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THE GROUP OF INVERTIBLE ELEMENTS IN A BANACH ALGEBRA

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Let A be a unital complex Banach algebra, $G(A)$ its group of invertible elements, $G_0(A)$ the connected component of the identity in $G(A)$, and $L(A) = G(A)/G_0(A)$ the quotient group. We give a simple construction which shows that $L(A)$ can be any finite abelian group.

In [2] Lorch proved that, in a commutative (complex) Banach algebra A, $L(A)$ has either infinitely many elements or only one. In [4] an example was given of a non-commutative Banach algebra A for which $L(A)$ was cyclic of order 2. Yuen's example relies on the rather deep fact that there are precisely two homotopy classes of maps from the 4-sphere to the 3-sphere, which makes it difficult to see exactly why the non-trivial element of $L(A)$ is of finite order. The purpose of this note is to exhibit for every n a Banach algebra A_n (actually a C^*-algebra) such that $L(A_n)$ is cyclic of order n. Although the existence of such algebras can be deduced from the homotopy theory of topological spaces, for the examples we construct the components of $L(A_n)$ are determined by a winding number (mod n), which we feel makes the reasons for their finite order more accessible. A consequence of our construction is that any finitely generated abelian group is $L(A)$ for some Banach algebra A. We begin by recalling a few elementary facts about the group of invertible elements in Banach algebras and C^*-algebras.

If A is a unital Banach algebra, $G(A)$ the group of invertible elements in A, and $G_0(A)$ the connected component of the identity, then $G_0(A)$ is a normal subgroup of $G(A)$ and is path connected (this follows since $G_0(A)$ is generated algebraically by $\exp(A)$) (see [5], Section 7). Thus, two elements of $G(A)$ are identified in the quotient group $L(A) = G(A)/G_0(A)$ if and only if they belong to the same path component. If A is a unital C^*-algebra [1], then we let $U(A)$ denote the group of unitary elements in A, and $U_0(A)$ the connected component of the identity in $U(A)$. Using the polar decomposition of elements in $G(A)$, one shows easily that $U_0(A) = U(A) \cap G_0(A)$, $U_0(A)$ is path connected, and the inclusion of $U(A)/U_0(A)$ into $G(A)/G_0(A)$ is a group isomorphism. Thus, two elements in $U(A)$ define the same element in $U(A)/U_0(A)$ if and only if they are in the same path component, and we may unambiguously set $U(A)/U_0(A) = L(A)$.

Let M_n denote the C^*-algebra of $n\times n$ complex matrices, U_n the $n\times n$ unitaries, and 1_n the $n\times n$ identity matrix. Let I denote the closed unit interval and set

$$SM_n = \{f\colon I \to M_n \mid f \text{ is continuous, } f(0) = \alpha 1_n, f(1) = \beta 1_n \text{ for } \alpha, \beta \text{ complex}\}.$$

If we define algebraic operations pointwise in SM_n and set

$$\|f\| = \sup_{t\in I} \|f(t)\|,$$

then we obtain a C^*-algebra (the unreduced suspension of M_n).

THEOREM. *$L(SM_n)$ is the cyclic group of order n.*

Proof. Given $f \in U(SM_n)$, let

$$f(0) = \exp(ix)\cdot 1_n \quad \text{and} \quad f(1) = \exp(iy)\cdot 1_n,$$

where x and y are real numbers. Set

$$g(t) = \exp\big(ix(t-1) - iyt\big)\cdot 1_n.$$

Then, clearly, $g \in U_0(SM_n)$ and $f\cdot g(0) = f\cdot g(1) = 1_n$. Thus, if we take the subgroup of $U(SM_n)$,

$$U_1 = \{f \in U(SM_n) \mid f(0) = f(1) = 1_n\},$$

then, by the above, $U_1/U_0(SM_n)\cap U_1$ is isomorphic to $L(SM_n)$ via the inclusion of U_1 into $U(SM_n)$. Hence, to prove the Theorem it is sufficient to construct a homomorphism φ from U_1 to Z_n, the cyclic group of order n, which is onto and such that $\ker\varphi = U_1 \cap U_0(SM_n)$.

Let S^1 denote the unit circle, $w(\cdot)$ the winding number of a continuous function from S^1 to S^1, and $\det(\cdot)$ the determinant of a matrix. If $f \in U_1$, then f can be regarded as a function from S^1 to U_n (which we shall frequently do), and thus $\det f$ is a function from S^1 to S^1. We set

$$\varphi(f) \equiv w(\det f) \pmod n.$$

This defines a homomorphism since $\det(\cdot)$ is multiplicative and $w(\cdot)$ is additive. Furthermore, for the diagonal function

$$g_k(t) = \begin{pmatrix} \exp(2\pi ikt) & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

we have $\varphi(g_k) \equiv w\big(\exp(2\pi ikt)\big) \equiv k \pmod{n}$, and so φ is onto Z_n.

Now we show that $\ker\varphi = U_1 \cap U_0(SM_n)$, i.e., $\varphi(f) \equiv 0 \pmod{n}$ if and only if f can be connected by a path in $U(SM_n)$ to the constant function 1_n. The set of equivalence classes of elements of U_1 which can be connected by paths in U_1, i.e., which are homotopic, is $\pi_1(U_n)$ which is isomorphic to the integers (by Theorem 25.2 in [3]). If f and g can be path connected in U_1, then $\det f$ and $\det g$ are homotopic maps into S^1, and so $w(\det f) = w(\det g)$. Thus, for a homotopy class of maps $[f] \in \pi_1(U_n)$ we may set $\theta([f]) = w(\det f)$ and this is well defined. Furthermore, recalling the operation in $\pi_1(U_n)$, one sees that θ is a homomorphism. But $\theta([g_k]) = k$, and so θ is onto, and hence an isomorphism. Thus, we see that two functions f and g in U_1 can be connected by a path in U_1 if and only if $w(\det f) = w(\det g)$. (This conclusion is actually contained in the proof of Theorem 25.2 in [3].)

Now, suppose $\varphi(f) \equiv 0 \pmod{n}$; then $w(\det f) = nk$ for some k. Thus, there is a path in U_1 from f to $g(t) = \exp(2\pi ikt)\cdot 1_n$, and setting $g_r(t) = g\big((1-r)t\big)$ defines a path through $U(SM_n)$ to the constant function 1_n. Hence $f \in U_1 \cap U_0(SM_n)$.

Conversely, suppose f can be connected by a path in $U(SM_n)$ to 1_n, i.e., there exists a continuous function $F\colon I\times I \to U_n$ such that $F(t,0) = f(t)$, $F(t,1) = 1_n$, and $F(0,r)$ and $F(1,r)$ are scalar unitaries for all r. Notice that F restricted to the boundary of $I\times I$ defines a function from S^1 to U_n and is homotopically trivial (restrict F to the boundary of $[r, 1-r]\times[r, 1-r]$ for the homotopy). Thus,

$$0 = w(\det F|_{\partial(I\times I)}) = w(\det f) + w\big(\det F(1,r)\big) + w(\det 1_n) + w\big(\det F(0,r)\big),$$

and since $F(1,r)$, $F(0,r)$, and 1_n are all scalar valued, the last three terms are divisible by n. Hence $w(\det f) \equiv 0 \pmod{n}$ and $f \in \ker\varphi$. This completes the proof of the Theorem.

If A and B are unital Banach algebras, then $L(A\oplus B)$ can be readily seen to be isomorphic to $L(A)\times L(B)$. Furthermore, if $A = C(S^1)$, the continuous functions on S^1, then $L(A)$ is isomorphic to Z. Thus, any finitely generated abelian group can be realized as $L(A)$, where A is a finite direct sum of the algebras $C(S^1)$ and SM_n.

Some slightly more general results can be obtained by considering infinite direct sums and direct limits, but the above algebras do not appear to be sufficiently flexible to allow one to obtain any abelian group. We do not know whether or not every abelian group can be $L(A)$ for some Banach algebra A. (**P 1266**)

In [4] an example is also given of a Banach algebra for which $L(A)$ is non-abelian, but it relies on very non-trivial topological facts. It would be interesting to know if somewhat more elementary examples can be constructed which yield some of the finite non-abelian groups.

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