ANNALES
POLONICI MATHEMATICI
XXX (1974)

On a functional equation related to the Cauchy equation

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Abstract. Equation (4) is considered for functions $f: X \to R$, where $(X, +)$ is a commutative group and $(R, +, \cdot)$ is a commutative integral domain with identity and of characteristic zero. $a, b \in R$ are constants. The general solution of (4) is described and, in particular, the problem of the equivalence of equations (4) and (1) is investigated.

Introduction. The following Cauchy equation

$$f(x+y) = f(x)+f(y) \tag{1}$$

has been studied extensively (cf. [1], [4]). The equivalence of (1) with the functional equation

$$f(x+y)^2 = [f(x)+f(y)]^2 \tag{2}$$

was investigated in [3], [5], [6], [7], [8]. Recently, the second author jointly with others [2] has studied the equivalence of (1) with the functional equation

$$f(x+y)[f(x+y)-f(x)-f(y)] = 0. \tag{3}$$

In this paper, along the same lines, we treat the functional equation

$$[f(x+y)-af(x)-bf(y)][f(x+y)-f(x)-f(y)] = 0, \tag{4}$$

which contains (2) and (3) as particular cases. We shall show that, in some cases, there exist solutions of (4) which are not solutions of (1).

1. Let $(X, +)$ be a commutative group and $(R, +, \cdot)$ a commutative integral domain with identity and of characteristic zero. Let $f: X \to R$ be a solution of (4).

First we note that if f is a constant, say $f(x) = c$, then $(a+b-1)c^2 = 0$, showing thereby that either $c = 0$ (in which case f is a solution of (1)) or $a+b = 1$. In the latter case every constant function satisfies (4). Thus in the sequel we may consider only the non-constant solutions of (4).

We prepare our final result by a sequence of lemmas.

LEMMA 1. *If f is a non-constant solution of* (4), *then the set*

$$K = \{x \in X : f(x) = 0\} \tag{5}$$

is a subgroup of X.

Proof. For $x, y \in K$ it is evident from (4) that $f(x+y)^2 = 0$ so that $x+y \in K$.

Now we will show that $f(0) = 0$.

With $x = y = 0$ (4) gives $(1-a-b)f(0)^2 = 0$. If we had $f(0) \neq 0$, then $a+b = 1$.

Now putting $y = 0$ in (4), (4) yields $b\big(f(x)-f(0)\big)f(0) = 0$. Since f is non-constant and $f(0) \neq 0$, we must have $b = 0$. On setting $x = 0$ in (4), we obtain $\big(f(y)-f(0)\big)f(0) = 0$, yielding $f(y) =$ constant, since $f(0) \neq 0$, which is a contradiction. Consequently $f(0) = 0$.

Take an arbitrary $x \in K$ and put $y = -x$ in (4). Then $bf(-x)^2 = 0$. Next, replace y by x and x by $-x$ in (4) to get $af(-x)^2 = 0$. Then either $-x \in K$, showing thereby that K is a subgroup, or $a = b = 0$ in which case (4) reduces to (3) and K is a subgroup as it has been shown in [2]. This completes the proof.

LEMMA 2. *If f is a non-constant solution of* (4), *then either f is odd, or $a = -b$ and f satisfies* (3).

Proof. Suppose that f is not odd. Then there is an $x_0 \in X$ such that $f(-x_0) \neq -f(x_0)$. So, by Lemma 1, $x_0 \notin K$ and $-x_0 \notin K$. From (4), on first setting $x = x_0$, $y = -x_0$, and then $x = -x_0$, $y = x_0$, we have $af(x_0)+$ $+bf(-x_0) = 0$ and $af(-x_0)+bf(x_0) = 0$. Hence

$$(a+b)[f(-x_0)+f(x_0)] = 0,$$

yielding $a = -b$.

Interchanging x and y in (4) we obtain

$$[f(x+y)-bf(x)-af(y)][f(x+y)-f(x)-f(y)] = 0. \tag{6}$$

Adding (4) and (6), since $a = -b$, we get

$$2f(x+y)[f(x+y)-f(x)-f(y)] = 0, \tag{7}$$

which implies (3). The proof of Lemma 2 is thus complete.

LEMMA 3. *Suppose that a non-constant solution f of* (4) *fulfils the condition*

$$f(x) \neq f(y) \quad \textit{implies } f(x+y) = f(x)+f(y). \tag{8}$$

Then (with K given by (5)), *either*

(i) *f is odd, K is of index 3, $a+b+1 = 0$, and f is given by*

$$f(x) = \begin{cases} 0 & \textit{for } x \in K, \\ c & \textit{for } x \in x_0+K, \\ -c & \textit{for } x \in -x_0+K, \end{cases} \tag{9}$$

where $x_0 \notin K$ and $c \neq 0$ is an arbitrary constant in R; or

(ii) *f is odd, K is of infinite index and f satisfies* (1); *or*

(iii) *f is not odd, K is of index 2, $a = -b$, and f is given by*

$$f(x) = \begin{cases} 0 & \text{for } x \in K, \\ c & \text{for } x \notin K, \end{cases} \tag{10}$$

where $c \neq 0$ is an arbitrary constant in R.

Proof. First we note that if f is odd (1), then the sets $\{x \in X : f(x) = \text{const}\}$ are the cosets of K. In fact, take $x, y \in X$ such that $f(x) = f(y) \neq 0$. (For $x, y \in K$ the relation $x - y \in K$ results from Lemma 1.) Then $f(-y) = -f(y) = -f(x) \neq f(x)$, and by (8) $f(x-y) = f(x) - f(y) = 0$, that is, $x - y \in K$.

Now we shall consider two cases.

I. First, let f be not odd. Then, by Lemma 2, $a = -b$ and f satisfies (3). As has been proved in [2], (3) implies that either f is a solution of (1), which is impossible since f is not odd, or the index of K is 2 and f is given by formula (10). Thus in this case we obtain (iii) above.

II. Now let f be odd. First we shall show that the index of K cannot be two. Supposing the contrary, take an arbitrary $x_0 \notin K$. Then also $-x_0 \notin K$, and hence both x_0, $-x_0$ belong to the same coset, $x_0 + K$. This means that $f(x_0) = f(-x_0) = -f(x_0)$, that is, $f(x_0) = 0$ and so $x_0 \in K$, a contradiction. So the index of $K \geqslant 3$.

Let the index of K be 3. Then f is given by (9). Since all groups of order 3 are isomorphic, $x \in x_0 + K$ implies $2x \in -x_0 + K$. Hence (4) with $x = y = x_0$ yields

$$(a+b+1)3c^2 = 0, \tag{11}$$

giving $a+b+1 = 0$. Thus in this case we obtain (i) above.

We will now prove that if the index of $K > 3$, then f satisfies (1) and the index of K should be infinite.

By (8) relation (1) holds whenever $f(x) \neq f(y)$. If $f(x) = f(y) = 0$, then (1) holds by virtue of Lemma 1. So let $f(x) = f(y) \neq 0$. Then there is a $u \in X$ such that $f(u)$ is neither 0, nor $f(x)$, nor $f(-x) = -f(x)$. Hence $f(-u) \neq f(x) = f(y)$ and by (8)

$$f(u+x) = f(u) + f(x) \quad \text{and} \quad f(y-u) = f(y) - f(u).$$

Hence

$$f(u+x) \neq f(y-u), \tag{12}$$

and again by (8),

$$f(x+y) = f(x+u+y-u) = f(x) + f(y).$$

Thus f satisfies (1) for all $x, y \in X$.

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(1) If f is not odd, this is also true (cf. (10)), but we shall not need this here

Since f is not constant, there exists an $x_0 \in X$ such that $f(x_0) = c \neq 0$. By (1), for every integer k, there exists a coset on which the value of f is kc. Since R is of characteristic zero, all these values are different, which means that the index of K is infinite. Thus in this case we obtain (ii) above and the proof of Lemma 3 is complete.

Now we will prove the following theorem which is the main result of the present paper.

THEOREM 1. *Let $(X, +)$ be a commutative group and $(R, +, \cdot)$ be an integral domain of characteristic zero. Further let $f: X \to R$ be a solution of* (4). *Then the set* (5) *(if non-empty) is a subgroup of X and we have the following four possibilities:*

(i) $a+b = 1$, $K = \emptyset$, *and f is constant*;

(ii) $a+b = 0$, *K is of index* 2, *and f is given by* (10);

(iii) $a+b = -1$, *K is of index* 3, *and f is given by* (9);

(iv) *a, b are arbitrary, f is a solution of* (1) *and the index of K is either* 1 *(in which case $f(x) \equiv 0$) or infinite.*

Proof. If f is constant, then we have either (i) or the case $f(x) \equiv 0$ of (iv), as it has been observed in the beginning of the paper. So let us assume that f is non-constant.

We must consider a few cases.

I. First, we will treat the case $a \neq b$. Interchanging x and y in (4) we get (6). Subtracting (6) from (4) we obtain

$$(b-a)[f(x)-f(y)][f(x+y)-f(x)-f(y)] = 0,$$

which implies (8). Now the theorem results from Lemma 3.

II. Next, we take the case $a = b = 0$. Then (4) reduces to (3) and the theorem follows from the results in [2].

III. Next, let $a = b = 1$. Then (4) yields immediately (1), that is, we obtain case (iv) above.

IV. Now, let $a = b = -1$. Then (4) reduces to (2), which in turn implies (1) (cf. [3], [5], [6], [7], [8]). Thus again we obtain case (iv) above.

V. Finally, we consider $a = b \neq 0, +1, -1$. By Lemma 2 f is odd. We will prove that (8) holds in this case, too.

Take $f(x_0) \neq f(y_0)$ and suppose that $f(x_0+y_0) \neq f(x_0)+f(y_0)$. Then by (4) results,

$$f(x_0+y_0) = a[f(x_0)+f(y_0)]. \tag{13}$$

Since f is odd, we get by (4) either

$$f(x_0) = f(x_0+y_0-y_0) = a[f(x_0+y_0)-f(y_0)] \tag{14}$$

or

$$f(x_0) = f(x_0+y_0-y_0) = f(x_0+y_0)-f(y_0). \tag{15}$$

Relation (15) implies immediately $f(x_0+y_0) = f(x_0)+f(y_0)$, contrary to the supposition. Thus we must have (14). But (14) and (13) yield, since $a \neq 1$,

$$a[f(x_0)+f(y_0)] = -f(x_0). \tag{16}$$

Similarly we arrive at

$$a[f(x_0)+f(y_0)] = -f(y_0). \tag{17}$$

Now (16) and (17) show that $f(x_0) = f(y_0)$, which is a contradiction. Thus (8) is true also in this case and the theorem results again from Lemma 3. This completes the proof of the theorem.

2. Now we shall discuss some examples.

1. Let $X = R = Z$, where Z is the set of integers with the usual addition and multiplication. Then, by Theorem 1, any solution $f: Z \to Z$ of equation (4) has one of the following forms (here $c \neq 0$ is an integer):

$$f(n) = c \tag{i}$$

(possible if and only if $a+b = 1$);

$$f(n) = \begin{cases} 0 & \text{for } n \text{ even}, \\ c & \text{for } n \text{ odd} \end{cases} \tag{ii}$$

(possible if and only if $a+b = 0$);

$$f(n) = \begin{cases} 0 & \text{for } n = 3k, \\ c & \text{for } n = 3k+1, \\ -c & \text{for } n = 3k+2, \end{cases} \quad \text{where } k = 0, \pm 1, \pm 2, \ldots \tag{iii}$$

(possible if and only if $a+b = -1$);

$$f(n) = cn \tag{iv}$$

(possible in all cases). The last formula results from (1), cf. [1].

2. Let $(X, +) = (E, \cdot)$, where $E = \{e_0, e_1, e_2, e_3\}$ and

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

and let $(R, +, \cdot)$ be the field of real numbers. Thus (1) and (4) become

$$f(xy) = f(x)+f(y), \tag{18}$$

and

$$[f(xy)-af(x)-bf(y)][f(xy)-f(x)-f(y)] = 0, \tag{19}$$

respectively. Then, by Theorem 1, if $f: E \to R$ is a solution of (19), then either f satisfies (18) and hence $f(x) \equiv 0$, or $a+b = 1$ and $f(x) \equiv c \neq 0$, or $a = -b$ and f has one of the following forms (cf. [2]):

$$f(x) = \begin{cases} 0 & \text{for } x = e_0, e_1, \\ c & \text{for } x = e_2, e_3; \end{cases}$$

$$f(x) = \begin{cases} 0 & \text{for } x = e_0, e_2, \\ c & \text{for } x = e_1, e_3; \end{cases}$$

$$f(x) = \begin{cases} 0 & \text{for } x = e_0, e_3, \\ c & \text{for } x = e_1, e_2. \end{cases}$$

3. Let $X = R$ be the field of real numbers. Then $(X, +) = (R, +)$ has no subgroups of finite index and, by Theorem 1, if $f: R \to R$ is a solution of equation (4), then either f satisfies the Cauchy equation (1), or $a+b = 1$ and f is constant.

3. From Theorem 1 we see that there may exist solutions f of (4) which do not satisfy (1). This can be avoided by assuming some weak regularity conditions. For this purpose we make use of the following result proved in [2].

LEMMA 4. *Let $(X, +)$ be a topological group. If $A, B \subset X$ are second category Baire sets, then the set $A+B = \{x = a+b : a \in A, b \in B\}$ has a non-void interior.*

THEOREM 2. *Let $(X, +)$ be a second category commutative topological group, and let $(R, +, \cdot)$ be an integral domain of characteristic zero. Further, let*

$$\bigcup_{n=1}^{\infty} nV = X, \tag{20}$$

for every neighbourhood V of 0 in X. If $f: X \to R$ is a solution of (4) such that (5) is a non-empty Baire set, then f is a solution of (1).

Proof. Case 1. Let K be of second category. Since K is a subgroup of X (Lemma 1), $K+K \subset K$. But, by Lemma 4, $K+K$ contains a non-empty open set and so K must contain a neighbourhood V of 0. Since K is a subgroup, $nV \subset K$ for all n, whence by (20) $K = X$. This means that f is identically zero and in particular satisfies (1).

Case 2. Let K be of first category. By Theorem 1 either f satisfies (1), or K is of index 2 or 3. If the index of K is 2, then $X = K \cup (x_0 + K)$ with an $x_0 \notin K$, that is, X is of first category contrary to the assumption. Similarly, if the index of K is 3, then $X = K \cup (x_0 + K) \cup (-x_0 + K)$ and again X is of first category, which is impossible. Thus we are left with the case that f is a solution of (1), which was to be proved.

Remark. Some remark is in order regarding the characteristic of the integral domain R. Lemma 1 is true for any characteristic of R. In order for Lemma 2 to be valid, the characteristic of R should be different from 2 (cf. (7)). In Lemma 3, first we note that the characteristic of R is to be different from 2 and 3 (cf. (11) and (12)); then assuming the characteristic of R to be p (> 3), we see that case I is valid and as for case II, we have that f satisfies (1) and for $x_0 \in X$ such that $f(x_0) = c \neq 0$, and for every integer k, there is a coset on which f takes value kc. Since R is of characteristic p, kc's are different for $0 \leqslant k \leqslant p-1$. Thus, we conclude that the index of K is either infinite or $\geqslant p$ (a multiple of $(p-1)$ plus one). So, Lemma 3 will be true for the characteristic of R to be p (> 3), when (ii) of Lemma 2 is replaced by (ii)$'$, f is odd, K is either of index infinite or of index $\geqslant p$ (a multiple of $(p-1)$ plus one) and f satisfies (1).

Now, if we assume the characteristic of R to be p (> 3), then

(a) Theorem 1 is valid, when (iv) is replaced by (iv)$'$. a, b arbitrary, f is a solution of (1) and the index of K is either 1 or infinite or $\geqslant p$ (a multiple of $(p-1)$ plus one);

(b) Theorem 2 is valid without change.

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Reçu par la Rédaction le 7. 7. 1972