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ON A PROBLEM OF NARKIEWICZ CONCERNING UNIFORM DISTRIBUTIONS OF SEQUENCES OF INTEGERS

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Narkiewicz [2] raises the following question: If $\mathcal{N}$ is a set of positive integers containing all divisors of its elements, does there exist a sequence (a_n) of positive integers such that (a_n) is uniformly distributed (u. d.) modulo M (in the sense of Niven [3]) if and only if $M \in \mathcal{N}$? The purpose of this note is to give an affirmative answer to this question.

Suppose G is a locally compact group. A closed subgroup H is said to be of *compact index* (or *syndetic*) if the quotient G/H is compact. If X is a compact Hausdorff space and μ is a regular Borel probability measure on X, then a sequence (x_n) is μ-u. d. if, for every continuous function f on X,

$$\lim \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_X f\, d\mu.$$

In particular, if X is a compact abelian group, μ normalized Haar measure on X, then Weyl's criterion yields that (x_n) is μ-u. d. if and only if, for every continuous character $\chi \neq 1$ on X, we have

$$\lim \frac{1}{N} \sum_{n=1}^{N} \chi(x_n) = 0.$$

It is known that μ-u. d. sequences exist whenever X is second countable and compact (in fact, μ-w. d. sequences exist — see [1]).

If G is a locally compact (additive) abelian group and H is a closed subgroup of compact index, then a sequence (g_n) in G is u. d. mod H if the sequence $(g_n + H)$ is u. d. in the group G/H. In this case, Weyl's criterion asserts that (g_n) is u. d. mod H if and only if, for every non-principal character χ of G which is 1 on H, we have

$$\lim \frac{1}{N} \sum_{n=1}^{N} \chi(g_n) = 0.$$

THEOREM. *Let G be a locally compact second countable abelian group. Let $\mathcal{S}$ ($\mathcal{S} \neq \emptyset$) and $\mathcal{T}$ be countable collections of closed subgroups of G such that*

(i) *finite intersections of members of $\mathcal{S} \cup \mathcal{T}$ are of compact index;*

(ii) *for each $S \in \mathcal{S}$ and $T \in \mathcal{T}$, we have $S \nsubseteq T$;*

(iii) *for each $T \in \mathcal{T}$ there exists a character χ_T of G such that χ_T is 1 on T but is non-trivial on each $S \in \mathcal{S}$.*

Then there exists a sequence (g_n) in G such that (g_n) is u. d. mod S *for every $S \in \mathcal{S}$ but is not u. d.* mod T *for $T \in \mathcal{T}$.*

Remarks. We notice that Condition (ii) is necessary for the conclusion of the theorem. When $\mathcal{S}$ and $\mathcal{T}$ are finite collections of subgroups of compact index, then it is possible to show that (ii) is sufficient for the conclusion, but the proof will not be given here. In the particular case, G being the integers, conditions (i) and (iii) hold in the context of Narkiewicz's questions and so we obtain the following result:

COROLLARY. *Let $\mathcal{N}$ be a non-empty subset of the positive integers Z^+ such that $n \in \mathcal{N}$, $m \in Z^+$, and $m|n \Rightarrow m \in \mathcal{N}$. Then there exists a sequence (x_k) of integers such that*

$$\lim \frac{1}{N} \quad (\# \text{ of } \; x_k \equiv j \pmod{M}) = \frac{1}{M} \qquad (j = 0, 1, \ldots, M-1)$$

for exactly those positive integers $M \in \mathcal{N}$.

Proof of the theorem. Let $\mathcal{S} = \{S_1, S_2, \ldots\}$, $\mathcal{T} = \{T_1, T_2, \ldots\}$. For each integer n, let

$$f_n = \tfrac{1}{2}\Big(2 + \sum_{j=1}^{n} (1/2^j)(\chi_{T_j} + \bar{\chi}_{T_j}\Big).$$

Let $U_n = S_1 \cap \ldots \cap S_n \cap T_1 \cap \ldots \cap T_n$ (where if $\mathcal{S}$ or $\mathcal{T}$ is finite, we just take the appropriate finite intersections). Then $G_n = G/U_n$ is a compact second countable abelian group and the groups G/T_i, and G/S_i ($i = 1, \ldots, n$) are quotients of G_n. Furthermore, we may consider f_n to be a continuous function on G_n. We notice also that $f_n \geqslant 0$. Let μ_n be the normalized Haar measure on G_n. Then, since the χ_{T_j} ($j = 1, \ldots, n$) are non-principal characters (considered as functions on G_n), we have

$$\int_{G_n} f_n d\mu_n = 1.$$

Let ν_n be the measure given by $d\nu_n = f_n d\mu_n$. Then ν_n is a probability measure on G_n. Therefore, there exists a sequence $(x_k^n)_k$ in G_n which is ν_n-u. d. In particular,

$$\lim \frac{1}{N} \sum_{r=1}^{N} \chi_{T_j}(x_r^n) = 1/2^j \qquad (1 \leqslant j \leqslant n)$$

and

$$\lim \frac{1}{N} \sum_{r=1}^{N} \chi(x_r^n) = 0$$

for every continuous character $\chi \neq 1$ on G which is trivial on S_i $(1 \leqslant i \leqslant n)$.

We enumerate the non-principal characters modulo the various S_n as $\chi_1, \chi_2, \ldots$ For each n, let $s(n)$ be an integer such that $N \geqslant s(n)$ implies that

(a) $$\left| \frac{1}{N} \sum_{k=1}^{N} \chi_j(x_k^n) \right| < 1/2^n$$

for all integers j $(1 \leqslant j \leqslant n)$ and

(b) $$\left| \frac{1}{N} \sum_{k=1}^{N} \chi_{T_j}(x_k^n) - 1/2^j \right| < 1/2^n \quad \text{for } 1 \leqslant j \leqslant n.$$

Weyl's criterion ensures the existence of such integers $s(n)$. Let $r(n) = 2^n s(n+1)$.

We now construct our sequence (g_k). (g_k) consists of the elements of various finite sequences $X_1, X_2, \ldots$ written in order, where X_n is the finite sequence $h_1^n, h_2^n, \ldots, h_r^n(n)$, where $h_i^n + U_n = x_i^n$. We claim that (g_k) is u.d. mod S_n, but not mod T_n. For let $\chi \neq 1$ be any character of G which is trivial on S_n. Then $\chi(h_i^m) = \chi(x_i^m)$ for $m \geqslant n$, whence, by (a),

$$\lim \frac{1}{N} \sum_{k=1}^{N} \chi(g_k) = 0.$$

On the other hand, in a similar manner, we see that (b) implies that

$$\lim \frac{1}{N} \sum_{k=1}^{N} \chi_{T_n}(g_k) = 1/2^n.$$

The result now follows by Weyl's criterion.

REFERENCES

[1] P. C. Baayen and Z. Hedrlin, *On the existence of well-distributed sequences in compact spaces*, Indagationes Mathematicae 27 (1965), p. 221-228.

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[4] L. A. Rubel, *Uniform distribution in locally compact groups*, Commentarii Mathematici Helvetici 39 (1965), p. 253-258.

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