ANNALES
POLONICI MATHEMATICI
XXXI (1975)

Some oscillation properties of third order linear homogeneous differential equations

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Abstract. This paper answers a certain question raised earlier concerning the oscillatory behaviour of solutions of third order linear homogeneous differential equations. We also determine a certain class of such equations with the property that each oscillatory equation has two oscillatory solutions such that a solution is oscillatory if and only if it is a non-trivial linear combination of these two solutions.

Consider the differential equation

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0, \tag{1}$$

where $p, q, r \in C[a, \infty)$. A non-trivial solution is said to be *oscillatory* if its set of zeros is not bounded above. Equation (1) is said to be *oscillatory* if it has oscillatory solutions. Non-trivial solutions which are not oscillatory are called *non-oscillatory*.

The following two definitions, introduced in [2], were mainly motivated by the work of Pólya [5].

DEFINITION 1. The differential equation (1) is said to have *property R* on $[a, \infty)$ if it has both oscillatory and non-oscillatory solutions, and further, it has two solutions u_1 and u_2 with $W(u_1, u_2)(x) \neq 0$, where $W(u_1, u_2)$ represents the wronskian of u_1 and u_2.

Remark 1. It follows that the solutions u_1 and u_2 are both oscillatory. For, suppose $u_1(x) \neq 0$ on some interval $[b, \infty)$. Then $u_1(x) \neq 0$ and $W(u_1, u_2)(x) \neq 0$ imply that no solution of (1) can have more than two zeros on $[b, \infty)$ (see [5]). Consequently, any linear combination of u_1 and u_2 is oscillatory since they are solutions of a second order linear homogeneous differential equation. It follows that if v is a non-oscillatory solution of (1), then $W(u_1, u_2, v)$ does not vanish anywhere.

DEFINITION 2. The differential equation (1) is said to have *property RO* if it has property R and a solution of (1) is oscillatory if and only if it is a non-trivial linear combination of u_1 and u_2, where u_1 and u_2 are the solutions of Definition 1. Equation (1) is said to have *property RN* if it has property R and every non-oscillatory solution of (1) is a constant multiple of a fixed non-oscillatory solution.

The following three theorems have been established in [2].

THEOREM 1. *The differential equation* (1) *has property* R *on* $[a, \infty)$ *if and only if its adjoint has property* R *on some interval* $[b, \infty)$.

THEOREM 2. *Suppose that* (1) *has solutions* u_1, u_2, *and* v *such that* $v(x) \neq 0$ *for* $x \geqslant a$, *and* u_1 *and* u_2 *are oscillatory with* $W(u_1, u_2)(x) \neq 0$ *for* $x \geqslant a$. *Then* (1) *has property RO if and only if*

$$\lim_{x\to\infty} \frac{u_1(x)}{v(x)} = \lim_{x\to\infty} \frac{u_2(x)}{v(x)} = 0.$$

THEOREM 3. *If* (1) *has property RO on* $[a, \infty)$, *then its adjoint has property RN on some interval* $[b, \infty)$, $b \geqslant a$.

An unresolved question raised in [2] was whether or not the converse of Theorem 3 holds. We give a counter-example to show that the answer is in the negative. In this paper, we also show that under certain reasonable assumptions on the coefficients, (1) has property *RO*.

EXAMPLE 1. Let $u_1 = \sin x^2$, $u_2 = \cos x^2$, and $v = (2+1/x)+(2-1/x)\times \times\cos 4x$, $x > 0$. It follows that v is non-oscillatory, $W(u_1, u_2)(x) = -2x < 0$ for $x > 0$. Furthermore, calculating $W(u_1, u_2, v)$, it can be verified that $W(u_1, u_2, v)(x) < 0$ on $[a, \infty)$ for a sufficiently large positive number a. Hence, there exists an equation of the form (1) with solutions u_1, u_2, and v. Thus, we may assume that u_1, u_2 and v are solutions of (1). Consider the adjoint

$$y''' - (py)'' + (qy)' - ry = 0 \tag{1'}$$

of (1). If

$$F(x) = e^{\int_a^x p(t)dt},$$

then $U_1 = F(x)W(u_1, v)(x)$, $U_2 = F(x)W(u_2, v)(x)$, and $V = F(x)W(u_1, u_2)(x)$ are solutions of (1′) (see [3]). Clearly, U_1 is oscillatory since u_1 is oscillatory and v is not. Similarly, U_2 is oscillatory and V is non-oscillatory. It is easy to verify that

$$\lim_{x\to\infty} \frac{U_1(x)}{V(x)}$$

does not exist. Hence, by Theorem 2, (1′) does not have property *RO*.

Let $y = c_1 \sin x^2 + c_2 \cos x^2 + (2+1/x) + (2-1/x)\cos 4x$. In order to show that (1) has property *RN*, we consider four exhaustive cases.

Case I. Suppose $c_1 \geqslant 0$, $c_2 \geqslant 0$, and $c_1^2+c_2^2 > 0$. Then y can be written as

$$y = \sqrt{c_1^2+c_2^2}\sin(x^2+a)+(2+1/x)+(2-1/x)\cos 4x, \qquad 0 \leqslant a \leqslant \pi/2.$$

Suppose y is non-oscillatory. Then for some positive number b, $y(x) > 0$ for $x > b$. For, there exist arbitrarily large values of x for which $y(x) > 0$ since $v(x) > 0$ and $x^2+a = \beta$ has solutions for any number $\beta \geqslant a$. We note that $v\big((2n+1)\pi/4\big) = 8/(2n+1)\pi$.

Consequently,

$$y\big((2n+1)\pi/4\big) = \sqrt{c_1^2+c_2^2}\left[\sin\big((2n+1)^2\pi^2/16+a\big)+\frac{8}{\sqrt{c_1^2+c_2^2}(2n+1)\pi}\right].$$

Let $\varepsilon = \sin\pi/16$, and let N be a number such that $N > b$, $(2N+1)\pi/4 > b$, and

$$\frac{8}{\sqrt{c_1^2+c_2^2}\,(2n+1)\pi} < \varepsilon$$

for all $n \geqslant N$. In order to obtain a contradiction to the assumption that $y(x) > 0$ for $x > b$, it is sufficient to show that $\sin\big((2n+1)^2\pi^2/16+a\big) < -\varepsilon$ for some integer n, $n > N > b$. Thus, it is sufficient to find integers n, $n > N > b$, such that $(2n+1)\pi/4 > b$,

$$\frac{8}{\sqrt{c_1^2+c_2^2}\,(2n+1)\pi} < \varepsilon,$$

and $\sin\big((2n+1)^2\pi^2/16+a\big) < -\sin\pi/16$. The latter inequality is satisfied if $17\pi/16+2k\pi < (2n+1)^2\pi^2/16+a < 31\pi/16+2k\pi$. Since a is between 0 and $\pi/2$, $a = m\pi/16$ for some real number m, $0 \leqslant m \leqslant 8$. Thus, it suffices to show the existence of arbitrarily large integers n satisfying

$$17+32k < (2n+1)^2\pi+m < 31+32k, \tag{2}$$

where k is an integer.

We assert that there exist arbitrarily large integers n such that $8n\pi$ can be written as $8n\pi = 32p+r$, where p is an integer and $|r| < 1/100$. This follows since for any positive integer N there exist integers a and b, $1 \leqslant b \leqslant 3200N$, satisfying the inequality $|b\pi-a| < 1/3200N$ (see e.g. p. 196 of [6]). Consequently, for arbitrarily large integers N there exist integers a and b, $1 \leqslant b \leqslant 3200N$ satisfying the inequality $|32bN\pi-32Na| < 1/100$. Thus, if we let $4bN = n$ and $Na = p$, then $8n\pi = 32p+r$, where $|r| < 1/100$.

It follows that for any choice of m, $0 \leqslant m \leqslant 8$, one of the numbers $x_i = (2i+1)^2\pi$, $i = n, n+1, \ldots, n+8$, satisfies inequality (2) for some integer k. To see this, let $x_n = (2n+1)^2\pi = 32k+r'$, where k is an integer and $0 < r' < 32$. Then, $x_{n+1} = \big((2n+1)+2\big)^2\pi = 32k+r'+8n\pi+8\pi$. Similarly, each x_{n+j}, $j = 2, 3, \ldots, 8$, can be written in the form $x_{n+j} =$

$= 32k + r' + p_j(8n\pi) + q_j(8\pi)$, where p_j and q_j are integers. Replacing $8n\pi$ by $32p + r$ and dividing $q_j(8\pi)$ by 32, we have

$$\begin{aligned} x_{n+1} &= 32k_1 + r + r' + 25.13272\ldots, \\ x_{n+2} &= 32k_2 + 2r + r' + 11.39816\ldots, \\ x_{n+3} &= 32k_3 + 3r + r' + 22.79632\ldots, \\ x_{n+4} &= 32k_4 + 4r + r' + 27.3272\ldots, \\ x_{n+5} &= 32k_5 + 5r + r' + 24.9908\ldots, \\ x_{n+6} &= 32k_6 + 6r + r' + 15.78712\ldots, \\ x_{n+7} &= 32k_7 + 7r + r' + 31.7161\ldots, \\ x_{n+8} &= 32k_8 + 8r + r' + 8.7719\ldots, \end{aligned}$$

where each k_i, $i = 1, 2, \ldots, 8$, is an integer, $|r| < 1/100$, and $0 < r' < 32$.

By dividing up the range of the values of m into subintervals $i \leqslant m \leqslant i+1$, $i = 0, 1, \ldots, 7$, one can verify that for each value of m in this range one of the numbers x_i, $i = n, n+1, \ldots, n+8$, satisfies inequality (2). For example, suppose that $0 \leqslant m \leqslant 1$.

Then if $r' \leqslant 7$, x_{n+3} satisfies (2). If $7 \leqslant r' \leqslant 14$, then x_{n+6} satisfies (2). For $14 \leqslant r' \leqslant 21$, x_{n+8} satisfies (2). If $21 \leqslant r' \leqslant 30$, then x_{n+7} satisfies (2). We note that for $21 \leqslant r' \leqslant 30$, x_{n+7} satisfies the inequality

$$17 + 32(k_7 + 1) < x_{n+7} + m < 31 + 32(k_7 + 1).$$

Finally, for $30 \leqslant r' \leqslant 32$, x_{n+5} satisfies the inequality

$$17 + 32(k_5 + 1) < x_{n+5} + m < 31 + 32(k_5 + 1).$$

Case II. Suppose $c_1 \geqslant 0$, $c_2 \leqslant 0$, and $c_1^2 + c_2^2 \neq 0$. Then we can write

$$y = \sqrt{c_1^2 + c_2^2}\cos(x^2 - a) + (2 + 1/x) + (2 - 1/x)\cos 4x,$$

$$\pi/2 \leqslant a \leqslant \pi.$$

If we let $-\varepsilon = \cos 9\pi/16$, the same reasoning as in Case I reduces our problem to showing the existence of arbitrarily large integers n satisfying

$$9 + 32k < (2n+1)^2\pi - m < 23 + 32k,$$

where k is an integer and m is a number such that $8 \leqslant m \leqslant 16$. As in Case I, it can be verified that for each value of m, some x_i, $i = n, n+1, \ldots, n+8$, satisfies the above inequality.

Case III. Suppose $c_1 \leqslant 0$, $c_2 \leqslant 0$, and $c_1^2 + c_2^2 \neq 0$. Then we can write

$$y = -\sqrt{c_1^2 + c_2^2}\sin(x^2 + a) + (2 + 1/x) + (2 - 1/x)\cos 4x, \qquad 0 \leqslant a \leqslant \pi/2.$$

If we let $\varepsilon = \sin\pi/16$, our problem reduces to showing the existence of arbitrarily large integers n satisfying

$$1+32k < (2n+1)^2\pi+m < 15+32k,$$

where k is an integer and m is a number such that $0 \leqslant m \leqslant 8$. Again, it can be verified that for each value of m, some x_i, $i = n, n+1, \ldots, n+8$, satisfies the above inequality.

Case IV. Suppose $c_1 \leqslant 0$, $c_2 \geqslant 0$, and $c_1^2+c_2^2 \neq 0$. Then y can be written as

$$y = -\sqrt{c_1^2+c_2^2}\cos(x^2-a)+(2+1/x)+(2-1/x)\cos 4x, \qquad \pi/2 \leqslant a \leqslant \pi.$$

Let $\varepsilon = \cos 7\pi/16$. It suffices to show that $-\cos(x^2-a) < -\varepsilon$. Thus it is sufficient to show the existence of arbitrarily large integers n satisfying one of the inequalities

$$32k < (2n+1)^2\pi-m < 7+32k$$

or

$$25+32k < (2n+1)^2\pi-m < 32+32k,$$

where k is an integer and m is a number such that $8 \leqslant m \leqslant 16$. Again, by considering values of m in subintervals of length one, it can be verified that, for each value of m, one of the x_i's satisfies one of the above two inequalities.

Now, we consider the differential equation

$$y''' = p(x)y'+q(x)y, \tag{3}$$

where $p, q \in C[a, \infty)$.

LEMMA 1. *Suppose $p, q \in C[a, \infty)$ with $p > 0$ and $q > 0$. If $p \in C'[a, \infty)$ with $p' \geqslant 0$, then all oscillatory solutions of* (3), *if there are any, are bounded on* $[a, \infty)$.

Proof. Let $y(x)$ be any oscillatory solution of (3), and let x_1 be a fixed zero of $y'(x)$. Let x_2 be any other zero of $y'(x)$, $x_2 > x_1$. If

$$\max_{[x_1, x_2]} [y(x)]^2 = [y(\bar{x})]^2,$$

$\bar{x} \in [x_1, x_2]$, then $y'(\bar{x}) = 0$. Define

$$F[y(x)] = [y'(x)]^2-2y(x)y''(x)+p(x)y^2(x). \tag{4}$$

By differentiation,

$$F[y(\bar{x})] = F[y(x_1)]+\int_{x_1}^{\bar{x}} p'(s)y^2(s)\,ds-2\int_{x_1}^{\bar{x}} q(s)y^2(s)\,ds.$$

If $\bar{x} = x_1$, then

$$\max_{[x_1,x_2]} [y(x)]^2 = y^2(x_1).$$

If $x_1 < \bar{x}$, then

$$(5) \qquad F[y(\bar{x})] = F[y(x_1)] + \int_{x_1}^{\bar{x}} p'(s)y^2(s)\,ds - 2\int_{x_1}^{\bar{x}} q(s)y^2(s)\,ds$$

$$\leqslant F[y(x_1)] + y^2(\bar{x}) \int_{x_1}^{\bar{x}} p'(s)\,ds$$

$$= F[y(x_1)] + y^2(\bar{x})[p(\bar{x}) - p(x_1)].$$

From (4) and the fact that $y'(\bar{x}) = 0$, we have

$$F[y(\bar{x})] = [y'(\bar{x})]^2 - 2y(\bar{x})y''(\bar{x}) + p(\bar{x})y^2(\bar{x})$$
$$= -2y(\bar{x})y''(\bar{x}) + p(\bar{x})y^2(\bar{x}).$$

Therefore,

$$-2y(\bar{x})y''(\bar{x}) + p(\bar{x})y^2(\bar{x}) \leqslant F[y(x_1)] + p(\bar{x})y^2(\bar{x}) - p(x_1)y^2(\bar{x}),$$

or

$$p(x_1)y^2(\bar{x}) - 2y(\bar{x})y''(\bar{x}) \leqslant F[y(x_1)].$$

Now, by Lemma 2.1 of [4], $y(\bar{x})y''(\bar{x}) \leqslant 0$. For, $y(\bar{x})y''(\bar{x}) \geqslant 0$ and $y'(\bar{x}) = 0$ would imply that y is non-oscillatory. Hence, $p(x_1)y^2(\bar{x}) \leqslant F[y(x_1)]$, or

$$y^2(\bar{x}) \leqslant \frac{F[y(x_1)]}{p(x_1)}.$$

Consequently,

$$\max_{[x_1,x_2]} [y(x)]^2 = y^2(\bar{x}) \leqslant y^2(x_1) + \frac{F[y(x_1)]}{p(x_1)},$$

and the lemma is proved.

THEOREM 4. *Assume the hypothesis of Lemma* 1. *Then if* (3) *is oscillatory, it has property RO.*

Proof. Using Lemma 2.1 of [4] and the technique used in the proof of Theorem 3 [1], it follows that (3) has two linearly independent oscillatory solutions u and v whose linear combinations are also oscillatory. It follows that $W(u, v)(x)$ does not vanish anywhere. For, otherwise, a linear combination of u and v would have a double zero and would, hence, be non-oscillatory by Lemma 2.1 of [4]. Let z be the solution of (3) defined by the initial conditions $z(a) = z'(a) = 0$, and $z''(a) = 1$. Then z is nonoscillatory. Let $y = c_1u + c_2v + c_3z$ be any solution of (3). By Lemma 1, u and v are bounded.

$\lim_{x\to\infty} z(x) = \infty$. Hence, y cannot be oscillatory unless $c_3 = 0$. This shows that (3) has property *RO*.

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Reçu par la Rédaction le 2. 3. 1973