4&0\\0&1\end{Vmatrix}\cdot\Phi(4)\cdot\begin{Vmatrix}4&0\\2d_2&1\end{Vmatrix},$$

i.e.

$$\begin{Vmatrix}16&0\\8d_2&1\end{Vmatrix}=\begin{Vmatrix}16&0\\2d_2&1\end{Vmatrix}.$$

Consequently $d_2=0$, which contradicts the assumption $d_2\neq 0$. Thus the subcase β) cannot occur.

3. $d_1\neq 0$, $d_2=0$. In this case our reasoning follows the same lines as in the preceding case, and finally we again obtain formula (67) as the general solution of equation (1) in case 3.

4. $d_1=d_2=0$. Then

$$H^*(x)=\begin{Vmatrix}d_0&0\\0&d_0\end{Vmatrix},\qquad d_0^2=1\,,$$

and

$$H(x)=Y\cdot H^*(x)\cdot Y^{-1}=Y\cdot\begin{Vmatrix}d_0&0\\0&d_0\end{Vmatrix}\cdot Y^{-1}\overset{\text{df}}{=\!=}H_0\,.$$

We have further, on account of lemma III and formula (43),

$$(69)\quad F(A)=H_0\cdot Y\cdot\begin{Vmatrix}\psi(a_{11})&0\\0&\varphi(a_{11})\end{Vmatrix}\cdot Y^{-1}\cdot M\cdot Y\cdot\begin{Vmatrix}\psi\left(\dfrac{\Delta}{a_{11}}\right)&0\\0&\varphi\left(\dfrac{\Delta}{a_{11}}\right)\end{Vmatrix}\cdot Y^{-1}\cdot H_0\cdot M\,.$$

Since H_0 commutes with every matrix and $H_0\cdot H_0=E$, making use of notation (61) we obtain from (69)

$$(70)\quad F^*(A)=\begin{Vmatrix}\psi(a_{11})&0\\0&\varphi(a_{11})\end{Vmatrix}\cdot Y^{-1}\cdot M\cdot Y\cdot\begin{Vmatrix}\psi\left(\dfrac{\Delta}{a_{11}}\right)&0\\0&\varphi\left(\dfrac{\Delta}{a_{11}}\right)\end{Vmatrix}\cdot Y^{-1}\cdot M\cdot Y\,.$$

We must again distinguish two subcases.

α) $Y^{-1}\cdot M\cdot Y=K\cdot J$. Then

$$(71)\qquad F^*(A)=\begin{Vmatrix}\psi(a_{11})\varphi\left(\dfrac{\Delta}{a_{11}}\right)&0\\0&\varphi(a_{11})\psi\left(\dfrac{\Delta}{a_{11}}\right)\end{Vmatrix}.$$

Let us choose $A = \begin{Vmatrix} 1 & u \\ 0 & 1 \end{Vmatrix}$, $B = \begin{Vmatrix} 1 & 0 \\ v & 1 \end{Vmatrix}$. Then

$$A \cdot B = \begin{Vmatrix} 1+uv & u \\ v & 1 \end{Vmatrix}.$$

We have further by (71)

$$F^*(A) = F^*(B) = \begin{Vmatrix} \psi(1)\varphi(1) & 0 \\ 0 & \varphi(1)\psi(1) \end{Vmatrix} = E,$$

$$F^*(A \cdot B) = \begin{Vmatrix} \psi(1+uv)\varphi\left(\frac{1}{1+uv}\right) & 0 \\ 0 & \varphi(1+uv)\psi\left(\frac{1}{1+uv}\right) \end{Vmatrix},$$

whence, since $F^*(A \cdot B) = F^*(A) \cdot F^*(B)$, we obtain (putting $1+uv = x$)

$$\frac{\psi(x)}{\varphi(x)} = 1,$$

i.e. $\psi(x) = \varphi(x)$. Thus we have finally from (71) $F^*(A) = \Phi(\Delta)$ and

$$F(A) = Y \cdot \Phi(\Delta) \cdot Y^{-1} = \Phi(\Delta),$$

for $\Phi(\Delta)$ commutes with every matrix. Therefore in this case we obtain no similar solutions. The solution obtained is a particular case of

$$F(A) = G(\Delta), \tag{72}$$

(where the function $G(x)$ satisfies equation (17)), which we obtained in cases (29) and (30).

β) $Y^{-1} \cdot M \cdot Y = M$ or $Y^{-1} \cdot M \cdot Y = N$. Then we have from (70)

$$F^*(A) = \begin{Vmatrix} \psi(\Delta) & 0 \\ 0 & \varphi(\Delta) \end{Vmatrix},$$

whence

$$F(A) = Y \cdot \begin{Vmatrix} \psi(\Delta) & 0 \\ 0 & \varphi(\Delta) \end{Vmatrix} \cdot Y^{-1}.$$

Taking into account also similar solutions we obtain hence

$$F(A) = C \cdot \begin{Vmatrix} \psi(\Delta) & 0 \\ 0 & \varphi(\Delta) \end{Vmatrix} \cdot C^{-1},$$

which is again a particular case of formula (72).

§ 6. Thus we have obtained formulae (67) and (72) as the general form of $F(A)$ for matrices A in which $a_{11} \neq 0$. It remains to determine the form of the function $F(A)$ for matrices A with $a_{11} = 0$.

Let

$$A = \begin{Vmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} \qquad (a_{12}a_{21} \neq 0)$$

and let us choose a non-singular matrix

$$P = \begin{Vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{Vmatrix}$$

in such a manner that

$$p_{11} \neq 0\,, \qquad p_{22} \neq 0\,, \qquad p_{11}p_{22} - p_{21}p_{12} \neq 0\,,$$
$$p_{12}p_{22}a_{21} - p_{11}p_{21}a_{12} - p_{12}p_{21}a_{22} \neq 0\,,$$

which can always be realized. We put

$$\bar{A} = \begin{Vmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{Vmatrix} \overset{\mathrm{df}}{=} P\cdot A\cdot P^{-1}. \tag{73}$$

We have

$$P^{-1} = \begin{Vmatrix} \dfrac{p_{22}}{\pi} & -\dfrac{p_{12}}{\pi} \\ -\dfrac{p_{21}}{\pi} & \dfrac{p_{11}}{\pi} \end{Vmatrix}$$

and consequently

$$\bar{a}_{11} = \frac{1}{\pi}(p_{12}p_{22}a_{21} - p_{11}p_{21}a_{12} - p_{12}p_{21}a_{22}) \neq 0\,,$$

where $\pi = \det P$.

Now we must distinguish two cases according to the two possible forms of $F(X)$.

I. If $F(X) = \Phi(\det X)\cdot C\cdot X\cdot C^{-1}$ for matrices X such that $x_{11} \neq 0$, then

$$F(P) = \Phi(\pi)\cdot C\cdot P\cdot C^{-1}\,, \qquad F(P^{-1}) = \Phi(\pi^{-1})\cdot C\cdot P^{-1}\cdot C^{-1}\,,$$
$$F(\bar{A}) = \Phi(\varDelta)\cdot C\cdot \bar{A}\cdot C^{-1}\,,$$

where $\varDelta = \det A = \det\bar{A}$. Thus we have from (73)

$$F(A) = F(P^{-1})\cdot F(\bar{A})\cdot F(P)\,,$$

whence

$$F(A) = \Phi(\pi)\cdot\Phi(\pi^{-1})\cdot\Phi(\varDelta)\cdot C\cdot P^{-1}\cdot C^{-1}\cdot C\cdot\bar{A}\cdot C^{-1}\cdot C\cdot P\cdot C^{-1}\,,$$

i.e.

$$F(A) = \Phi(\varDelta)\cdot C\cdot P^{-1}\cdot\bar{A}\cdot P\cdot C^{-1}\,.$$

Hence on account of (73)

$$F(A) = \Phi(\varDelta)\cdot C\cdot A\cdot C^{-1}\,,$$

which means that $F(A)$ has form (67) also for matrices A in which $a_{11} = 0$.

II. Similarly, if

$$F(X) = G(\det X)$$

for matrices X such that $x_{11} \neq 0$, then

$$F(P) = G(\pi), \quad F(P^{-1}) = G(\pi^{-1}), \quad F(\bar{A}) = G(\Delta)$$

and on account of (73) and (17)

$$F(A) = G(\pi^{-1}) \cdot G(\Delta) \cdot G(\pi) = G(\Delta),$$

which means that $F(A)$ has form (72) also for matrices A in which $a_{11} = 0$.

IV. The result

Summing up the results obtained we have the following

THEOREM. *Let us assume that a function $F(A)$ (the arguments as well as the values of which are 2×2 matrices with real elements) satisfies the functional equation*

$$F(A \cdot B) = F(A) \cdot F(B) \tag{1}$$

for all non-singular matrices A and B. Then we have

$$F(A) = \Theta \tag{10}$$

or

$$F(A) = \Phi(\Delta) \cdot C \cdot \Xi \cdot C^{-1}, \tag{13}$$

or

$$F(A) = \Phi(\Delta) \cdot C \cdot A \cdot C^{-1}, \tag{67}$$

or

$$F(A) = G(\Delta). \tag{72}$$

In formulae (10), (13), (67) *and* (72) *Δ denotes the determinant of the matrix A, $\Phi(x)$ is given by*

$$\Phi(x) = \begin{Vmatrix} \varphi(x) & 0 \\ 0 & \varphi(x) \end{Vmatrix}, \tag{3}$$

where $\varphi(x) \not\equiv 0$ is an arbitrary multiplicative function, i.e. satisfying the equation

$$\varphi(xy) = \varphi(x)\varphi(y), \tag{4}$$

$G(x)$ is an arbitrary (matrix-valued) function satisfying the equation

$$G(xy) = G(x) \cdot G(y) \tag{17}$$

and the condition

$$G(1) = E = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}. \tag{18}$$

Θ and Ξ denote the matrices

$$\Theta = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}, \quad \Xi = \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}, \tag{2}$$

and C is an arbitrary constant non-singular matrix (and thus playing in formulae (13) *and* (67) *the role of a parameter). Formulae* (10) *and* (13) *give the singular solutions of equation* (1), *while solutions* (67) *and* (72) *are non-singular.*

On the other hand, as is easy to verify, each of the functions (10), (13), (67), (72) actually satisfies equation (1).

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