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On differentiable solutions of Böttcher's functional equation

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We shall consider Böttcher's functional equation

$$\varphi[f(x)] = \{\varphi(x)\}^p, \quad p > 1, \tag{1}$$

where φ is the unknown function.

For a fixed number a, $0 < a \leqslant \infty$, we denote by U^a (for every real a) the class of functions which are defined and positive in $(0, a)$ and for which there exists a $\lim\limits_{x\to 0+} x^{-a} f(x)$ and the limit is positive.

In paper [3] Kuczma has proved under some assumptions on the function $f(x)$ (which are certainly fulfilled in our case) that there exists in $\langle 0, a)$ exactly one solution of equation (1) that belongs to the class U^a. This solution is given by the formula

$$\chi(x) = x^a \cdot e^{\psi(x)}, \tag{2}$$

where $\psi(x)$ is a certain function continuous in $\langle 0, a)$.

Giving the example with the function $f(x) = x^2$ and $p = 2$, Kuczma has proved (also in [3]) the lack of uniqueness of C^r solutions of equation (1) in $\langle 0, a)$.

It turns out, however, that this is a general situation.

We shall denote by H^r the following set of hypotheses:

(H^r) *$f \in U^p$, $p > 1$, $f(x)$ is of class C^r in $\langle 0, a)$, $0 < a \leqslant \infty$, $f'(x) > 0$ and $f(x) \neq x$ in $(0, a)$ and $x^{-p} f(x)$ (defined for $x = 0$ as its limit) is of class C^r in $\langle 0, a)$.*

THEOREM 1. *Suppose that hypotheses* (H^1) *are fulfilled. Then there exists a C^1 solution of equation* (1) *in $\langle 0, a)$ depending on an arbitrary function.*

Proof. Let us take an arbitrary $x_0 \in (0, a)$ and let us put:

$$x_1 = f(x_0), \quad x_{n+1} = f(x_n), \quad n > 0.$$

The sequence $\{x_n\}$ is strictly decreasing and converges to zero. We obviously have:

$$(0, x_0\rangle = \bigcup_{k=0}^{\infty} (x_{k+1}, x_k\rangle .$$

Let $\varphi_0(x)$ be an arbitrary function in $\langle x_1, x_0\rangle$ fulfilling the following conditions:

(3) $$\varphi_0(x) \text{ is of class } C^1 \text{ in } \langle x_1, x_0\rangle ,$$

(4) $$0 \leqslant \varphi_0(x) \leqslant \chi(x) \quad \text{in } \langle x_1, x_0\rangle ,$$

(5) $$\varphi_0(x_1) = \{\varphi_0(x_0)\}^p ,$$

(6) $$\varphi_0'(x_1) = \frac{1}{f'(x_0)} p[\varphi_0(x_0)]^{p-1}\varphi_0'(x_0) .$$

$\chi(x)$ denotes here a fixed function (2) with an $a > 1$.

It follows from a result of Choczewski [1] that under hypotheses (3), (5) and (6) there exists exactly one C^1 solution $\varphi(x)$ of equation (1) in $(0, a)$, which is an extension of the function $\varphi_0(x)$ onto the whole interval $(0, a)$.

It turns out that the inequality $0 \leqslant \varphi(x) \leqslant \chi(x)$ holds in the whole interval $(0, x_0\rangle$.

In fact, this inequality holds in $(x_1, x_0\rangle$ by (4). Assuming it true in $(x_k, x_{k-1}\rangle$, $k \geqslant 1$, we have for $x \in (x_{k+1}, x_k\rangle$, $f^{-1}(x) \in (x_k, x_{k-1}\rangle$, and, since both $\varphi(x)$ and $\chi(x)$ satisfy (1),

$$\begin{aligned}\varphi(x) = \varphi\big(f[f^{-1}(x)]\big) &= \{\varphi[f^{-1}(x)]\}^p \leqslant \{\chi[f^{-1}(x)]\}^p \\ &= \chi\big(f[f^{-1}(x)]\big) = \chi(x);\end{aligned}$$

on the other hand,

$$\varphi(x) = \varphi\big(f[f^{-1}(x)]\big) = \{\varphi[f^{-1}(x)]\}^p \geqslant 0 .$$

Thus $0 \leqslant \varphi(x) \leqslant \chi(x)$ in $\bigcup_{k=0}^{\infty} (x_{k+1}, x_k\rangle = (0, x_0\rangle$. Putting

$$\varphi(0) = \lim_{x\to 0+} \varphi(x) = 0$$

(cf. (2)), we have the inequalities

(7) $$0 \leqslant \varphi(x) \leqslant \chi(x) \quad \text{for } x \in \langle 0, x_0\rangle .$$

Since $a > 1$, we can easily obtain in view of (7):

$$\varphi'(0) = \lim_{h\to 0+} \frac{\varphi(h) - \varphi(0)}{h} = \lim_{h\to 0+} \frac{\varphi(h)}{h} = 0 .$$

It remains to prove that the derivative $\varphi'(x)$ is continuous at zero. We differentiate equation (1) and we write it in the form:

$$\varphi'[f(x)] = \frac{p[\varphi(x)]^{p-1}}{f'(x)}\varphi'(x), \quad p > 1. \tag{8}$$

In equation (8) we can treat φ as the known function and φ' as the unknown one: then (8) is a linear equation.

For $x \in (0, x_0\rangle$ we have the inequalities

$$0 \leqslant \frac{p[\varphi(x)]^{p-1}}{f'(x)} \leqslant \frac{p[\chi(x)]^{p-1}}{f'(x)} = \frac{px^{a(p-1)}e^{(p-1)\psi(x)}}{f'(x)}. \tag{9}$$

It follows from hypotheses (H^1) that $f(x) = x^p g(x)$, where $g(x)$ is a function of class C^1 in $\langle 0, a)$ and there exists a limit $\lim\limits_{x\to 0+} g(x) > 0$. Since $f'(x) = x^{p-1}\big(pg(x)+xg'(x)\big)$ and $a > 1$, the right-hand side of (9) tends to zero as $x \to 0+$. Thus in virtue of a theorem due to Kordylewski and Kuczma (cf. [2], and [4] Theorem 2.8) we obtain $\lim\limits_{x\to 0+} \varphi'(x) = 0$, which was to be proved.

Lemma. *Let the functions appearing in* (1) *be of class* C^r *and* $f'(x) \neq 0$ *in an interval. Differentiating* (1) k *times,* $k \leqslant r$, *we may write equation* (1) *in the form*:

$$\varphi^{(k)}[f(x)] = \frac{p[\varphi(x)]^{p-1}}{[f'(x)]^k}\varphi^{(k)}(x) + \frac{[\varphi(x)]^{p-1}}{[f'(x)]^{2k-1}}L(\varphi, \varphi', \ldots, \varphi^{(k-1)}, f', \ldots, f^{(k)}),$$

where L *is a polynomial in the variables* $\varphi, \varphi', \ldots, f^{(k)}$.

We omit the simple inductive proof of this lemma.

Theorem 2. *Suppose that hypotheses* (H^r), $r \geqslant 1$, *are fulfilled and* $r < p$. *Then there exists a* C^r *solution of equation* (1) *in* $\langle 0, a)$ *depending on an arbitrary function.*

Proof. As in the proof of Theorem 1, let $\varphi_0(x)$ be an arbitrary function in $\langle x_1, x_0\rangle$ fulfilling conditions (4), (5) and

$$\varphi_0(x) \text{ is of class } C^r \text{ in } \langle x_1, x_0\rangle, \tag{10}$$

$$\varphi_0^{(k)}(x_1) = F_k(x_0), \quad k = 1, \ldots, r, \tag{11}$$

where

$$F_1(s) \stackrel{\text{df}}{=} \frac{1}{f'(s)} p[\varphi_0(s)]^{p-1}\varphi_0'(s),$$

$$F_{k+1}(s) \stackrel{\text{df}}{=} \frac{1}{f'(s)} \cdot \frac{d}{ds}F_k(s), \quad k = 1, \ldots, r-1.$$

In condition (4) $\chi(x)$ denotes a fixed function (2) with an

$$a > \frac{(2r-1)(p-1)}{p-r}.$$

From paper [1] we know that under hypotheses (5), (10), (11) there exists exactly one C^r solution $\varphi(x)$ of equation (1) in $(0, a)$, which is an extension of the function $\varphi_0(x)$ onto the whole interval $(0, a)$.

It remains to prove that all the derivatives $\varphi^{(k)}(x)$, $k = 1, \ldots, r$, exist and are continuous at zero.

We shall show by induction that if $k \leqslant r < p$, then

$$\lim_{x\to 0+} \varphi^{(k)}(x) = 0 = \varphi^{(k)}(0). \tag{12}$$

From Theorem 1 it follows that $\lim_{x\to 0+} \varphi'(x) = 0 = \varphi'(0)$. Suppose that (12) holds for a certain $k < r$.

We differentiate (1) $k+1$ times. Applying the lemma in $(0, a)$ we obtain:

$$\varphi^{(k+1)}[f(x)] = \frac{p[\varphi(x)]^{p-1}}{[f'(x)]^{k+1}}\varphi^{(k+1)}(x) + \frac{[\varphi(x)]^{p-k-1}}{[f'(x)]^{2k+1}} L_1(\varphi, \varphi', \ldots, \varphi^{(k)}, f', \ldots, f^{(k+1)})$$

where L_1 is a polynomial.

Since by (12) $\varphi(x)$ is of class C^k in $\langle 0, a)$ and on account of hypotheses (H^r), $f(x)$ is of class C^r in $\langle 0, a)$ and $L_1(\varphi(x), \varphi'(x), \ldots, \varphi^{(k)}(x), f'(x), \ldots, f^{(k+1)}(x))$ is at least of class C^0 in $\langle 0, a)$.

If we show that

$$\frac{[\varphi(x)]^{p-k-1}}{[f'(x)]^{2k+1}} \to 0 \quad \text{as } x \to 0+,$$

then

$$\frac{\varphi^{p-k-1}}{(f')^{2k+1}} L_1 \to 0 \quad \text{and} \quad \frac{p\varphi^{p-1}}{(f')^{k+1}} \to 0$$

a fortiori.

For $x \in (0, x_0\rangle$ we have

$$0 \leqslant \frac{[\varphi(x)]^{p-k-1}}{[f'(x)]^{2k+1}} \leqslant \frac{[\chi(x)]^{p-k-1}}{[f'(x)]^{2k+1}} = \frac{x^{a(p-k-1)} e^{(p-k-1)\psi(x)}}{[f'(x)]^{2k+1}}. \tag{13}$$

Since $f'(x) = x^{p-1}(pg(x) + xg'(x))$ and, moreover, $p > k+1$ and

$$a > \frac{(2k+1)(p-1)}{p-k-1},$$

we see that the right-hand side of (13) tends to zero as $x \to 0+$. As before, we obtain

$$\lim_{x\to 0+} \varphi^{(k+1)}(x) = 0 ,$$

and since by (12) $\varphi(x)$ is of class C^k in $\langle 0, a)$ and of class C^r in $(0, a)$, $\varphi^{(k+1)}(0)$ exists and

$$\varphi^{(k+1)}(0) = \lim_{x\to 0+} \varphi^{(k+1)}(x) = 0 .$$

This completes the proof.

References

[1] B. Choczewski, *On differentiable solutions of a functional equation*, Ann. Polon. Math. 13 (1963), pp. 133-138.

[2] J. Kordylewski and M. Kuczma, *On some linear functional equations*, ibidem 9 (1960), pp. 119-136.

[3] M. Kuczma, *Sur l'équation fonctionnelle de Böttcher*, Mathematica, Cluj, 8 (31), (1966), pp. 279-285.

[4] — *Functional equations in a single variable*, Warszawa 1968.

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