ANNALES
POLONICI MATHEMATICI
XXI (1969)

On finite summation, recurrence relations and identities of H-functions

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§ 1. In this paper we have summed a number of finite series of H-functions by expressing the H-function as the Mellin-Barnes type integral, interchanging the order of integration and summation, using Gauss' multiplication theorem for gamma functions along with its deductions and then evaluating the summation inside the integral with the help of known results. Also various interesting identities and recurrence relations for the H-function have been deduced.

Fox [8], p. 480, introduced the H-function in the form of a Mellin-Barnes type integral, which has been symbolically denoted by Gupta and Jain [9] as

$$(1.1)\quad H_{p,q}^{m,n}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}x^s ds,$$

where $\{(f_r,\gamma_r)\}$ stands for the set of the parameters $(f_1,\gamma_1),\dots,(f_r,\gamma_r)$; x is not equal to zero and the empty product is interpreted as unity; p,q,m and n are integers satisfying $1\leqslant m\leqslant q$, $0\leqslant n\leqslant p$; α_j $(j=1,2,\dots,p)$, β_j $(j=1,2,\dots,q)$ are positive numbers and a_j $(j=1,2,\dots,p)$, b_j $(j=1,2,\dots,q)$ are complex numbers such that no pole of $\Gamma(b_h-\beta_h s)$ $(h=1,2,\dots,m)$ coincides with any pole of $\Gamma(1-a_i+\alpha_i s)$ $(i=1,2,\dots,n)$, i.e.

$$(1.2)\quad \alpha_i(b_h+\nu)\neq\beta_h(a_i-\eta-1)\qquad (\nu,\eta=0,1,\dots;\ h=1,\dots,m;\ i=1,\dots,n).$$

Moreover, we assume that (see [4], pp. 239-240)

$$(1.3)\quad \mu=\sum_{j=1}^{q}(\beta_j)-\sum_{j=1}^{p}(\alpha_j)\geqslant 0$$

and that the relation

$$(1.4)\qquad 0 < |x| < \prod_{j=1}^{p} \alpha_j^{-\alpha_j} \prod_{j=1}^{q} \beta_j^{\beta_j} \quad \text{for } \mu = 0$$

holds T is a contour in the complex s-plane such that the points $s = (b_j+\nu)/\beta_j$ $(j = 1, 2, \ldots, m; \nu = 0, 1, \ldots)$ resp. $s = (a_j-1-\nu)/\alpha_j$ $(j = 1, \ldots, n; \nu = 0, 1, \ldots)$ lie to the right resp. left of T, while further T runs from $s = \infty - ik$ to $s = \infty + ik$. Here k is a constant with $k > |\mathrm{Im}\, b_j|/\beta_j$ $(j = 1, 2, \ldots, m)$. The conditions for the contour T can be fulfilled on account of (1.2).

The behaviour of the H-function has been given by Braaksma [4], p. 279 (6.5), and p. 246 (2.16),

$$(1.5)\qquad H_{p,q}^{m,n}\left[x \middle| \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] = o(|x|^{a}) \quad \text{for small } x,$$

where $\sum_1^q (\beta_j) - \sum_1^p (\alpha_j) \geqslant 0$ and $a = \mathrm{Re}(b_h/\beta_h)$ $(h = 1, 2, \ldots, m)$ and

$$(1.6)\qquad H_{p,q}^{m,n}\left[x \middle| \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] = o(|x|^{\beta}) \quad \text{for large } x,$$

where

$$\sum_1^q (\beta_j) - \sum_1^p (\alpha_j) > 0, \qquad \sum_1^n (\alpha_j) - \sum_{n+1}^p (\alpha_j) + \sum_1^m (\beta_j) - \sum_{m+1}^q (\beta_j) \equiv \Phi > 0,$$

$$|\arg x| < \tfrac{1}{2}\Phi\pi \quad \text{and} \quad \beta = \mathrm{Re}\left(\frac{a_i - 1}{\alpha_i}\right) \quad (i = 1, 2, \ldots, n).$$

The multiplication formula for the gamma functions ([2], p. 4 (11)) is

$$(1.7)\qquad \Gamma(mz) = (2\pi)^{1/2-m/2} m^{mz-1/2} \prod_{i=0}^{m-1} \Gamma\left(z + \frac{i}{m}\right)$$

where m is a positive integer. Hence when r is a positive integer, we have

$$(1.8)\qquad \prod_{i=0}^{m-1} \Gamma\left(\frac{a+r+i}{m}\right) = m^{-r}(a)_r \prod_{i=0}^{m-1} \Gamma\left(\frac{a+i}{m}\right),$$

$$(1.9)\qquad \prod_{i=0}^{m-1} \Gamma\left(\frac{a+r-i}{m}\right) = m^{-r}(a-m+1)_r \prod_{i=0}^{m-1} \Gamma\left(\frac{a-i}{m}\right),$$

and

$$(1.10)\qquad \prod_{i=0}^{m-1} \Gamma\left(\frac{a-r+i}{m}\right) = \frac{(-m)^r}{(1-a)_r} \prod_{i=0}^{m-1} \Gamma\left(\frac{a+i}{m}\right),$$

where $(a)_r$ is a factorial function

$$(a)_r = a(a+1)\dots(a+r-1)\,.$$

The relation between H and G-functions is

$$\text{(1.11)}\qquad H_{p,q}^{m,n}\left[x\left|\begin{matrix}\{(a_p,1)\}\\ \{(b_q,1)\}\end{matrix}\right.\right] = G_{p,q}^{m,n}\left(x\left|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right.\right).$$

§ 2. In what follows the symbol $(\Delta(\lambda, a), h)$ represents the set of the parameters

$$\left(\frac{a}{\lambda}, h\right), \left(\frac{a+1}{\lambda}, h\right), \dots, \left(\frac{a+\lambda-1}{\lambda}, h\right).$$

Also, because of the large number of parameters, the notation

$$\left(\Delta\left(\lambda, a+\left|\begin{matrix}r_1\\ \vdots\\ r_n\end{matrix}\right|\right), h\right)$$

will stand for the set of the parameters $(\Delta(\lambda, a+r_1), h), \dots, (\Delta(\lambda, a+r_n), h)$.

FIRST SUMMATION.

$$\text{(2.1)}\qquad \sum_{r=0}^{n}(-1)^{n+r}\,{}^{n}c_r\,H_{p+\lambda,q+\lambda}^{l,u+\lambda}\left[x\left|\begin{matrix}(\Delta(\lambda, a-r), h), \{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}, (\Delta(\lambda, \beta-r), h)\end{matrix}\right.\right]$$

$$= \frac{\Gamma(1-a+\beta)}{\lambda^n\Gamma(1-a+\beta-n)}\,H_{p+\lambda,q+\lambda}^{l,u+\lambda}\left[x\left|\begin{matrix}(\Delta(\lambda, a), h), \{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}, (\Delta(\lambda, \beta-n), h)\end{matrix}\right.\right]$$

provided λ, n, r are positive integers, $h > 0$, $\mathrm{Re}(\beta-a) < n$ and

$$\left[\sum_{1}^{u}(\alpha_j) - \sum_{u+1}^{p}(\alpha_j) + \sum_{1}^{l}(\beta_j) - \sum_{l+1}^{q}(\beta_j)\right] > 0\,.$$

Proof. Substituting on the left from (1.1), changing the order of summation and integration and using (1.9), we have a series equal to

$$\frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{l}\Gamma(b_j-\beta_j s)\prod_{j=1}^{u}\Gamma(1-a_j+\alpha_j s)\prod_{i=0}^{\lambda-1}\Gamma\left(1-\frac{a+i}{\lambda}+hs\right)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=u+1}^{p}\Gamma(a_j-\alpha_j s)\prod_{i=0}^{\lambda-1}\Gamma\left(1-\frac{\beta+i}{\lambda}+hs\right)}(-1)^n x^s I\,ds\,,$$

where

$$I = {}_2F_1\left[\begin{matrix}-n, 1-a+\lambda hs; 1\\ 1-\beta+\lambda hs\end{matrix}\right].$$

Using Gauss' theorem ([10], p. 144),

$$\frac{(-1)^n}{(a)_n}=\frac{\Gamma(1-a-n)}{\Gamma(1-a)},$$

Gauss' multiplication theorem for gamma functions and (1.1), the definition of the H-function, we get (2.1).

Proceeding as above, we can easily obtain the following summations:

SECOND SUMMATION.

$$(2.2)\quad \sum_{r=0}^{n}\frac{(-1)^{n+r}\lambda^{r-n}\,{}^nc_r}{\Gamma(\beta+r)}H^{l,u+\lambda}_{p+\lambda,q}\left[x\left|\begin{matrix}(\Delta(\lambda,a-r),h),\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]$$

$$=\frac{1}{\Gamma(\beta+n)}H^{l,u+2\lambda}_{p+2\lambda,q+\lambda}\left[x\left|\begin{matrix}(\Delta(\lambda,a),h),(\Delta(\lambda,a+\beta-1),h),\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\},(\Delta(\lambda,a+\beta+n-1),h)\end{matrix}\right.\right],$$

where λ, n *and* r *are positive integers,* $h>0$, $\operatorname{Re}(a+\beta+n)>1$ *and*

$$\left[\sum_{1}^{u}(\alpha_j)-\sum_{u+1}^{p}(\alpha_j)+\sum_{1}^{l}(\beta_j)-\sum_{l+1}^{q}(\beta_j)\right]>0\,.$$

THIRD SUMMATION.

$$(2.3)\quad \sum_{r=0}^{n}(-1)^{n+r}\lambda^{-r}\,{}^nc_r\,\Gamma(a-\beta+n+r)\times$$

$$\times H^{l+\lambda,u}_{p,q+2\lambda}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\}\\ (\Delta(\lambda,a),h),\{(b_q,\beta_q)\},(\Delta(\lambda,\beta-r),h)\end{matrix}\right.\right]$$

$$=\Gamma(a-\beta+n)H^{l+\lambda,u}_{p,q+2\lambda}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\}\\ (\Delta(\lambda,a+n),h),\{(b_q,\beta_q)\},(\Delta(\lambda,\beta-n),h)\end{matrix}\right.\right]$$

provided λ, n *and* r *are positive integers,* $h>0$, $\operatorname{Re}(a)<1$ *and*

$$\left[\sum_{1}^{u}(\alpha_j)-\sum_{u+1}^{p}(\alpha_j)+\sum_{1}^{l}(\beta_j)-\sum_{l+1}^{q}(\beta_j)\right]>0\,.$$

FOURTH SUMMATION.

$$(2.4)\quad \sum_{r=0}^{n}{}^nc_r\,\lambda^n H^{l,u}_{p+\lambda-1,q+\lambda-1}\left[x\left|\begin{matrix}\{(a_{p-1},\alpha_{p-1})\},(\Delta(\lambda,a+r),h)\\ \{(b_{q-1},\beta_{q-1})\},(\Delta(\lambda,\beta+r),h)\end{matrix}\right.\right]$$

$$=\frac{\Gamma(a-\beta+n+\lambda-1)}{\Gamma(a-\beta+\lambda-1)}H^{l,u}_{p+\lambda-1,q+\lambda-1}\left[x\left|\begin{matrix}\{(a_{p-1},\alpha_{p-1})\},(\Delta(\lambda,a+n),h)\\ \{(b_{q-1},\beta_{q-1})\},(\Delta(\lambda,\beta),h)\end{matrix}\right.\right],$$

where λ, n *and* r *are positive integers,* $h>0$, $\operatorname{Re}(a-\beta+n)>0$, $q>l$, $p>u$ *and*

$$\left[\sum_{1}^{u}(\alpha_j)-\sum_{u+1}^{p-1}(\alpha_j)+\sum_{1}^{l}(\beta_j)-\sum_{l+1}^{q-1}(\beta_j)-2\lambda h\right]>0 .$$

FIFTH SUMMATION.

$$\text{(2.5)}\quad \sum_{r=0}^{n}\frac{(-1)^{n+r}\,{}^{n}c_r\lambda^r}{\Gamma(1+b-a_p+r)}H^{l+\lambda,u}_{p,q+\lambda}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\}\\ \big(\Delta(\lambda,b+r),h\big),\{(b_q,\beta_q)\}\end{matrix}\right.\right]$$

$$=\frac{\lambda^n}{\Gamma(1+b-a_p+n)}H^{l+2\lambda,u}_{p+\lambda,q+2\lambda}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\},\big(\Delta(\lambda,a_p-n),h\big)\\ \big(\Delta(\lambda,a_p),h\big),\big(\Delta(\lambda,b),h\big),\{(b_q,\beta_q)\}\end{matrix}\right.\right]$$

provided λ, n *and* r *are positive integers,* $\operatorname{Re}(a_p)<(n+1)$ *and*

$$\left[\sum_{1}^{u}(\alpha_j)-\sum_{u+1}^{p}(\alpha_j)+\sum_{1}^{l}(\beta_j)-\sum_{l+1}^{q}(\beta_j)+\lambda h\right]>0 .$$

SIXTH SUMMATION.

$$\text{(2.6)}\quad \sum_{r=0}^{n}(-1)^{n+r}\lambda^{-r}\,{}^{n}c_r\Gamma(\beta-a+n+r)\times$$

$$\times H^{l,u+\lambda}_{p+2\lambda,q}\left[x\left|\begin{matrix}\big(\Delta(\lambda,a),h\big),\{(a_p,\alpha_p)\},\big(\Delta(\lambda,\beta+r),h\big)\\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]$$

$$=\Gamma(\beta-a+n)H^{l,u+\lambda}_{p+2\lambda,q}\left[x\left|\begin{matrix}\big(\Delta(\lambda,a-n),h\big),\{(a_p,\alpha_p)\},\big(\Delta(\lambda,\beta+n),h\big)\\ \{(b_q,\beta_q)\}\end{matrix}\right.\right],$$

where λ, n *and* r *are positive integers,* $h>0$, $\operatorname{Re}(a)>0$ *and*

$$\left[\sum_{1}^{u}(\alpha_j)-\sum_{u+1}^{p}(\alpha_j)+\sum_{1}^{l}(\beta_j)-\sum_{l+1}^{q}(\beta_j)\right]>0 .$$

SEVENTH SUMMATION.

$$\text{(2.7)}\quad \sum_{r=0}^{n}\frac{{}^{n}c_r\lambda^{2r}}{\Gamma(1+k+r)\Gamma(1-k-n+r)}\times$$

$$\times H^{l,u+\lambda}_{p+\lambda,q+\lambda}\left[x\left|\begin{matrix}\big(\Delta(\lambda,c-r),h\big),\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\},\big(\Delta(\lambda,c+r),h\big)\end{matrix}\right.\right]$$

$$=\frac{\lambda^{2n}}{\Gamma(1-k)\Gamma(1+k+n)}H^{l,u+2\lambda}_{p+2\lambda,q+2\lambda}\left[x\left|\begin{matrix}\left(\Delta\left(\lambda,c+\left|\begin{matrix}k\\-k-n\end{matrix}\right|\right),h\right),\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\},\left(\Delta\left(\lambda,c+\left|\begin{matrix}-k\\k+n\end{matrix}\right|\right),h\right)\end{matrix}\right.\right],$$

where λ, n and r are positive integers, $h>0$ and

$$\left[\sum_{1}^{u}(a_j)-\sum_{u+1}^{p}(a_j)+\sum_{1}^{l}(\beta_j)-\sum_{l+1}^{q}(\beta_j)\right]>0 .$$

Proof. On the left-hand side, expressing the H-function as a Mellin-Barnes type integral, using Gauss' multiplication formula for gamma functions, changing the order of integration and summation and applying the relation

$$\frac{(-1)^n}{(a)_n}=\frac{\Gamma(1-a-n)}{\Gamma(1-a)} \tag{2.8}$$

we reduce the series

$$\frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{l}\Gamma(b_j-\beta_j s)\prod_{j=1}^{u}\Gamma(1-a_j+a_j s)x^s I\,ds}{\prod_{j=l+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=u+1}^{p}\Gamma(a_j-a_j s)\Gamma(1+k)\Gamma(1-k-n)},$$

where

$$I={}_3F_2\left[\begin{matrix}\lambda-c+\lambda hs,\ 1-\lambda+c-\lambda hs,\ -n;\ 1\\ 1+k,\ 1-k-n\end{matrix}\right].$$

Using Saalschutz's theorem ([10], p. 360), (2.8), (1.7) and (1.1), we obtain the result.

EIGHTH SUMMATION.

$$\sum_{r=0}^{n}\frac{{}^nc_r\lambda^{2r}}{\Gamma(1+k+r)\Gamma(1-k-n+r)}\times \tag{2.9}$$

$$\times H^{l+\lambda,u}_{p+\lambda,q+\lambda}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\},\ (\Delta(\lambda,a-r),h)\\ (\Delta(\lambda,a+r),h),\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]$$

$$=\frac{\lambda^{2n}}{\Gamma(1-k)\Gamma(1+k+n)}H^{l+2\lambda,u}_{p+2\lambda,q+2\lambda}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\},\ \left(\Delta\left(\lambda,a+\left|\begin{matrix}-k-n\\ k\end{matrix}\right|\right),h\right)\\ \left(\Delta\left(\lambda,a+\left|\begin{matrix}-k\\ k+n\end{matrix}\right|\right),h\right),\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]$$

provided λ, n and r are positive integers, $h>0$ and

$$\left[\sum_{1}^{u}(a_j)-\sum_{u+1}^{p}(a_j)+\sum_{1}^{l}(\beta_j)-\sum_{l+1}^{q}(\beta_j)\right]>0 .$$

Proof. Substituting on the left from (1.1), changing the order of summation and integration, using (1.8) and (1.10) we have a series equal to

$$\frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{l}\Gamma(b_j-\beta_j s)\prod_{j=1}^{u}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=u+1}^{p}\Gamma(a_j-\alpha_j s)\Gamma(1+k)\Gamma(1-k-n)}\, x^s I\, ds\,,$$

where

$$I={}_3F_2\left[\begin{matrix}a-\lambda hs,\, 1-a+\lambda hs,\, -n\,;\, 1\\ 1+k,\, 1-k-n\end{matrix}\right].$$

Using Saalschutz's theorem ([10], p. 360), (2.8), (1.7) and (1.1), we get the required result.

NINTH SUMMATION. *Proceeding as above and using a modified form of Whipple's theorem as given by Dzrbasjan* [7], *i.e.*

$${}_3F_2\left[\begin{matrix}a,\, 1-a,\, -n\,;\, 1\\ f,\, 1-2n-f\end{matrix}\right]=\frac{2^{2n}(\frac{1}{2}a+\frac{1}{2}f)_n(\frac{1}{2}a-n-\frac{1}{2}f+\frac{1}{2})_n}{(f)_n(1-2n-f)_n},$$

we can easily obtain the following summation:

$$(2.10)\qquad \sum_{r=0}^{n}\frac{{}^n c_r(2\lambda)^{2r}}{\Gamma(2\beta+r)\Gamma(1-2\beta-2n+r)}\times$$
$$\times H^{l+2\lambda,u}_{p+2\lambda,q+2\lambda}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\},\, (\Delta(2\lambda,2a-r),h)\\ (\Delta(2\lambda,2a+r),h),\, \{(b_q,\beta_q)\}\end{matrix}\right.\right]$$
$$=\frac{(2\lambda)^{2n}}{\Gamma(2\beta+n)\Gamma(1-2\beta-n)}\times$$
$$\times H^{l+2\lambda,u}_{p+2\lambda,q+2\lambda}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\},\, (\Delta(\lambda,a+\beta),h),\, (\Delta(\lambda,a-\beta-n+\frac{1}{2}),h)\\ (\Delta(\lambda,a+\beta+n),h),\, (\Delta(\lambda,a-\beta+\frac{1}{2}),h),\, \{(b_q,\beta_q)\}\end{matrix}\right.\right],$$

where λ, n *and* r *are positive integers*, $h>0$,

$$\left|\sum_{1}^{u}(\alpha_j)-\sum_{u+1}^{p}(\alpha_j)+\sum_{1}^{l}(\beta_j)-\sum_{l+1}^{q}(\beta_j)\right|>0\,.$$

TENTH SUMMATION.

$$(2.11)\qquad \sum_{r=0}^{n}\frac{{}^n c_r\Gamma(\beta+n+r)(-\lambda)^r}{\Gamma(1+\beta+2r)}H^{l+\lambda,u}_{p+\lambda,q+\lambda}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\},\, (\Delta(\lambda,a+r),h)\\ (\Delta(\lambda,a+2r),h),\, \{(b_q,\beta_q)\}\end{matrix}\right.\right]$$
$$=\frac{(-\lambda)^n}{(\beta+2n)}H^{l+\lambda,u}_{p+\lambda,q+\lambda}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\},\, (\Delta(\lambda,a-\beta-n),h)\\ (\Delta(\lambda,a-\beta),h),\, \{(b_q,\beta_q)\}\end{matrix}\right.\right]$$

provided λ, n and r are positive integers, $h > 0$,

$$\left|\sum_{1}^{u}(\alpha_j) - \sum_{u+1}^{p}(\alpha_j) + \sum_{1}^{l}(\beta_j) - \sum_{l+1}^{q}(\beta_j)\right| > 0 .$$

Proof. Substituting on the left from (1.1), changing the order of integration and summation, using (1.8) and the relation

$$(a)_{2n} = 2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n ,$$

we have a series equal to

$$\frac{\Gamma(\beta+n)}{\Gamma(1+\beta)}\cdot\frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{l}\Gamma(b_j-\beta_j s)\prod_{j=1}^{u}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=l+1}^{u}\Gamma(1-b_j+\beta_j s)\prod_{j=u+1}^{p}\Gamma(a_j-\alpha_j s)}\, x^s I\, ds ,$$

where

$$I = {}_4F_3\left[\begin{matrix}-n,\ \tfrac{1}{2}(a-\lambda hs),\ \tfrac{1}{2}(a-\lambda hs+1),\ \beta+n\ ;\ 1\\ a-\lambda hs,\ \tfrac{1}{2}(\beta+1),\ \tfrac{1}{2}\beta+1\end{matrix}\right].$$

Using Carlitz's theorem [5], that is

$${}_4F_3\left[\begin{matrix}-n,\ \tfrac{1}{2}(a+1),\ \tfrac{1}{2}a+1,\ \beta+n\ ;\ 1\\ a+1,\ \tfrac{1}{2}(\beta+1),\ \tfrac{1}{2}\beta+1\end{matrix}\right] = \frac{\beta(\beta-a)_n}{(\beta+2n)(\beta)_n} ,$$

we have

$$I = \frac{\beta(\beta-a+1+\lambda hs)_n}{(\beta+2n)(\beta)_n} ;$$

now applying $(a)_n = \Gamma(a+n)/\Gamma(a)$, (2.8), (1.7) and (1.1), we obtain the result.

Eleventh summation.

$$(2.12)\qquad \sum_{r=0}^{n}\frac{(-1)^r\,{}^nc_r(a+2r)\,\Gamma(a+r)}{\Gamma(a+n+r+1)}\times$$

$$\times H^{l+4\lambda,u}_{p+4\lambda,q+4\lambda}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\},\ \left(\Delta\left(\lambda,\ -r+\left|\frac{\gamma}{\varepsilon}\right|\right),h\right),\ \left(\Delta\left(\lambda,\ a+r+\left|\frac{\gamma}{\varepsilon}\right|\right),h\right)\\ \left(\Delta\left(\lambda,\ r+\left|\frac{\beta}{\delta}\right|\right),h\right),\ \left(\Delta\left(\lambda,\ -a-r+\left|\frac{\beta}{\delta}\right|\right),h\right),\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]$$

$$= \left(\frac{2}{\lambda^3}\right)^n (a-\beta+\gamma)_n(a+\gamma-\delta)_n \times$$

$$\times H^{l+b\lambda,u}_{p+b\lambda,q+b\lambda}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\},\ (\Delta(\lambda,\gamma),h),\ \left(\Delta\left(\lambda,a+n+\left|\begin{matrix}\gamma\\ \varepsilon\end{matrix}\right|\right),h\right),\\ (\Delta(\lambda,\varepsilon),h),\ (\Delta(2\lambda,\beta+\delta-a-n),h)\\ (\Delta(\lambda,\beta),h),\ \left(\Delta\left(\lambda,-a-n+\left|\begin{matrix}\beta\\ \delta\end{matrix}\right|\right),h\right),\ (\Delta(\lambda,\delta),h),\\ (\Delta(2\lambda,\beta+\delta-a),h),\ \{(b_q,\beta_q)\}\end{matrix}\right.\right],$$

where λ, n *and* r *are positive integers,* $h>0$, $2a+\gamma+\varepsilon+n=\beta+\delta+1$ *and*

$$\Big[\sum_1^u(\alpha_j)-\sum_{u+1}^p(\alpha_j)+\sum_1^l(\beta_j)-\sum_{l+1}^q(\beta_j)\Big]>0\,.$$

Proof. Proceeding as before and noting that $a+2r=a(\frac{1}{2}a+1)_r/(\frac{1}{2}a)_r$, using (1.8), (1.10) and $\Gamma(a+n)/\Gamma(a)=(a)_n$, we get a series equal to

$$\frac{\Gamma(a+1)}{\Gamma(a+n+1)}\cdot\frac{1}{2\pi i}\times$$

$$\times\int_T\frac{\prod_{j=1}^{l}\Gamma(b_j-\beta_js)\prod_{j=1}^{u}\Gamma(1-a_j+\alpha_js)\prod_{i=0}^{\lambda-1}\Gamma\left(\frac{\beta+i}{\lambda}-hs\right)\prod_{i=0}^{\lambda-1}\Gamma\left(\frac{\delta+i}{\lambda}-hs\right)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+\beta_js)\prod_{j=u+1}^{p}\Gamma(a_j-\alpha_js)\prod_{i=0}^{\lambda-1}\Gamma\left(\frac{\gamma+i}{\lambda}-hs\right)\prod_{i=0}^{\lambda-1}\Gamma\left(\frac{\varepsilon+i}{\lambda}-hs\right)}\times$$

$$\times\frac{\prod_{i=0}^{\lambda-1}\Gamma\left(\frac{\beta-a+i}{\lambda}-hs\right)\prod_{i=0}^{\lambda-1}\Gamma\left(\frac{\delta-a+i}{\lambda}-hs\right)}{\prod_{i=0}^{\lambda-1}\Gamma\left(\frac{a+\gamma+i}{\lambda}-hs\right)\prod_{i=0}^{\lambda-1}\Gamma\left(\frac{a+\varepsilon+i}{\lambda}-hs\right)}x^sI\,ds\,,$$

where

$$I=$$

$${}_7F_6\left[\begin{matrix}a,\ \frac{1}{2}a+1,\ \beta-\lambda hs,\ 1-\gamma+\lambda hs,\ \delta-\lambda hs,\ 1-\varepsilon+\lambda hs,\ -n;1\\ \frac{1}{2}a,\ a-\beta+1+\lambda hs,\ a+\gamma-\lambda hs,\ a-\delta+1+\lambda hs,\ a+\varepsilon-\lambda hs,\ a+n+1\end{matrix}\right].$$

Using Dougall's first theorem ([10], p. 371; [11], p. 23 (3)), (2.8), we have

$$I=(a+1)_n(a-\beta+\gamma)_n(a+\gamma-\delta)_n\times$$

$$\times\frac{\Gamma(\beta+\delta-a-2\lambda hs)\Gamma(\beta-a-n-\lambda hs)\Gamma(a+\gamma-\lambda hs)}{\Gamma(\beta+\delta-a-n-2\lambda hs)\Gamma(\beta-a-\lambda hs)\Gamma(a+\gamma+n-\lambda hs)}\times$$

$$\times\frac{\Gamma(\delta-a-n-\lambda hs)\Gamma(\beta+\delta-a-\gamma-n+1-\lambda hs)}{\Gamma(\delta-a-\lambda hs)\Gamma(\beta+\delta-a-\gamma+1-\lambda hs)};$$

the result follows by (1.7) and (1.1).

TWELFTH SUMMATION.

$$(2.13)\qquad \sum_{r=0}^{n} \frac{{}^{n}c_{r}(-4\lambda\mu)^{r}}{(-2n)_{r}t^{r}} \times$$

$$\times H^{l+2\lambda,u+2\mu}_{p+\lambda+\mu,q+2\lambda}\left[x\left|\begin{matrix}(\Delta(2\mu,2a-r),h),\{(a_p,\alpha_p)\},(\Delta(t,1-a+b+r),h)\\(\Delta(2\lambda,2b+r),h),\{(b_q,\beta_q)\}\end{matrix}\right.\right]$$

$$=\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-n\right)(2\pi)^{2\mu-1}\left(\frac{\lambda\mu}{t}\right)^{n}\times$$

$$\times H^{l+2\lambda,u+\lambda+\mu}_{p+2\lambda+2\mu,q+3\lambda+\mu}\left[x\left|\begin{matrix}(\Delta(2\mu,2a),h),(\Delta(t,1-a+b+n),h),\\ \{(a_p,\alpha_p)\},(\Delta(\lambda,\tfrac{1}{2}+b),h),(\Delta(\mu,a-n),h)\\(\Delta(2\lambda,2b),h),\{(b_q,\beta_q)\},(\Delta(\lambda,\tfrac{1}{2}+b+n),h),\\(\Delta(\mu,a),h)\end{matrix}\right.\right],$$

where n, λ, μ, t and r are positive integers, $h>0$, $t=\lambda-\mu$, $\mathrm{Re}(a-b)>n$ *and*

$$\left[\sum_{1}^{u}(\alpha_j)-\sum_{u+1}^{p}(\alpha_j)+\sum_{1}^{l}(\beta_j)-\sum_{l+1}^{q}(\beta_j)+th+4\mu h\right]>0\ .$$

Proof. Substituting on the left from (1.1), using (1.8), (1.9) and interchanging the order of summation and integration, we have a series equal to

$$\frac{1}{2\pi i}\times$$

$$\int_T \frac{\prod_{j=1}^{l}\Gamma(b_j-\beta_j s)\prod_{j=1}^{u}\Gamma(1-a_j+\alpha_j s)\prod_{i=0}^{2\lambda-1}\Gamma\left(\frac{2b+i}{2\lambda}-hs\right)\prod_{i=0}^{2\mu-1}\Gamma\left(1-\frac{2a+i}{2\mu}-hs\right)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=u+1}^{p}\Gamma(a_j-\alpha_j s)\prod_{i=0}^{t-1}\Gamma\left(\frac{1+b-a+i}{t}-hs\right)}\,x^{s}I\,ds,$$

where

$$I={}_3F_2\left[\begin{matrix}2b-2\lambda hs,\,1-2a+2\mu hs,\,-n;\,1\\1+b-a-ths,\,-2n\end{matrix}\right].$$

From Whipple's theorem ([10], p. 363) we have

$$I=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-n)\Gamma(1+b-a-ths)\Gamma(a-b-n+ths)}{\Gamma(\frac{1}{2}+b-\lambda hs)\Gamma(1-a+\mu hs)\Gamma(\frac{1}{2}-b-n+\lambda hs)\Gamma(a-n-\mu hs)}\ .$$

Hence, using Gauss' multiplication formula for gamma functions and (1.1) the definition of the H-function, we get the result.

§ 3. In this section, we obtain some recurrence formulae for the H-function by specializing the parameters in the results of § 2.

(i) *Taking* $\lambda = 1, a_1 = b_q, h = \beta_q = \alpha_1, a = a_1, \beta = b_q$ *and* $n = 1$ *in* (2.1), *we get the recurrence relation*:

$$(3.1)\quad H_{p,q}^{l,u}\left[x\left|\begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\(b_1, \beta_1), \ldots, (b_{q-1}, \beta_{q-1}), (b_q, \alpha_1)\end{matrix}\right.\right]$$
$$= H_{p,q}^{l,u}\left[x\left|\begin{matrix}(a_1-1, \alpha_1), (a_2, \alpha_2), \ldots, (a_p, \alpha_p)\\(b_1, \beta_1), \ldots, (b_{q-1}, \beta_{q-1}), (b_q-1, \alpha_1)\end{matrix}\right.\right] -$$
$$-(b_q - a_1) H_{p,q}^{l,u}\left[x\left|\begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\(b_1, \beta_1), \ldots, (b_{q-1}, \beta_{q-1}), (b_q-1, \alpha_1)\end{matrix}\right.\right].$$

(ii) *In* (2.2), *putting* $\lambda = 1, a = a_1$ *and* $\beta = 1 - a_1 + b_q$; *replacing* q *by* $q+1$ *and taking* $a_1 = b_{q+1}, \alpha_1 = \beta_{q+1}, \beta_q = h = \alpha_1$ *and* $n = 1$ *we get*

$$(3.2)\quad (b_q - a_1 + 1) H_{p,q}^{l,u}\left[x\left|\begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\(b_1, \beta_1), \ldots, (b_{q-1}, \beta_{q-1}), (b_q, \alpha_1)\end{matrix}\right.\right]$$
$$= H_{p,q}^{l,u}\left[x\left|\begin{matrix}(a_1-1, \alpha_1), (a_2, \alpha_2), \ldots, (a_p, \alpha_p)\\(b_1, \beta_1), \ldots, (b_{q-1}, \beta_{q-1}), (b_q, \alpha_1)\end{matrix}\right.\right] +$$
$$+ H_{p,q}^{l,u}\left[x\left|\begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\(b_1, \beta_1), \ldots, (b_{q-1}, \beta_{q-1}), (b_q+1, \alpha_1)\end{matrix}\right.\right]$$

(*which is the relation* ([1], (3.3)) *with* $\beta_q = \alpha_1$).

(iii) *In* (2.3), *setting* $\lambda = 1, b_1 = a_p, \beta_1 = \alpha_p, a = b_1, h = \beta_1, a_1 = b_q, \alpha_1 = \beta_q, \beta = b_q$ *and replacing* u *by* $u+1$, p *by* $p+2$ *and* (a_{j+1}, α_{j+1}) *by* (a_j, α_j) $(j = 1, 2, \ldots, p)$ *and putting* $n = 1$, *we obtain*

$$(3.3)\quad H_{p,q}^{l,u}\left[x\left|\begin{matrix}\{(a_p, \alpha_p)\}\\\{(b_{q-1}, \beta_{q-1})\}, (b_q, \beta_1)\end{matrix}\right.\right]$$
$$= (1 + b_1 - b_q) H_{p,q}^{l,u}\left[x\left|\begin{matrix}\{(a_p, \alpha_p)\}\\\{(b_{q-1}, \beta_{q-1})\}, (b_q-1, \beta_1)\end{matrix}\right.\right] -$$
$$- H_{p,q}^{l,u}\left[x\left|\begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\(1+b_1, \beta_1), (b_2, \beta_2), \ldots, (b_{q-1}, \beta_{q-1}), (b_q-1, \beta_1)\end{matrix}\right.\right].$$

(*With* b_q *replaced by* b_q+1 *we get* ([1], (3.6)) *with* $\beta_q = \beta_1$).

(iv) (2.4) *with* $\lambda = 1, a = a_p, \beta = b_q, h = \alpha_p$ *and* $n = 1$ *reduces to*

$$(3.4)\quad H_{p,q}^{l,u}\left[x\left|\begin{matrix}\{(a_p, \alpha_p)\}\\\{(b_{q-1}, \beta_{q-1})\}, (b_q, \alpha_p)\end{matrix}\right.\right] + H_{p,q}^{l,u}\left[x\left|\begin{matrix}\{(a_{p-1}, \alpha_{p-1})\}, (a_p+1, \alpha_p)\\\{(b_{q-1}, \beta_{q-1})\}, (b_q+1, \alpha_p)\end{matrix}\right.\right]$$
$$= (a_p - b_q) H_{p,q}^{l,u}\left[x\left|\begin{matrix}\{(a_{p-1}, \alpha_{p-1})\}, (a_p+1, \alpha_p)\\\{(b_{q-1}, \beta_{q-1})\}, (b_q, \alpha_p)\end{matrix}\right.\right].$$

(*Replacing* a_p *by* $a_p - 1$ *we obtain* ([1], (3.5)) *with* $\beta_q = \alpha_p$.)

(v) *In* (2.5), *replacing* p *by* $p+1$, *taking* $\lambda = 1, h = \alpha_p, b_1 = a_{p+1}$, $\beta_1 = \alpha_{p+1}$, $b = b_1$ *and* $n = 1$, *we get*

$$(3.5)\qquad (1+b_1-a_p)H^{l,u}_{p,q}\left[x\left|\begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\(b_1, \alpha_p), (b_2, \beta_2), \ldots, (b_q, \beta_q)\end{matrix}\right.\right]$$

$$= H^{l,u}_{p,q}\left[x\left|\begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\(b_1+1, \alpha_p), (b_2, \beta_2), \ldots, (b_q, \beta_q)\end{matrix}\right.\right]-$$

$$-H^{l,u}_{p,q}\left[x\left|\begin{matrix}(a_1, \alpha_1), \ldots, (a_{p-1}, \alpha_{p-1}), (a_p-1, \alpha_p)\\(b_1, \alpha_p), (b_2, \beta_2), \ldots, (b_q, \beta_q)\end{matrix}\right.\right]$$

(*which is the result* ([1], (3.4)) *with* $\beta_1 = \alpha_p$).

(vi) *In* (2.6), *setting* $\lambda = 1, a_1 = b_q, \alpha_1 = \beta_q, a_p = b_1, \alpha_p = \beta_1, a = a_1$, $\beta = a_p$ *and* $h = \alpha_1$; *replacing* l *by* $l+1$, q *by* $q+2$; (b_{j+1}, β_{j+1}) *by* (b_j, β_j) $(j = 1, 2, \ldots, q)$ *and putting* $n = 1$, *we have*

$$(3.6)\qquad H^{l,u}_{p,q}\left[x\left|\begin{matrix}(a_1, \alpha_1), \ldots, (a_{p-1}, \alpha_{p-1}), (a_p, \alpha_1)\\(b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right.\right]$$

$$=(a_p-a_1+1)H^{l,u}_{p,q}\left[x\left|\begin{matrix}\{(a_{p-1}, \alpha_{p-1})\}, (a_p+1, \alpha_1)\\\{(b_q, \beta_q)\}\end{matrix}\right.\right]-$$

$$-H^{l,u}_{p,q}\left[x\left|\begin{matrix}(a_1-1, \alpha_1), (a_2, \alpha_2), \ldots, (a_{p-1}, \alpha_{p-1}), (a_p+1, \alpha_1)\\(b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right.\right]$$

(*when we replace* a_p *by* a_p-1 *we get* ([1], (3.1)) *with* $\alpha_p = \alpha_1$).

§ 4. In this section we derive some identities for the H-functions from the summations evaluated in § 2.

(2.7), (2.9), (2.10) and (2.11) with $\lambda = 1$ and $n = 1$ respectively reduce to the identities:

$$(4.1)\qquad H^{l,u+2}_{p+2,q+2}\left[x\left|\begin{matrix}(c+k, h), (c-k-1, h), \{(a_p, \alpha_p)\}\\\{(b_q, \beta_q)\}, (c-k, h), (c+k+1, h)\end{matrix}\right.\right]$$

$$= H^{l,u+1}_{p+1,q+1}\left[x\left|\begin{matrix}(c-1, h), \{(a_p, \alpha_p)\}\\\{(b_q, \beta_q)\}, (c+1, h)\end{matrix}\right.\right]-k(k+1)H^{l,u}_{p,q}\left[x\left|\begin{matrix}\{(a_p, \alpha_p)\}\\\{(b_q, \beta_q)\}\end{matrix}\right.\right],$$

$$(4.2)\qquad H^{l+2,u}_{p+2,q+2}\left[x\left|\begin{matrix}\{(a_p, \alpha_p)\}, (a-k-1, h), (a+k, h)\\(a-k, h), (a+k+1, h), \{(b_q, \beta_q)\}\end{matrix}\right.\right]$$

$$= H^{l+1,u}_{p+1,q+1}\left[x\left|\begin{matrix}\{(a_p, \alpha_p)\}, (a-1, h)\\(a+1, h), \{(b_q, \beta_q)\}\end{matrix}\right.\right]-k(k+1)H^{l,u}_{p,q}\left[x\left|\begin{matrix}\{(a_p, \alpha_p)\}\\\{(b_q, \beta_q)\}\end{matrix}\right.\right],$$

$$(4.3)\qquad H^{l+2,u}_{p+2,q+2}\left[x\left|\begin{matrix}(\{a_p, \alpha_p)\}, (a+\beta, h), (a-\beta-\frac{1}{2}, h)\\(a+\beta+1, h), (a-\beta+\frac{1}{2}, h), \{(b_q, \beta_q)\}\end{matrix}\right.\right]$$

$$= H^{l+2,u}_{p+2,q+2}\left[x\left|\begin{matrix}\{(a_p, \alpha_p)\}, (a-\frac{1}{2}, h), (a, h)\\(a+\frac{1}{2}, h), (a+1, h), \{(b_q, \beta_q)\}\end{matrix}\right.\right]-\beta(\beta+\tfrac{1}{2})H^{l,u}_{p,q}\left[x\left|\begin{matrix}\{(a_p, \alpha_p)\}\\\{(b_q, \beta_q)\}\end{matrix}\right.\right],$$

$$(4.4)\qquad H^{l+1,u}_{p+1,q+1}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\},(a-\beta-1,h)\\(a-\beta,h),\{(b_q,\beta_q)\}\end{matrix}\right.\right]$$

$$=H^{l+1,u}_{p+1,q+1}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\},(a+1,h)\\(a+2,h),\{(b_q,\beta_q)\}\end{matrix}\right.\right]-(\beta+2)H^{l,u}_{p,q}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\}\\\{(b_q,\beta_q)\}\end{matrix}\right.\right].$$

§ 5. PARTICULAR CASES. *For* $\alpha_i=\beta_j=h=1$ $(i=1,2,\dots,p;j=1,2,\dots,q)$, (2.9), (2.10), (2.11), (2.12) *and* (2.13) *in view of* (1.11) *respectively reduce to the known results* ([3], p. 15 (4.6); [6], (3.1); [6], (3.3); [6], (3.6) *and* [3], p. 16 (4.7)).

I wish to express my sincere thanks to Dr. R. K. Saxena of G. S. Technological Institute, Indore, for his kind help and guidance during the preparation of this paper.

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Reçu par la Rédaction le 26. 7. 1967