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LOCALLY CONFORMAL ALMOST COSYMPLECTIC MANIFOLDS

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1. Introduction. The aim of the present paper is to study the nature of the curvature of almost contact metric manifolds which are locally conformal (shortly, l.c.) to almost cosymplectic manifolds. We start with this in Section 3, where preliminary properties of such manifolds and their analytical characterization are given. In Section 4 basic identities for various types of curvatures of l.c. almost cosymplectic manifolds will be proved. Having these identities in mind we examine in Section 5 almost α-Kenmotsu manifolds being either $C(\lambda)$-manifolds or conformally flat manifolds. Finally, we investigate the pointwise constant φ-holomorphic sectional curvature condition in a certain class of l.c. almost cosymplectic manifolds.

2. Preliminaries. Let M be a $(2n+1)$-dimensional almost contact manifold (cf., e.g., [2]). Denote by (φ, ξ, η) the almost contact structure of M. Thus, φ is a $(1, 1)$-tensor field, ξ is a vector field, and η a 1-form on M such that

$$\varphi^2 = -I+\eta\otimes\xi, \quad \eta(\xi) = 1.$$

Then also $\varphi\xi = 0$ and $\eta(\varphi X) = 0$ for any $X \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. Define an almost complex structure J on the product manifold $M \times \boldsymbol{R}$ by

$$J\left(X, \lambda\frac{d}{ds}\right) = \left(\varphi X - \lambda\xi, \eta(X)\frac{d}{ds}\right),$$

where X and $\lambda(d/ds)$ are vectors tangent to M and $\boldsymbol{R}$, respectively, $\boldsymbol{R}$ being the real line with coordinates s. The manifold M is said to be *normal* (cf. [14] or [2]) if the almost complex structure J is integrable (i.e., J arises from a complex structure on $M \times \boldsymbol{R}$). The necessary and sufficient condition for M to be normal is

$$[\varphi, \varphi]+2\xi\otimes d\eta = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ defined by

$$[\varphi, \varphi](X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y]$$

for any $X, Y \in \mathscr{X}(M)$, $[\cdot, \cdot]$ being the Lie bracket of vector fields.

Let g be a Riemannian metric on M compatible with (φ, ξ, η), that is, such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any $X, Y \in \mathscr{X}(M)$. Thus the manifold M is almost contact metric, and the quadruple (φ, ξ, η, g) is its almost contact metric structure. Clearly, we have $\eta(X) = g(X, \xi)$ for $X \in \mathscr{X}(M)$. Denote by Φ the fundamental 2-form of M defined by

$$\Phi(X, Y) = g(\varphi X, Y), \qquad X, Y \in \mathscr{X}(M).$$

It can be easily noted that for any function σ on M the D-conformal change of g given by

$$g' = e^{2\sigma} g + (1 - e^{2\sigma})\eta \otimes \eta$$

is also compatible with (φ, ξ, η). If $\dim M = 3$, then any Riemannian metric g' compatible with (φ, ξ, η) can be obtained by a D-conformal change of the metric g.

The manifold M is said to be *almost cosymplectic* if the forms η and Φ are closed, i.e., $d\eta = 0$ and $d\Phi = 0$, where d is the operator of exterior differentiation. With our terminology we follow Goldberg and Yano [5]. However, it could be also noted that Bouzon (see [3]) has studied manifolds of this kind under the name of *almost co-Kähler manifolds*. If M is almost cosymplectic and normal, then it is called *cosymplectic* (cf. [2]). It is well known that the almost contact metric manifold is cosymplectic if and only if $\nabla\varphi$ vanishes identically, where ∇ is the Levi-Cività connection.

The products of almost Kähler manifolds and $\boldsymbol{R}$ are the simplest examples of almost cosymplectic manifolds. In general, an almost cosymplectic manifold is not such a product, even locally. On the contrary, a cosymplectic manifold is locally the product of a Kähler manifold and $\boldsymbol{R}$. Many examples of almost cosymplectic manifolds are given in [4], [11] and [12]. Curvature properties of such manifolds are studied in [5], [8], [11] and [12].

In the present paper we are especially interested in (local) conformal deformations of almost cosymplectic manifolds. Under some special conformal deformations we obtain α-Kenmotsu or, more generally, almost α-Kenmotsu manifolds. Thus we need to recall the definitions of these manifolds.

The almost contact metric manifold M is, by the definition, *almost* α-

Kenmotsu if

$$d\eta = 0 \quad \text{and} \quad d\Phi = 2\alpha\eta \wedge \Phi,$$

α being a non-zero real constant (cf. [6]). A normal almost α-Kenmotsu manifold is an *α-Kenmotsu manifold*. The necessary and sufficient condition for the almost contact metric manifold M to be α-Kenmotsu is

$$(\nabla_X \varphi) Y = \alpha \{g(\varphi X, Y)\xi - \eta(Y)\varphi X\}$$

for any $X, Y \in \mathscr{X}(M)$ (cf. [6]). The warped products $\boldsymbol{R} \times_f N$ in the sense of [1] or [9] (N being a Kähler manifold and $f(s) = ce^s$, where c is a positive constant and $s \in \boldsymbol{R}$) provide examples of Kenmotsu (i.e., 1-Kenmotsu) manifolds (cf. [7]). If (φ, ξ, η, g) and $(\varphi', \xi', \eta', g')$ are almost contact metric structures related by the homothetic transformation

$$\varphi' = \varphi, \quad \xi' = \alpha\xi, \quad \eta' = \frac{1}{\alpha}\eta, \quad g' = \frac{1}{\alpha^2}g, \quad \alpha = \text{const} \neq 0, \tag{2.1}$$

then (φ, ξ, η, g) is almost Kenmotsu (resp., Kenmotsu) if and only if $(\varphi', \xi', \eta', g')$ is almost α-Kenmotsu (resp., α-Kenmotsu). The class of Kenmotsu manifolds is one of the three classes which occur in a classification theorem of connected almost contact metric manifolds for which the automorphism group has maximal dimension $(n+1)^2$ (see [15]).

In the sequel, $X, Y, Z, \ldots$ always denote arbitrary smooth vector fields on their appropriate domains if it is not otherwise stated.

3. L.c. almost cosymplectic manifolds. Let M be an almost contact metric manifold and (φ, ξ, η, g) its almost contact metric structure. The manifold M (and the structure (φ, ξ, η, g)) is said to be *l.c. almost cosymplectic* if M has an open covering $\{U_t\}$ endowed with smooth functions $\sigma_t \colon U_t \to \boldsymbol{R}$ such that over each U_t the almost contact metric structure $(\varphi_t, \xi_t, \eta_t, g_t)$ defined by

$$\varphi_t = \varphi, \quad \xi_t = \exp\{\sigma_t\}\xi, \quad \eta_t = \exp\{-\sigma_t\}\eta, \quad g_t = \exp\{-2\sigma_t\}g \tag{3.1}$$

is almost cosymplectic. If in the above definition the structures $(\varphi_t, \xi_t, \eta_t, g_t)$ are cosymplectic, then M is called *l.c. cosymplectic*.

With the definition just quoted we follow Vaisman [16], although almost cosymplectic manifolds are called *metric cosymplectic* in [16].

Conformal transformations of almost contact metric structures are also considered in another context (see, e.g., [13]).

The following theorem provides a characterization of l.c. almost cosymplectic manifolds which was announced by Vaisman in [16].

THEOREM 3.1. *An almost contact metric manifold M is l.c. almost cosymplectic if and only if there exists a 1-form ω on M such that*

$$d\Phi = 2\omega \wedge \Phi, \quad d\eta = \omega \wedge \eta, \quad d\omega = 0. \tag{3.2}$$

If the form ω verifying (3.2) exists, then it is unique. Thus this is a characteristic form of an l.c. almost cosymplectic manifold.

COROLLARY 3.1. *An almost contact metric manifold is almost α-Kenmotsu if and only if it is l.c. almost cosymplectic with* $\omega = \alpha\eta$, $\alpha = \mathrm{const} \neq 0$.

For a characterization of l.c. cosymplectic manifolds see Theorem 3.2.

Suppose that M is an l.c. almost cosymplectic manifold. Thus identities (3.2) are satisfied for a certain 1-form ω. For any t, over open set U_t the structure $(\varphi_t, \xi_t, \eta_t, g_t)$ given by (3.1) is almost cosymplectic and $d\sigma_t = \omega$. Let ∇ and ∇^t be the Levi-Cività connections associated with the metrics g and g_t, respectively. As is well known, they are connected by

$$\nabla^t_X Y = \nabla_X Y - \omega(X)Y - \omega(Y)X + g(X, Y)B, \tag{3.3}$$

where B is the vector field defined by $g(B, X) = \omega(X)$. By definition, B is a globally defined vector field on M. However, $B = \operatorname{grad}\sigma_t$ over any U_t.

Introduce a (1, 1)-tensor field h on M taking

$$hX = \nabla_X \xi - \omega(\xi)X + \eta(X)B. \tag{3.4}$$

Using (3.1) and (3.3), we can see that on each U_t

$$\exp\{-\sigma_t\}\, \nabla^t_X \xi_t = hX, \tag{3.5}$$

which implies that $h = 0$ if M is l.c. cosymplectic.

LEMMA 3.1. *The linear operator h has the properties*

$$\begin{gathered} h\varphi + \varphi h = 0, \quad h\xi = 0, \quad \operatorname{tr} h = 0, \\ g(hX, Y) = g(hY, X) \quad (\textit{i.e., } h \textit{ is a symmetric operator}). \end{gathered} \tag{3.6}$$

Proof. By identities (2.11) and (2.12) from [12], we have

$$\nabla^t_{\varphi_t X}\xi_t + \varphi_t \nabla^t_X \xi_t = 0, \qquad \nabla^t_{\xi_t}\xi_t = 0,$$

which, with the help of (3.5) and (3.1), yields $h\varphi + \varphi h = 0$ and $h\xi = 0$. Then also $\operatorname{tr} h = 0$. The symmetry of h follows from $d\eta = \omega \wedge \eta$ and (3.4).

Note that in view of (3.3) the covariant derivatives $\nabla^t \varphi_t$ and $\nabla\varphi$ are related by

$$(\nabla^t_X \varphi_t)Y = (\nabla_X \varphi)Y - \omega(\varphi Y)X + \omega(Y)\varphi X + g(X, \varphi Y)B - g(X, Y)\varphi B. \tag{3.7}$$

LEMMA 3.2. *The covariant derivative $\nabla\varphi$ satisfies the identity*

$$\begin{aligned} (\nabla_{\varphi X}\varphi)\varphi Y + (\nabla_X \varphi)Y = {} & -\eta(Y)\{\varphi hX + \omega(\xi)\varphi X - \eta(X)\varphi B\} \\ & + 2\omega(\varphi^2 Y)\varphi X - 2\omega(\varphi Y)\varphi^2 X + \omega(\varphi Y)\eta(X)\xi \\ & + 2g(\varphi X, \varphi Y)\varphi B - 2g(X, \varphi Y)B. \end{aligned} \tag{3.8}$$

Proof. By identities (2.7) and (2.11) from [12], we have

$$(\nabla^t_{\varphi_t X}\varphi_t)\varphi_t Y + (\nabla^t_X\varphi_t)Y = -\eta_t(Y)\varphi_t\nabla^t_X\xi_t,$$

whence using relations (3.7), (3.1) and (3.5) we obtain (3.8).

LEMMA 3.3. *For the covariant derivative $\nabla\varphi$ we have*

$$(3.9)\qquad \begin{cases} \text{(a)}\ (\nabla_\xi\varphi)\xi = \varphi B, \quad \text{(b)}\ \sum\limits_{i=0}^{2n}(\nabla_{E_i}\varphi)E_i = (2n-1)\varphi B,\\ \text{(c)}\ \sum\limits_{i=0}^{2n}(\nabla_{E_i}\varphi)\varphi E_i = (2n-2)B + 2\omega(\xi)\xi,\\ \text{(d)}\ (\nabla_\xi\varphi)Y = \eta(Y)\varphi B + \omega(\varphi Y)\xi, \end{cases}$$

where $\{E_i,\ 0 \leqslant i \leqslant 2n\}$ is an orthonormal frame.

Proof. Taking $X = \xi$ into (3.8) and using (3.6) we find (d). Equality (a) follows from (d). Let

$$\{E_0 = \xi,\ E_a,\ E_{a+n} = \varphi E_a,\ 1 \leqslant a \leqslant n\}$$

be an adapted orthonormal frame. By (3.8) we get

$$\sum_{i=1}^{2n}(\nabla_{E_i}\varphi)E_i = (2n-2)\varphi B,$$

which together with (a) gives (b). Since $\varphi^2 E_i = -E_i + \delta_{i0}\xi$, we derive

$$(\nabla_{E_i}\varphi)\varphi E_i + \varphi(\nabla_{E_i}\varphi)E_i - g(\nabla_{E_i}\xi, E_i)\xi = \delta_{i0}\nabla_{E_i}\xi$$

for $0 \leqslant i \leqslant 2n$. The last relation, together with (3.4), (3.6) and (b), allows us to obtain (c).

Next, we prove the following two theorems which will be used later.

THEOREM 3.2. *An almost contact metric manifold M is l.c. cosymplectic if and only if there exists a 1-form ω on M such that $d\omega = 0$ and*

$$(3.10)\qquad (\nabla_X\varphi)Y = \omega(\varphi Y)X - \omega(Y)\varphi X - g(X, \varphi Y)B + g(X, Y)\varphi B,$$

where B is the vector field defined by $g(B, X) = \omega(X)$.

Proof. This in fact is a consequence of Theorem 3.1 and equality (3.7).

COROLLARY 3.2. *An almost contact metric manifold is α-Kenmotsu if and only if it is l.c. cosymplectic with $\omega = \alpha\eta$, $\alpha = \text{const} \neq 0$.*

THEOREM 3.3. *For an almost contact metric manifold M the following conditions are mutually equivalent:*

(a) *the manifold is normal l.c. almost cosymplectic,*

(b) *the manifold is l.c. cosymplectic with $\omega = f\eta$,*

(c) $(\nabla_X\varphi)Y = f\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\}$,

where f is a function such that $df \wedge \eta = 0$.

Proof. Our first observation is that if $d\sigma_t = \omega = f\eta$, then M is normal if and only if the structure $(\varphi_t, \xi_t, \eta_t, g_t)$ given in (3.1) is normal for each t. Indeed, in this case we have

$$[\varphi_t, \varphi_t] + 2\xi_t \otimes d\eta_t = [\varphi, \varphi] + 2\xi \otimes d\eta.$$

Next, recall that a normal almost contact metric manifold satisfies $\nabla_{\varphi X} \xi = \varphi \nabla_X \xi$. Thus, if M is normal l.c. almost cosymplectic, then considering this relation, (3.4) and (3.6), we conclude that $h = 0$ and $B = f\xi$, and consequently $\omega = f\eta$. The above facts imply immediately that (a)$\Leftrightarrow$(b). Now, note that (c) implies $\nabla_X \xi = -f\varphi^2 X$, and therefore $d\eta = 0$. This and Theorem 3.2 give the equivalence (b)$\Leftrightarrow$(c).

Finally, we prove a lemma which will be needed in what follows.

LEMMA 3.4. *For any l.c. almost cosymplectic manifold M we have*

$$|\nabla\varphi|^2 - (8n-6)|B|^2 + (4n-6)\big(\omega(\xi)\big)^2 \geqslant 0, \tag{3.11}$$

$$|h|^2 = |\nabla\xi|^2 - (2n-1)\big(\omega(\xi)\big)^2 - |B|^2 \geqslant 0. \tag{3.12}$$

The equality in (3.11) *holds if and only if the manifold M is l.c. cosymplectic, and in this case equality in* (3.12) *also holds true.*

Proof. Using (3.9), by direct computation we get

$$\begin{aligned} 0 &\leqslant \sum_{i,j=0}^{2n} |(\nabla_{E_i}\varphi)E_j - \omega(\varphi E_j)E_i + \omega(E_j)\varphi E_i \\ &\quad + g(E_i, \varphi E_j)B - g(E_i, E_j)\varphi B|^2 \\ &= |\nabla\varphi|^2 - (8n-6)|B|^2 + (4n-6)\big(\omega(\xi)\big)^2. \end{aligned}$$

On the other hand, using (3.4) and (3.6), we claim

$$\text{(a)}\ \nabla_\xi \xi = \omega(\xi)\xi - B, \qquad \text{(b)}\ \operatorname{div}\xi = \operatorname{tr}\{X \mapsto \nabla_X \xi\} = 2n\omega(\xi). \tag{3.13}$$

Thus, in view of (3.13) and (3.4) we find

$$\begin{aligned} 0 \leqslant |h|^2 &= |\nabla\xi|^2 - 2\omega(\xi)\operatorname{div}\xi + 2g(\nabla_\xi \xi, B) + (2n-1)\big(\omega(\xi)\big)^2 + |B|^2 \\ &= |\nabla\xi|^2 - (2n-1)\big(\omega(\xi)\big)^2 - |B|^2. \end{aligned}$$

The remaining assertions of our lemma follow easily by applying Theorem 3.2 and relation (3.5).

Corollaries 3.1, 3.2 and Lemma 3.4 imply

COROLLARY 3.3. *For any almost α-Kenmotsu manifold we have*

$$|\nabla\varphi|^2 - 4n\alpha^2 \geqslant 0, \qquad |\nabla\xi|^2 - 2n\alpha^2 \geqslant 0,$$

and the manifold is α-Kenmotsu if and only if $|\nabla\varphi|^2 - 4n\alpha^2 = 0$.

4. Curvature identities. In this section we obtain basic identities for various types of curvatures of l.c. almost cosymplectic manifolds.

Let M be an l.c. almost cosymplectic manifold. We are keeping here the whole formalism of local conformal transformations (see (3.1) and other formulas) used in the previous section.

Define a symmetric (0, 2)-tensor field P on M by

$$P(X, Y) = (\nabla_X \omega) Y + \omega(X)\omega(Y) - \tfrac{1}{2}|B|^2 g(X, Y). \tag{4.1}$$

Thus, on an arbitrary U_t we have

$$P(X, Y) = -(\nabla_X Y)\sigma_t + XY\sigma_t + (X\sigma_t)(Y\sigma_t) - \tfrac{1}{2}|B|^2 g(X, Y).$$

Note also that, with the help of (3.13) (a), from (4.1) it can be derived

$$\text{(a) } P(\xi; \xi) = \xi\omega(\xi) + \tfrac{1}{2}|B|^2, \quad \text{(b) } \operatorname{tr} P = \operatorname{div} B - \tfrac{1}{2}(2n-1)|B|^2, \tag{4.2}$$

which will be used in the sequel.

For the Riemann curvature we use the convention

$$R_{XYZW} = g(R_{XY} Z, W), \quad \text{where } R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

The Ricci curvature tensor S and the scalar curvature r are given by

$$S(X, Y) = \sum_{i=0}^{2n} R_{E_i XYE_i}, \quad r = \sum_{i=0}^{2n} S(E_i, E_i),$$

respectively, $\{E_i, 0 \leqslant i \leqslant 2n\}$ being an orthonormal frame with respect to g. Moreover, we consider the so-called Ricci *-curvature tensor S^* and the scalar *-curvature r^* defined by

$$S^*(X, Y) = \sum_{i=0}^{2n} R_{E_i X\varphi Y\varphi E_i}, \quad r^* = \sum_{i=0}^{2n} S^*(E_i, E_i).$$

The same curvatures corresponding to the metric g_t will be denoted by R^t, S^t, r^t, S^{t*} and r^{t*}, respectively.

The Riemann curvatures are related by

$$\begin{aligned} \exp\{2\sigma_t\} R^t_{XYZW} = R_{XYZW} &+ g(X, W)P(Y, Z) - g(X, Z)P(Y, W) \\ &+ g(Y, Z)P(X, W) - g(Y, W)P(X, Z). \end{aligned} \tag{4.3}$$

From (4.3) for the Ricci curvatures we obtain

$$S^t(X, Y) = S(X, Y) + (2n-1)P(X, Y) + (\operatorname{tr} P)g(X, Y). \tag{4.4}$$

Applying (4.3), it can be also shown for the Ricci *-curvatures that

$$S^{t*}(X, Y) = S^*(X, Y) - P(X, \varphi^2 Y) + P(\varphi X, \varphi Y). \tag{4.5}$$

For the scalar curvatures, using (4.2), from (4.4) and (4.5) we obtain

$$\exp\{-2\sigma_t\} r^t = r + 4n \operatorname{div} B - 2n(2n-1)|B|^2, \tag{4.6}$$

$$\exp\{-2\sigma_t\} r^{t*} = r^* - 2\xi\omega(\xi) + 2 \operatorname{div} B - 2n|B|^2. \tag{4.7}$$

PROPOSITION 4.1. *The Ricci tensor of an l.c. almost cosymplectic manifold fulfils the identity*

$$(4.8)\qquad S(\xi, \xi)+|\nabla\xi|^2+\operatorname{div} B+(2n-1)\,\xi\omega(\xi)-|B|^2-(2n-1)\big(\omega(\xi)\big)^2 = 0.$$

Proof. In virtue of identity (4.11) from [12], we have

$$(4.9)\qquad S^t(\xi_t, \xi_t)+\|\nabla^t\xi_t\|^2 = 0,$$

$\|\cdot\|$ being the norm of tensors with respect to g_t. Now, using (4.4), (3.1) and (4.2), we get

$$(4.10)\qquad \exp\{-2\sigma_t\}\, S^t(\xi_t, \xi_t) = S(\xi, \xi)+(2n-1)\,\xi\omega(\xi)+\operatorname{div} B.$$

From (3.4), (3.5) and (3.13) it can be found that

$$(4.11)\qquad \exp\{-2\sigma_t\}\,\|\nabla^t\xi_t\|^2 = |\nabla\xi|^2-(2n-1)\big(\omega(\xi)\big)^2-|B|^2.$$

Thus, relation (4.8) follows from (4.9) by (4.10) and (4.11).

PROPOSITION 4.2. *The scalar curvatures r and r^* of an l.c. almost cosymplectic manifold satisfy*

$$(4.12)\qquad r-r^*+|\nabla\xi|^2+\tfrac{1}{2}|\nabla\varphi|^2+(4n-2)\operatorname{div} B+2\xi\omega(\xi)$$
$$-(4n^2-2)|B|^2-2\big(\omega(\xi)\big)^2 = 0.$$

Moreover,

$$(4.13)\qquad r^*-r+4n(n-1)|B|^2-(4n-2)\operatorname{div} B-2\xi\omega(\xi) \geqslant 0,$$

and the equality holds if and only if the manifold is l.c. cosymplectic.

Proof. Relations (4.3) and (4.11), proved in [12], for the structure $(\varphi_t, \xi_t, \eta_t, g_t)$ imply

$$(4.14)\qquad r^t-r^{t*}+\tfrac{1}{2}\|\nabla^t\varphi_t\|^2+\|\nabla^t\xi_t\|^2 = 0.$$

Taking into account (3.7) and (3.9), we can compute

$$(4.15)\qquad \exp\{-2\sigma_t\}\,\|\nabla^t\varphi_t\|^2 = |\nabla\varphi|^2-(8n-6)|B|^2+(4n-6)\big(\omega(\xi)\big)^2.$$

Therefore, (4.12) follows from (4.14) in view of (4.6), (4.7), (4.11) and (4.15). The second part of our assertion can be deduced from Lemma 3.4 and (4.12).

Using Corollary 3.1, Propositions 4.1 and 4.2 and relation (3.13) (b) we get

COROLLARY 4.1. *For an almost α-Kenmotsu manifold we have*

$$S(\xi, \xi) = -|\nabla\xi|^2 \quad \textit{and} \quad r-r^*+|\nabla\xi|^2+\tfrac{1}{2}|\nabla\varphi|^2+4n(n-1)\alpha^2 = 0.$$

Moreover, $r-r^*+4n^2\alpha^2 \leqslant 0$, and the equality holds if and only if the manifold is α-Kenmotsu.

As we already know, the Ricci tensor of an almost cosymplectic manifold fulfils the identity

$$S(\xi, \xi) = -|\nabla\xi|^2,$$

and the scalar curvatures r and r^* fulfil the identity

$$r - r^* + |\nabla\xi|^2 + \tfrac{1}{2}|\nabla\varphi|^2 = 0$$

(cf. equalities (4.3) and (4.11) in [12]). We shall prove that any of these identities characterizes almost cosymplectic manifolds among certain l.c. almost cosymplectic manifolds.

THEOREM 4.1. *Let M be a compact l.c. almost cosymplectic manifold for which the function $\omega(\xi)$ is constant along any trajectory of the vector field ξ. If*

(i) $S(\xi, \xi) = -|\nabla\xi|^2$

or

(ii) $r - r^* + |\nabla\xi|^2 + \frac{1}{2}|\nabla\varphi|^2 = 0,$

then M is almost cosymplectic.

Proof. Let $\int$ be the integral over M with respect to the natural volume element arising from the metric g.

(i) Integrating equality (4.8), using Green's theorem and $\xi\omega(\xi) = 0$, we get

$$\int \{S(\xi, \xi) + |\nabla\xi|^2\} = \int \{|B|^2 + (2n-1)(\omega(\xi))^2\}.$$

Hence, under our assumption, we obtain $B = 0$. Hence $\omega = 0$. This clearly, in view of (3.2), gives the assertion.

(ii) The idea of this proof is similar to that of (i). We integrate the equality (4.12) and use the assumptions.

Remark 4.1. In [4], almost cosymplectic and non-cosymplectic structures on compact quotients of the generalized Heisenberg groups are constructed. It is clear that the product of a circle and a compact almost Kähler manifold admits an almost cosymplectic structure. Thus, the existence of compact l.c. almost cosymplectic manifolds is obvious. However, almost α-Kenmotsu manifolds cannot be compact. This can be easily deduced from the fact that $\operatorname{div}\xi = 2n\alpha = \text{const} \neq 0$ (use (3.13) (b) and $\omega = \alpha\eta$).

In the next proposition we prove two curvature identities for l.c. almost cosymplectic manifolds satisfying additionally $d\eta = 0$ or else $\omega = f\eta$ with $f = \omega(\xi)$. In this situation $df \wedge \eta = 0$. Hence $df = f'\eta$ for certain f'. Thus, $Xf = f'\eta(X)$ and $f' = \xi f$.

PROPOSITION 4.3. *The curvature tensor of an l.c. almost cosymplectic manifold with $\omega = f\eta$ satisfies*

$$(4.16)\quad R_{XY\varphi Z\xi} - R_{\varphi X\varphi Y\varphi Z\xi} - R_{\varphi XYZ\xi} - R_{X\varphi YZ\xi} + 2\eta(X)\{(f^2+f')g(Y, \varphi Z) - fg(Y, \varphi hZ)\} - 2\eta(Y)\{(f^2+f')g(X, \varphi Z) - fg(X, \varphi hZ)\} - 2(\nabla_{hZ}\Phi)(X, Y) = 0,$$

(4.17) $$R_{\xi\varphi Y\varphi Z\xi}+R_{\xi YZ\xi}+2(f^2+f')g(\varphi Y, \varphi Z)+2g(h^2 Y, Z) = 0.$$

Proof. First, note that from (3.4) we can find

$$(\nabla_X \eta)(Y) = g(hX, Y)+fg(X, Y)-f\eta(X)\eta(Y).$$

Therefore, the tensor P defined in (4.1) takes the form

(4.18) $$P(X, Y) = fg(hX, Y)+\tfrac{1}{2}f^2 g(X, Y)+f'\eta(X)\eta(Y).$$

Consequently, using (4.3), (3.1), (4.18) and (3.6), we obtain

(4.19) $$\exp\{\sigma_t\} R^t_{XYZ\xi_t} = R_{XYZ\xi}+\eta(X)\{fg(hY, Z)+(f^2+f')g(Y, Z)\} -\eta(Y)\{fg(hX, Z)+(f^2+f')g(X, Z)\}.$$

Moreover, using (3.5), (3.1) and (3.7) we derive

(4.20) $$\exp\{\sigma_t\}(\nabla^t_{\nabla^t_Z\xi_t}\Phi_t)(X, Y) = (\nabla_{hZ}\Phi)(X, Y)+f\eta(X)g(\varphi hZ, Y) -f\eta(Y)g(\varphi hZ, X).$$

Recall the identity (4.9) from [12], which for the structure $(\varphi_t, \xi_t, \eta_t, g_t)$ leads to

$$R^t_{XY\varphi_t Z\xi_t}-R^t_{\varphi_t X\varphi_t Y\varphi_t Z\xi_t}-R^t_{\varphi_t XYZ\xi_t}-R^t_{X\varphi_t YZ\xi_t}-2(\nabla^t_{\nabla^t_Z\xi_t}\Phi_t)(X, Y) = 0.$$

It can be seen that (4.16) follows from the above identity after using (4.19), (4.20) and (3.6).

On the other hand, by (3.4) and (3.6) we can compute

$$(\nabla_{hZ}\Phi)(\xi, \varphi Y) = -g(\nabla_{hZ}\xi, Y) = -g(h^2 Z, Y)-fg(hZ, Y).$$

Finally, taking ξ and φY instead of X and Y into (4.16) and applying the above equality and (3.6), we obtain (4.17).

5. L.c. almost cosymplectic manifolds under additional conditions. Janssens and Vanhecke initiated in [6] investigations of almost $C(\lambda)$-manifolds, λ being a real number. An almost contact metric manifold is said to be an *almost $C(\lambda)$-manifold* if its Riemann curvature tensor has the following property:

(5.1) $$R_{XYZW} = R_{XY\varphi Z\varphi W}+\lambda\{g(X, W)g(Y, Z)-g(X, Z)g(Y, W) -g(X, \varphi W)g(Y, \varphi Z)+g(X, \varphi Z)g(Y, \varphi W)\}.$$

A normal almost $C(\lambda)$-manifold is called a *$C(\lambda)$-manifold*. It was observed in [6] (Theorem 2.8) that an α-Kenmotsu manifold is a $C(-\alpha^2)$-manifold. Concerning the above we prove

THEOREM 5.1. *Let M be an almost α-Kenmotsu manifold of dimension $\geqslant 5$ which is additionally an almost $C(\lambda)$-manifold. Then $\lambda = -\alpha^2$ and M is α-Kenmotsu.*

Proof. By Corollary 3.1 we have $\omega = \alpha\eta$ and $B = \alpha\xi$, $\alpha = \text{const} \neq 0$. By (5.1) we get

$$R_{XYZ\xi} = \lambda\{\eta(X)g(Y, Z)-\eta(Y)g(X, Z)\},$$

which applied to (4.17) gives

$$h^2 = (\lambda+\alpha^2)\varphi^2. \tag{5.2}$$

Hence, since h is a symmetric linear operator, we see that $\lambda+\alpha^2 \leqslant 0$, and $\lambda+\alpha^2 = 0$ if and only if $h = 0$. Moreover, from (5.1) it follows that

$$r = r^*+4n^2\lambda. \tag{5.3}$$

Assume that $\lambda+\alpha^2 < 0$. Just as (5.2), we obtain from (4.16)

$$(\nabla_{hZ}\varphi)X = (\lambda+\alpha^2)\{\eta(X)\varphi Z-g(X, \varphi Z)\xi\}-\alpha\{\eta(X)\varphi hZ-g(X, \varphi hZ)\xi\}.$$

Taking in the above equality hZ instead of Z and using (5.2) and $\nabla_\xi\varphi = 0$ (see (3.9) (d)) we find

$$(\nabla_Z\varphi)X = -\eta(X)(\varphi h+\alpha\varphi)Z+g\big(X, (\varphi h+\alpha\varphi)Z\big)\xi.$$

Hence, with the help of (3.6), it can be derived that

$$|\nabla\varphi|^2 = 2\operatorname{tr} h^2+4n\alpha^2.$$

But, by (5.2), $\operatorname{tr} h^2 = -2n(\lambda+\alpha^2)$. Thus $|\nabla\varphi|^2 = -4n\lambda$. By (5.1) we have also $S(\xi, \xi) = 2n\lambda$, and, by (3.13) (b), $\operatorname{div} B = 2n\alpha^2$. Therefore, (4.8) leads to $|\nabla\xi|^2 = -2n\lambda$. Now, using the above identities and (5.3) in (4.12), we obtain

$$(n-1)(\lambda+\alpha^2) = 0,$$

a contradiction. Consequently, we must have $\lambda+\alpha^2 = 0$. But in this case Corollary 4.1 together with (5.3) implies that M is α-Kenmotsu.

Applying Theorem 5.1, we can state the following

COROLLARY 5.1. *An almost α-Kenmotsu manifold of dimension $\geqslant 5$ and of constant curvature K is α-Kenmotsu and $K = -\alpha^2$.*

It has been proved in [7], Proposition 11, that a conformally flat Kenmotsu manifold of dimension $\geqslant 5$ is of constant curvature (-1). By using the homothetic transformation (2.1), it can be deduced from this that a conformally flat α-Kenmotsu manifold of dimension $\geqslant 5$ is of constant curvature $(-\alpha^2)$. We would like to add that the existence of conformally flat almost α-Kenmotsu and non α-Kenmotsu manifolds is still an open problem. At this moment, we can prove the following theorem:

THEOREM 5.2. *Let M be a conformally flat almost α-Kenmotsu manifold with $\dim M \geqslant 5$. Then the scalar curvature of M satisfies*

$$r \leqslant -2n(2n+1)\alpha^2.$$

If $r = -2n(2n+1)\alpha^2$, then M is α-Kenmotsu and of constant curvature $(-\alpha^2)$.

Proof. Conformal flatness yields

$$r^* = \frac{1}{2n-1}\{r-2S(\xi,\xi)\}$$

(cf., e.g., [12], p. 249), which together with identities from Corollary 4.1 leads to

$$\frac{2n-3}{2n-1}(|\nabla\xi|^2-2n\alpha^2)+\frac{1}{2}(|\nabla\varphi|^2-4n\alpha^2) = -\frac{2n-2}{2n-1}\{r+2n(2n+1)\alpha^2\}.$$

This and Corollary 3.3 prove the required result.

Next we study l.c. almost cosymplectic manifolds of pointwise constant φ-holomorphic sectional curvature. Recall the necessary definitions and results.

Let M be an almost contact metric manifold. A 2-plane Q of T_pM is called a *φ-holomorphic plane* if $Q \perp \xi$ and $\varphi(Q) = Q$. We say that M is of *pointwise constant φ-holomorphic sectional curvature* if at any point $p \in M$ the sectional curvature $K(Q)$ is independent of the choice of the φ-holomorphic plane Q of T_pM, and in this instance the function H defined by $H(p) = K(Q)$, where $p \in M$ and Q is a φ-holomorphic plane of T_pM, is called the *φ-holomorphic sectional curvature* of M. In the case where $H = c =$ const, M is said to be of *constant φ-holomorphic sectional curvature c*.

Remark 5.1. Regarding cosymplectic manifolds of dimension ≥ 5, it is known (cf. [6] and [10]) that if such a manifold is of pointwise constant φ-holomorphic sectional curvature, then it is of constant φ-holomorphic sectional curvature, say c, and its Riemann curvature tensor is given by

$$R_{XYZW} = \frac{c}{4}\{g(\varphi X,\varphi W)g(\varphi Y,\varphi Z)-g(\varphi X,\varphi Z)g(\varphi Y,\varphi W)$$
$$+g(X,\varphi W)g(Y,\varphi Z)-g(X,\varphi Z)g(Y,\varphi W)$$
$$-2g(X,\varphi Y)g(Z,\varphi W)\}.$$

Our main result in this field is the following

THEOREM 5.3. *A normal l.c. almost cosymplectic manifold M of dimension ≥ 5 is of pointwise constant φ-holomorphic sectional curvature if and only if its Riemann curvature is of the form*

$$(5.4)\qquad R_{XYZW} = \frac{H-3f^2}{4}\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\}$$
$$+\frac{H+f^2}{4}\{g(X,\varphi W)g(Y,\varphi Z)-g(X,\varphi Z)g(Y,\varphi W)$$
$$-2g(X,\varphi Y)g(Z,\varphi W)\}$$

$$-\left(\frac{H+f^2}{4}+f'\right)\{g(X, W)\eta(Y)\eta(Z)-g(X, Z)\eta(Y)\eta(W)$$
$$+g(Y, Z)\eta(X)\eta(W)-g(Y, W)\eta(X)\eta(Z)\},$$

where f is the function such that $\omega = f\eta$, $f' = \xi f$, and H is the φ-holomorphic sectional curvature of M.

Proof. Let M be normal l.c. almost cosymplectic and of pointwise constant φ-holomorphic sectional curvature. By Theorem 3.3, it is l.c. cosymplectic and $\omega = f\eta$. Hence $h = 0$, and the equality (4.18) takes the form

$$P(X, Y) = \tfrac{1}{2}f^2 g(X, Y)+f'\eta(X)\eta(Y). \tag{5.5}$$

Consider the cosymplectic structure $(\varphi_t, \xi_t, \eta_t, g_t)$ defined as in (3.1) on an open subset U_t. If Q is a φ-holomorphic plane at a point $p\in U_t$, then applying (4.3) and (5.5) we can find that

$$K^t(Q) = \exp\{2\sigma_t\}(K(Q)+f^2).$$

Consequently, the structure $(\varphi_t, \xi_t, \eta_t, g_t)$ is of pointwise constant φ_t-holomorphic sectional curvature on U_t and $H^t = \exp\{2\sigma_t\}(H+f^2)$ is its φ_t-holomorphic sectional curvature. Now, using relations (4.3), (5.5) and (3.1) and Remark 5.1, after some long calculations we get (5.4). The converse statement of this theorem is a straightforward calculation.

COROLLARY 5.2. *A normal l.c. almost cosymplectic manifold of pointwise constant φ-holomorphic sectional curvature is η-Einstein, that is,*

$$S(X, Y) = \lambda g(X, Y)+\mu\eta(X)\eta(Y),$$

λ and μ being scalar functions. Its scalar curvature is given by

$$r = n(n+1)H-n(3n+1)f^2-4nf'.$$

To get a concrete manifold occurring in Theorem 5.3 we can take the product $M = N\times \boldsymbol{R}$, where N is a complex space form, and σ as a function depending only on the coordinate of $\boldsymbol{R}$. Then transforming conformally the cosymplectic structure of M we have the desired manifold. The problem of the existence of l.c. almost cosymplectic and non-normal manifolds of pointwise constant φ-holomorphic sectional curvature is still unsolved. Meanwhile, we prove the following theorem concerning these manifolds:

THEOREM 5.4. *Let M be an l.c. almost cosymplectic manifold with $\omega = f\eta$, which is additionally of pointwise constant φ-holomorphic sectional curvature. Then its φ-holomorphic sectional curvature H satisfies the inequality*

$$n(n+1)H \geqslant r+n(3n+1)f^2+4nf'.$$

The equality holds if and only if M is normal.

Proof. By assumption we have

$$4n(n+1)H-3r^{*}-r+2S(\xi,\xi)=0. \tag{5.6}$$

For details see, e.g., the proof of Theorem 6.3 in [12]. On the other hand, since $\omega = f\eta$, $B = f\xi$ and $\operatorname{div} B = f'+2nf^2$, from (4.8) and (4.12) we obtain

$$S(\xi,\xi) = -|\nabla\xi|^2-2nf',$$

$$r^{*} = r+|\nabla\xi|^2+\tfrac{1}{2}|\nabla\varphi|^2+4nf'+4n(n-1)f^2,$$

respectively, which used in (5.6) yields

$$n(n+1)H-r-n(3n+1)f^2-4nf' = \tfrac{5}{4}(|\nabla\xi|^2-2nf^2)+\tfrac{3}{8}(|\nabla\varphi|^2-4nf^2).$$

This, in virtue of Lemma 3.4 and Theorem 3.3, gives the assertion.

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