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ON LEAST SQUARE ESTIMATION OF SECOND ORDER STOCHASTIC PROCESSES WITH DISCRETE TIME

1. Introduction. Classical least square linear prediction theory is concerned with a stationary stochastic process, i.e. with a family X_n ($n = 0, 1, -1, \ldots$) of complex-valued random variables on a probability space $(\Omega, \mathcal{B}, P)$ which have zero means and finite covariances $\mathrm{E}X_n\bar{X}_m$ depending only on $n-m$. One accomplishment of this theory is the analytical characterization of regular and singular processes. In the one-dimensional theory of stationary processes the family of random variables forms a Hilbert space and, consequently, Hilbert space methods play there a key role. The idea occurred again for non-stationary processes of second order.

The attempt to extend the classical prediction theory to non-stationary processes, developed by Cramer [2], has attracted the attention of several mathematicians, e.g. Abdrabbo and Priestley [1], Mandrekar [4] and others.

This paper is devoted to the study of second order processes with discrete time (being not necessarily stationary). In Section 2 we give results on the Wold decomposition and I_∞-regularity. In the last section we show how the classical results may be extended to oscillatory processes.

2. J-regularity and Wold decomposition. Let $(\Omega, \mathcal{B}, P)$ be a probability space, and T — the set of all integers. $X = \{X_t, t \in T\}$ is called a ***second order process*** if, for every $t \in T$, $X_t \in L_2(\Omega, \mathcal{B}, P)$. Let $H(X, A)$ and $H(X)$ be closed subspaces of $H = L_2(\Omega, \mathcal{B}, P)$ generated by X_t, $t \in A$, and by all X_t, respectively, where A is an arbitrary subset of T.

Definition 1. Let J be an arbitrary family of non-empty subsets of T. The process X is

(a) *J-regular* if

$$\bigcap_{A \in J} H(X, A) = \{0\},$$

(b) *J-singular* if

$$\bigcap_{A\in J} H(X, A) = H(X).$$

THEOREM 1 (the Wold decomposition). *Let J be an arbitrary family of non-empty subsets of T. Then:*

(a) *For any given second order process X_t there exists a decomposition $X_t = Y_t + W_t$ having properties* (i)-(v):

(i) *Y and W are second order processes on T;*

(ii) *for all t, $s \in T$, Y_t is orthogonal to W_s;*

(iii) *the process Y is J-regular, and the process W is J-singular;*

(iv) *for all $A \in J$, $H(Y, A) \subset H(X, A)$ and $H(W, A) \subset H(X, A)$;*

(v) *$H(Y) \subset H(X)$ and $H(W) \subset H(X)$.*

(b) *If the family J satisfies the condition*

$$(*) \qquad \forall_{t\in T} \exists_{A\in J} \; t \in A,$$

then the components of the Wold decomposition are uniquely determined.

Proof. (a) The proof of the first part of this theorem is similar to that in the classical case. Let S be an intersection (over all $A \in J$) of the subspaces $H(X, A)$. Let W_t be the orthogonal projection of X_t onto S and let Y_t be the orthogonal projection onto its orthogonal complement $S^{\perp}$. One can easily show that conditions (i)-(v) are satisfied.

(b) Suppose that $X_t = Y_t + W_t$ for all $t \in T$ and that conditions (i)-(v) are satisfied. We prove that Y and W are the same processes which we have constructed in part (a). From conditions (ii) and (iv) it follows that for all $A \in J$

$$H(X, A) = H(Y, A) \oplus H(W, A).$$

If $Y_t \in H(Y, A)$, then $Y_t \perp H(W, A)$. We have

$$\begin{aligned} S &= \bigcap_{A\in J} H(X, A) = \bigcap_{A\in J} [H(Y, A) \oplus H(W, A)] \\ &= \Big[\bigcap_{A\in J} H(Y, A)\Big] \oplus \Big[\bigcap_{A\in J} H(W, A)\Big]. \end{aligned}$$

Using the J-regularity of Y and the J-singularity of W we may write this formula in the following form:

$$S = H(W, B) \quad \text{for all } B \in J.$$

By (*), for any $t \in T$ there exists a $B \in J$ such that $W_t \in H(W, B)$ and $Y_t \in H(Y, B)$. Then for all $t \in T$ we have $W_t \in S$, $Y_t \perp S$, and $X_t = Y_t + W_t$. Hence

$$W_t = \mathrm{Proj}_S X_t \quad \text{and} \quad Y_t = \mathrm{Proj}_{S^\perp} X_t.$$

Remark 1. If the process X is stationary and J is closed over translations, then Y and W are stationary and condition (*) is satisfied.

Example 1. Let $U_0, U_1, \ldots$ be an orthonormal system in $L_2(\Omega, \mathscr{B}, P)$. Consider now the processes $\{V_t\}$, $t \in T$, and $\{X_t\}$, $t \in T$, where

$$V_0 = U_1, \quad V_1 = U_2, \quad V_{-1} = U_3, \quad \ldots,$$
$$X_t = U_0 + V_t + V_{t-1}.$$

We define a family of subsets of T as follows:

$$J = \{A \subset T : A \cap \{0, 1\} = \emptyset\}.$$

We want to find the Wold decomposition of the process X.

By the construction used in Theorem 1 we have $Y_t = V_t + V_{t-1}$ and $W_t = U_0$, since

$$\bigcap_{A \in J} H(X, A) = [\![U_0]\!].$$

On the other hand, the processes

$$W_t' = \begin{cases} U_0, & t \neq 0, 1, \\ U_0 + V_0, & t = 0, 1, \end{cases}$$

$$Y_t' = \begin{cases} V_t + V_{t-1}, & t \neq 0, 1, \\ V_{-1}, & t = 0, \\ V_1, & t = 1, \end{cases}$$

satisfy conditions (i)-(v) of Theorem 1.

We remark that X is stationary.

Definition 2. Let $\hat{X}(t, A)$ be an element of $H(X, A)$ which satisfies the following condition:

$$\|X_t - \hat{X}(t, A)\| = \min_{y \in H(X,A)} \|X_t - y\|, \quad \text{i.e.,} \quad \hat{X}(t, A) = \mathrm{Proj}_{H(X,A)} X_t.$$

THEOREM 2. *Let I_∞ be $\{A_t = \{s \in T : s \leqslant t\}, t \in T\}$. Then:*

(a) *The process X is I_∞-regular iff in $H(X)$ there exists a complete orthogonal system $\{V_s, s \in T\}$ such that*

$$\forall_{t \in T}\ X_t = \sum_{s=-\infty}^{t} a(t, s) V_s \quad \text{and} \quad \sum_{s=-\infty}^{t} |a(t, s)|^2 < \infty.$$

(b) *If X is an I_∞-regular process, then there exists a complete orthogonal system $\{V_s\}$, $s \in T$, such that*

$$\forall_{m > 0}\ \hat{X}(t+m, A_t) = \sum_{s=-\infty}^{t} a(t+m, s) V_s.$$

Proof. (a) Sufficiency. We have

$$H(X, A_t) = [\![X_s, s \leqslant t]\!] \subset [\![V_s, s \leqslant t]\!].$$

Therefore,

$$\bigcap_{t\in T} H(X, A_t) \subset \bigcap_{t\in T} [\![V_s, s \leqslant t]\!] = \{0\},$$

since the system $\{V_s, s \in T\}$ is complete in $H(X)$. Hence X is I_∞-regular.

Necessity. Let $D(X, t)$ be the orthogonal complement of $H(X, A_{t-1})$ in $H(X, A_t)$. We note that

$$\forall_{t\in T}\ H(X, A_t) = [\bigoplus_{s=-\infty}^{t} D(X, s)] \oplus [\bigcap_{s=-\infty}^{t} H(X, A_s)] = \bigoplus_{s=-\infty}^{t} D(X, s),$$

since the process X is I_∞-regular. We have also

$$D(X, t) = \mathrm{Proj}_{D(X,t)} H(X, A_t) = \mathrm{Proj}_{D(X,t)} [\![X_t]\!],$$

since $X_s \perp D(X, t)$ for each $s < t$. It follows that $D(X, t)$ is either a one-dimensional space or $D(X, t) = \{0\}$. (In the case of a stationary process, $D(X, t)$ is exactly one-dimensional.)

Let V_s be an element of $D(X, s)$ which has norm 1 if $D(X, t) \neq \{0\}$. If $D(X, s) = \{0\}$, let $V_s = 0$.

From this definition it follows immediately that the set $\{V_s : V_s \neq 0, s \leqslant t\}$ is either empty (and then $H(X, A_t) = \{0\}$) or forms a complete orthonormal system in $H(X, A_t)$. In both cases, for each $w \in H(X, A_t)$ (in particular, for X_t) we have

$$w = \sum_{s=-\infty}^{t} (w, V_s) V_s$$

($(\ ,\ )$ denotes the inner product in $L_2(\Omega, \mathscr{B}, P)$), where

$$\sum_{s=-\infty}^{t} |(w, V_s)|^2 < \infty.$$

(b) This follows directly from part (a) of this theorem if we observe that

$$X_{t+m} = \sum_{s=-\infty}^{t+m} a(t+m, s) V_s = \sum_{s=-\infty}^{t} a(t+m, s) V_s + \sum_{s=t+1}^{t+m} a(t+m, s) V_s,$$

where the first sum belongs to $H(X, A_t)$ and the second one is orthogonal to $H(X, A_t)$.

The error of prediction $\|X_{t+m} - \hat{X}(t+m, A_t)\|^2$ is given by

$$\sum_{s=t+1}^{t+m} |a(t+m, s)|^2.$$

As an application of the results above we consider simple examples of the autoregressive process and the moving-average process.

Example 2. Let X be an autoregressive process, i.e., X_t is given by $X_t - a_t X_{t-1} = V_t$, where V_t is an orthogonal system in $L_2(\Omega, \mathscr{B}, P)$ and the norm of V_t is either 1 or 0. The process X_t may be written in the form

$$X_t = V_t + a_t V_{t-1} = V_t + a_t V_{t-1} + a_t a_{t-1} X_{t-2} = \ldots$$

Thus

$$\forall_{t \in T} \forall_{s \leqslant t} \; X_t = a_t a_{t-1} \ldots a_{t-s+1} V_s +$$
$$+ [V_t + a_t V_{t-1} + \ldots + a_t a_{t-1} \ldots a_{t-s+2} V_{s-1} + X_{s-1}],$$

and the second term is orthogonal to V_s. Hence

$$(X_t, V_s) = \begin{cases} a_t a_{t-1} \ldots a_{t-s+1}, & s < t, \\ 1, & s = t, \\ 0, & s > t. \end{cases}$$

From Theorem 2 (b) it follows that

$$\hat{X}(t+m, t) = \sum_{s=-\infty}^{t} (X_{t+m}, V_s) V_s = \sum_{s=-\infty}^{t} a_{t+m} a_{t+m-1} \ldots a_{t+m-s} V_s$$
$$= a_{t+m} a_{t+m-1} \ldots a_{m+1+t-s} \Big[V_t + \sum_{s=-\infty}^{t-1} a_t a_{t-1} \ldots a_{t-s+1} V_s \Big]$$
$$= a_{t+m} a_{t+m-1} \ldots a_{m+1} X_t .$$

We note that $\hat{X}(t+m, t)$ depends only on X_t.

Remark 2. If $\hat{X}(t+m, t)$ depends only on X_t for each $m > 0$, then X must satisfy the condition

$$X_{t+1} - b_{t+1} X_t = V_{t+1};$$

$\{V_t\}$ forms a complete orthogonal system in $H(X)$.

Indeed, for all $t \in T$ we have $\hat{X}(t+1, t) = b_{t+1} X_t$. Thus

$$X_{t+1} - b_{t+1} X_t = V_{t+1} \in D(X, t+1).$$

$\{V_t\}$ is a complete system, since $X_t \perp D(X, t-2)$.

Example 3. Let X be a moving-average process such that $X_t = a_t V_t - b_t V_{t-1}$, where $\{V_t\}$ forms an orthogonal system in $L_2(\Omega, \mathscr{B}, P)$ and the norm of V_t is either 1 or 0. We want to find $\hat{X}(t+m, t)$.

We have

$$X_{t+m} = a_{t+m} V_{t+m} - b_{t+m} V_{t+m-1} + \sum_{s=-\infty}^{t+m-2} 0 \cdot V_s$$

and, by Theorem 2 (a), X is I_∞-regular. From Theorem 2 (b) it follows immediately that

$$\hat{X}(t+m, t) = \sum_{s=-\infty}^{t} (X_{t+m}, V_s) V_s = \begin{cases} 0, & m > 1, \\ -b_{t+1} V_t, & m = 1. \end{cases}$$

Remark 3. If $\hat{X}(t+m, t) = 0$ for $m > 1$, then

$$H(X, t) \ominus H(X, t-2) = H(X, t).$$

Since $H(X, t) = D(X, t) \oplus D(X, t-1) \oplus H(X, t-2)$, we have

$$H(X, t) = D(X, t) \oplus D(X, t-1).$$

Hence $X_t = a_t V_t + b_t V_{t-1}$, where $V_t \in D(X, t)$, $V_{t-1} \in D(X, t-1)$.

3. Oscillatory processes.

Definition 3 (after [4]). The second order process $\{X_t, t \in T\}$ is called *oscillatory* if it has the representation

$$X_t = \int_0^{2\pi} e^{itu} a_t(u) dZ(u) \quad \text{for all } t \in T,$$

where Z is an orthogonal measure defined on the Borel subsets of $[0, 2\pi$ with values in H and

$$\int_0^{2\pi} |a_t(u)|^2 F(du) < \infty \quad \text{for all } t \in T,$$

where $F(A) = \|Z(A)\|^2$.

PROPOSITION 1. *If $\{X_t, t \in T\}$ is an oscillatory process, then there exists an inner product preserving isomorphism l between $H(X)$ and some closed subspace of $L_2(F)$.*

Proof. We put

$$[l(X_t)](u) = a_t(u) e^{itu}.$$

Obviously, $l(X_t)$ is an element of $L_2(F)$. We have

$$\begin{aligned}(X_t, X_s)_H &= \Big(\int_0^{2\pi} a_t(u) e^{itu} dZ(u), \int_0^{2\pi} a_s(u) e^{isu} dZ(u)\Big)_H \\ &= \int_0^{2\pi} a_t(u) e^{itu} \overline{a_s(u) e^{isu}} dF(u) = \big(a_t(\cdot) e^{it\cdot}, a_s(\cdot) e^{is\cdot}\big)_{L_2(F)} \\ &= \big(l(X_t), l(X_s)\big)_{L_2(F)}.\end{aligned}$$

Let $G = [\![l(X_t), t \in T]\!]$. The mapping l may be extended uniquely to the inner product preserving isomorphism between $H(X)$ and G. Theorems 3 and 4 are based on the idea of [1].

THEOREM 3. *Let $\{X_t, t \in T\}$ be an oscillatory process of the form*

$$X_t = \int_0^{2\pi} a_t(u) e^{itu} dZ(u), \quad F(A) = \|Z(A)\|^2.$$

Suppose F to be absolutely continuous with respect to the Lebesgue measure and such that

$$\frac{dF}{du} = f, \quad f(u) = |\psi(u)|^2, \quad \text{where } \psi(u) = \sum_{n=0}^{\infty} e^{-iun} g(n),$$

and

$$a_t(u) = \sum_{n=0}^{\infty} e^{-iun} g(t, n), \quad t \in T.$$

Then

$$X_t = \sum_{n=0}^{\infty} h(t, n) V_{t-n},$$

where $\{V_t\}$ is a complete orthonormal system in $H(X)$.

Proof. Since $\psi(u) \neq 0$ a.s. (with respect to the Lebesgue measure), we can write

$$X_t = \int_0^{2\pi} e^{itu} a_t(u) \frac{\psi(u)}{\psi(u)} dZ(u).$$

From the forms of the functions ψ and a_t it follows that

$$a_t(u)\psi(u) = \sum_{n=0}^{\infty} e^{-iun} h(t, n).$$

We put

$$V_t = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \frac{e^{itu}}{\psi(u)} dZ(u).$$

We show that $\{V_t, t \in T\}$ forms a complete orthonormal system in $H(X)$. We have

$$\|V_t\|^2 = \left(\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \frac{e^{itu}}{\psi(u)} dZ(u), \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \frac{e^{itu}}{\psi(u)} dZ(u) \right)_H$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{dF(u)}{|\psi(u)|^2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{|\psi(u)|^2}{|\psi(u)|^2} du = 1.$$

Let $s \neq t$. Then

$$(V_t, V_s)_H = \left(\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\frac{e^{itu}}{\psi(u)}\,dZ(u), \frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\frac{e^{isu}}{\psi(u)}\,dZ(u)\right)_H = \frac{1}{2\pi}\int_0^{2\pi} e^{iu(t-s)}\,du = 0.$$

Thus the completeness of $\{V_t\}$ is obvious.

Coming back to X_t we evaluate

$$X_t = \int_0^{2\pi}\frac{e^{itu}}{\psi(u)}\Big(\sum_{n=0}^{\infty} e^{-iun}h(t, n)\Big)dZ(u).$$

Since

$$\infty > \|X_t\|^2 = \sum_{n=0}^{\infty}|h(t, n)|^2,$$

we can write

$$X_t = \sum_{n=0}^{\infty} h(t, n)\int_0^{2\pi}\frac{e^{iu(t-n)}}{\psi(u)}\,dZ(u) = \sqrt{2\pi}\sum_{n=0}^{\infty} h(t, n)\,V_{t-n}.$$

The corollary follows immediately from Theorem 2 (a):

COROLLARY. *Under the assumptions of Theorem* 3 *the process* $\{X_t\}$ *is* I_∞*-regular.*

THEOREM 4. *Suppose that* $\{V_t, t \in T\}$ *is an orthonormal system in* $H(X)$. *Let*

$$X_t = \sum_{n=0}^{\infty} h(t, n)\,V_{t-n}.$$

Then X_t *is an oscillatory process and*

(i) $\int_0^{2\pi} \ln|a_t(u)|^2\,du > -\infty$,

(ii) F *is absolutely continuous with respect to the Lebesgue measure and*

$$\int_0^{2\pi} \ln\frac{dF}{du(u)}\,du > -\infty.$$

Proof. By the assumptions, $\{V_t\}$ is a stationary I_∞-regular process with random measure Z and spectral measure F. It is known (see [6]) that F satisfies (ii).

We have

$$X_t = \sum_{n=0}^{\infty} h(t, n) \int_0^{2\pi} e^{iu(t-n)} dZ(u).$$

Since

$$\infty > \|X_t\|^2 = 2\pi \sum_{n=0}^{\infty} |h(t, n)|^2,$$

we get

$$X_t = \int_0^{2\pi} e^{iut} \Big(\sum_{n=0}^{\infty} h(t, n) e^{-iun}\Big) dZ(u).$$

Putting $a_t(u) = \sum_{n=0}^{\infty} h(t, n) e^{-iun}$, we obtain

$$X_t = \int_0^{2\pi} e^{iut} a_t(u) dZ(u).$$

From the form of a_t it follows that (i) must be satisfied. Thus the proof is completed.

Now we consider a family J_0 of complements of all singletons of T.

For the rest of this paper let us suppose that $\{X_t\}$ is an oscillatory process and $a_t(u) = \psi_t(u) e^{-itu}$, where $\{\psi_t, t \in T\}$ forms an orthonormal complete system in $L_2(du)$. Let l be as in Proposition 1. We put

$$\hat{X}_s = \mathrm{Proj}_{[X_t, t \neq s]} X_s.$$

THEOREM 5. *Let F be absolutely continuous with respect to the Lebesgue measure and let $dF/du = f > 0$ a.s. Then for all $s \in T$ there exists $c(s)$ such that*

$$X_t = l^{-1}\left[\psi_s - \frac{c(s)\psi_s}{f}\right].$$

Proof. Let s be fixed and let $\varphi = l(\hat{X}_s)$. We know that $\varphi \in [\![\psi_t, t \neq s]\!]$. It follows that $\psi_s - \varphi \perp [\![\psi_t, t \neq s]\!]$. Hence

$$(1) \qquad \int_0^{2\pi} [\psi_s(u) - \varphi(u)] \overline{\psi_t(u)} f(u) du = 0, \quad s \neq t.$$

Let us put

$$(2) \qquad c(s) = \int_0^{2\pi} [\psi_s(u) - \varphi(u)] \overline{\varphi_s(u)} f(u) du$$

and consider the functions $[\psi_s(u)-\varphi(u)]f(u)$ and $c(s)\psi_s(u)$. According to (1) and (2) both functions have the same Fourier coefficients with respect to $\{\psi_t, t \in T\}$ and, therefore, they coincide.

Proposition 2. *Under the assumptions of Theorem* 5 *we have*

$$c(s) = \|X_s - \hat{X}_s\|^2.$$

Proof. We have

$$c(s) = (\psi_s - \varphi, \psi_s)_{L_2(F)}$$

and

$$(\psi_s - \varphi, \psi_s)_{L_2(F)} = (\psi_s - \varphi, \psi_s)_{L_2(F)} - (\psi_s - \varphi, \varphi)_{L_2(F)}$$

(because φ is orthogonal to $\psi_s - \varphi$). Hence

$$c(s) = \|\psi_s - \varphi\|^2_{L_2(F)} = \|X_s - \hat{X}_s\|^2.$$

Theorem 6. *Under the assumptions of Theorem* 5 *we have*

$$\|X_s - \hat{X}_s\|^2 = \left[\int_0^{2\pi} \frac{|\psi_s(u)|^2}{f(u)}\, du\right]^{-1}.$$

The proof of this result is essentially the same as in the classical case (see [6], p. 42-47) if we put ψ_t instead of $e^{2\pi itu}$ and $[0, 2\pi]$ instead of $[0, 1]$.

Example 4. We consider a Haar system in $L_2[0, 2\pi]$. Let

$$\varphi_0^{(0)}(u) = 1/\sqrt{2\pi},$$

$$\varphi_1^{(0)}(u) = \begin{cases} 1/\sqrt{2\pi}, & u \in [0, \pi), \\ 0, & u = \pi, \\ -1/\sqrt{2\pi}, & u \in (\pi, 2\pi], \end{cases}$$

and, for $m = 1, 2, \ldots$ and $k = 1, 2, \ldots, 2^m$,

$$\varphi_m^{(k)}(u) = \begin{cases} \sqrt{2^m}/\sqrt{2\pi}, & (k-1)\pi/2^{m-1} < u < (k-2^{-1})\pi/2^{m-1}, \\ -\sqrt{2^m}/\sqrt{2\pi}, & (k-2^{-1})\pi/2^{m-1} < u < k\pi/2^{m-1}, \\ 0, & \text{otherwise}. \end{cases}$$

The set of functions $\{\varphi_0^{(0)}, \varphi_1^{(0)}, \varphi_m^{(k)}, \ m = 1, 2, \ldots; \ k = 1, 2, \ldots, 2^m\}$ is called a *Haar system* in $L_2[0, 2\pi]$ (as we know, it is a complete orthonormal system; see [3], p. 194).

Let us put

$$\psi_0 = \varphi_0^{(0)}, \quad \psi_1 = \varphi_1^{(0)}, \quad \psi_{-1} = \varphi_1^{(1)}, \quad \psi_2 = \varphi_2^{(1)}, \quad \psi_{-2} = \varphi_2^{(2)}, \quad \ldots$$

We consider the oscillatory process $\{X_t, t \in T\}$ for which $a_t(u) = \psi_t(u)e^{-itu}$ and the spectral measure F has the density $f(x) = x^2$.

To verify if $\{X_t\}$ is J_0-regular we evaluate

$$\int_0^{2\pi} \frac{|\psi_0(u)|^2}{f(u)} du = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_0^{2\pi} \frac{1}{u^2} du = \infty.$$

From Theorem 6 it follows that $\|X_0 - \hat{X}_0\|^2 = 0$, and hence $\{X_t\}$ is not J_0-regular.

To see that $\{X_t\}$ is not J_0-singular we calculate

$$\int_0^{2\pi} \frac{|\psi_{-2}(u)|^2}{f(u)} du = \int_{\pi/2}^{\pi} \frac{4}{2\pi} \frac{1}{u^2} du = \frac{2}{\pi^2}.$$

By virtue of Theorem 6 we have $\|X_{-2} - \hat{X}_{-2}\|^2 = \pi^2/2 > 0$ and it follows that $\{X_t\}$ is not J_0-singular.

We note that $\{X_t\}$ is not stationary, since, e.g., $(X_0, X_1) \neq (X_{-1}, X_0)$.

Example 5. Let $\{\varphi_s\}$ be as in Example 4. Let $\{X_t\}$ be an oscillatory process of the form

$$X_t = \int_0^{2\pi} \psi_t(u) e^{itu} e^{-itu} dZ(u)$$

with the spectral measure F which has the density

$$f(u) = \left[\psi_0(u) + \frac{1}{2}\psi_1(u)\right]^{-1}.$$

To find $\hat{X}_0$ we evaluate

$$\int_0^{2\pi} \frac{|\psi_0(u)|^2}{f(u)} du = \frac{1}{\sqrt{2\pi}}.$$

By virtue of Theorem 6 we have $\|X_0 - \hat{X}_0\|^2 = \sqrt{2\pi}$. From Theorem 5 and Proposition 2 we obtain

$$\begin{aligned}\varphi(u) &= \left\{\psi_0(u) - \sqrt{2\pi}\,\psi_0(u)\left[\psi_0(u) + \frac{1}{2}\psi_1(u)\right]\right\} \\ &= \psi_0(u) - \psi_0(u) - \frac{1}{2}\psi_1(u) = -\frac{1}{2}\psi_1(u)\end{aligned}$$

and

$$X_0 = l^{-1}(\varphi) = l^{-1}\left(-\frac{1}{2}\psi_1\right) = -\frac{1}{2}X_1.$$

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ZOFIA ŁAWNICZAK (Wrocław)

O ŚREDNIOKWADRATOWEJ ESTYMACJI PROCESÓW STOCHASTYCZNYCH DRUGIEGO RZĘDU Z CZASEM DYSKRETNYM

STRESZCZENIE

Praca poświęcona jest badaniu procesów stochastycznych drugiego rzędu z czasem dyskretnym (niekoniecznie stacjonarnych). Otrzymuje się rozkład Wolda i reprezentację w postaci średniej ruchomej. Dla procesów oscylujących otrzymuje się analityczną charakteryzację I_∞-regularnych i J_0-singularnych procesów oraz postać ich liniowej prognozy.
