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# Estimation of the sixth coefficient in the class of bounded univalent functions with real coefficients

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**Abstract.** Let $S_R(M)$, $M > 1$, denote the family of functions of the form

(1) $$F(z) = z + \sum_{n=2}^{\infty} A_{nF} z^n$$

holomorphic and univalent in the disc $K = \{z: |z| < 1\}$ satisfying the conditions

1° $|F(z)| \leqslant M$ for $z \in K$,

2° $A_{nF} = \bar{A}_{nF}$ for $n = 2, 3, \ldots$

In the present paper it is proved that there exists a constant $M_6$, $M_6 > 1$, such that, for all $M > M_6$ and every function $F \in S_R(M)$, the estimation

$$A_{6F} \leqslant P_6(M)$$

holds, where $P_6(M)$ is the sixth coefficient in Taylor expansion (1) of the function $w = P(z, M)$ given by the equation

$$w(1-wM^{-1})^{-2} = z(1-z)^{-2}, \quad P(0, M) = 0.$$

The proof avoids complicated integration of the differential-functional equation involved.

**1. Introduction.** Let $S$ denote the class of functions

(1) $$F(z) = z + \sum_{n=2}^{\infty} A_{nF} z^n$$

holomorphic and univalent in the disc $K = \{z: |z| < 1\}$. Bieberbach [1] obtained the estimation

(2) $$|A_{2F}| \leqslant 2$$

for functions $F$ of class $S$, equality in (2) holding only for the Koebe function

(3) $$F(z) = z/(1-e^{it} z)^2, \quad t \in \langle 0, 2\pi).$$

The fact that function (3) is extremal also with respect to some other functionals (e.g. $|F(z)|$, $|F'(z)|$, $z \in K$, $F \in S$) and its $n$-th coefficient is equal to $n$ allowed to formulate the supposition that the coefficients of every

$F \in S$ satisfy the estimate

$$|A_{nF}| \leqslant n, \quad n = 2, 3, \ldots \tag{4}$$

Many authors have been concerned with the verification of this hypothesis. In particular, positive results were obtained e.g. for $n = 3, 4, 5, 6$ ([12], [5], [2], [16], [15], [14]). In some subclasses of the family $S$ it was possible to obtain estimation (4) for every $n$. And so, let $S_K \subset S$ denote the subclass consisting of functions with real coefficients, i.e. $A_{nF} = \bar{A}_{nF}$ for $n = 2, 3, \ldots$ J. Dieudonné [3] proved that estimations (4) hold in the family $S_K$, equality taking place only for function (3) with real coefficients.

The above mentioned of Dieudonné is a premise for our further considerations. Another premise comes from some previous results concerning properties of the subclass $S_K(M)$, $M > 1$, of bounded functions ([4], [17], [18], [13], [8], [7], [19], [20], [26], [9]).

Let $S_K(M)$, $M > 1$, denote the family of functions of form (1) which are holomorphic and univalent in the disc $K$ and satisfy the conditions

1° $|F(z)| \leqslant M$ for $z \in K$,

2° $A_{nF} = \bar{A}_{nF}$ for $n = 2, 3, \ldots$

Of course, for every $M > 1$, $S_K(M) \subset S_K$.

It is known that (e.g. [17], [19], [26]): for any fixed $M > 1$ and every function $F \in S_K(M)$

$$A_{2F} \leqslant 2(1-M^{-1}); \tag{5}$$

for any $M > 11$ and every function $F \in S_K(M)$

$$A_{4F} \leqslant 2(2-10M^{-1}+15M^{-2}-7M^{-3}). \tag{6}$$

Equality in estimations (5), (6) holds only for the Pick function $w = P(z, M)$ given by the equation

$$w(1-wM^{-1})^{-2} = z(1-z)^{-2} \tag{7}$$

$(P(0, M) = 0,\ P(z, M) \in S_K(M))$. Moreover, estimation (6) is not true when $M$ is sufficiently close to unity (cf. [21]–[24]).

Z. J. Jakubowski has raised the following question: does there exist, for every $N = 2, 4, 6, \ldots$, a constant $M_N$, $M_N > 1$, such that for all $M > M_N$ and every function $F \in S_K(M)$ the estimation

$$A_{NF} \leqslant P_N(M) \tag{8}$$

takes place, where $P_N(M)$ is the $N$-th coefficient in Taylor expansion (1) of the function $P(z, M)$.

In the present paper this hypothesis is verified for $N = 6$. Its solution for an arbitrary even $N$ will be made the object of the next publication.

Let us notice that for functions of class $S_K(M)$, $M > 1$, the estimation

$$A_{3F} \leqslant 1+2\lambda^2-4\lambda M^{-1}+M^{-2} \quad \text{for } e \leqslant M < +\infty$$

is well known ([8], [11], [18], [25]), where $\lambda$ is the greater root of the equation $\lambda \log \lambda = -M^{-1}$. As is seen, the Pick function is not extremal in this case, and so hypothesis (8) is not true for an arbitrary odd $N$. Consequently, an analogous problem for $N$ odd requires separate consideration.

**2. The equation of extremal functions.** Let us consider in the family $S_K(M)$, $M > 1$, the functional

$$H(F) = A_{6F}. \tag{9}$$

This functional is continuous and the class is compact; thus there exists at least one function for which functional (9) attains its maximum. Denote by $\mathscr{F}_M$ the family of all functions maximal with respect to functional (9) in the class $S_K(M)$.

It follows immediately from the main theorem ([4], p. 8) that if $F(z) = z+\sum_{n=2}^{\infty} A_{nF}z^n$ is an arbitrary function of the family $\mathscr{F}_M$, then the function

$$w = f(z) = \frac{1}{M}F(z) = \sum_{n=1}^{\infty} a_{nf}z^n,$$
$$a_{1f} = 1/M, \quad a_{nf} = (1/M)A_{nF}, \quad n = 2, 3, \ldots,$$

satisfies the differential-functional equation

$$(zw'/w)^2\,\mathfrak{M}(w) = \mathfrak{N}(z), \quad 0 < |z| < 1, \tag{10}$$

where

$$\begin{aligned}
\mathfrak{M}(w) &= \sum_{j=2}^{6} a_{6f}^{(j)}\left(w^{j-1}+\frac{1}{w^{j-1}}\right)-\mathfrak{P},\\
\mathfrak{N}(z) &= 5a_{6f}+\sum_{j=2}^{6}(7-j)a_{7-j,f}\left(z^{j-1}+\frac{1}{z^{j-1}}\right)-\mathfrak{P},\\
f^m(z) &= \sum_{n=m}^{\infty} a_{nf}^{(m)}z^n, \quad m = 2, 3, \ldots,\\
\mathfrak{P} &= \min_{0\leqslant x\leqslant 2\pi}\left[\sum_{j=2}^{6} 2a_{6f}^{(j)}\cos(j-1)x\right].
\end{aligned} \tag{11}$$

**3. Auxiliary theorems.** Before we proceed to the investigation of the equation, we prepare a few lemmas.

Let us introduce the one-parameter family $\mathscr{P} = (P(z, M))_{M\in(1,+\infty)}$ of functions $w = P(z, M)$, $z \in K$, satisfying equation (7) and the condition $P(0, M) = 0$, $M \in (1, +\infty)$. Each function of the family $\mathscr{P}$ can be represented in the form

$$(12)\qquad P(z, M) = \frac{2Mz+M^2(1-z)^2-M(1-z)\sqrt{M[4z+M(1-z)^2]}}{2z},$$
$$z \in K,\ M \in (1, +\infty),$$

where the branch of the root has been fixed so that at the point $z = 0$ the function assumes value $M$. It is known that for every $M$, $M \in (1, +\infty)$, $P(z, M) \in S_K(M)$.

Denote

$$P(z, M) = z+\sum_{n=2}^{\infty} P_n(M)z^n, \quad z \in K,\ M \in (1, +\infty),$$

$$(13)\qquad F_0(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n, \quad z \in K.$$

From formulae (12), (13) follows

LEMMA 1. *Let $(M_h)_{h=1,2,\dots}$ be an arbitrary sequence of real numbers, $M_h > 1$, $h = 1, 2, \dots$, such that $\lim_{h\to+\infty} M_h = +\infty$, and let $(P(z, M_h))_{h=1,2,\dots}$ be the sequence of functions of family $\mathscr{P}$, corresponding to it. Then, for every number $\varepsilon > 0$ and any closed set $\Delta \subset K$, there exists an $h_0$ such that for all $h > h_0$ and $z \in \Delta$ we have*

$$|P(z, M_h)-F_0(z)| < \varepsilon.$$

Remark 1. It follows immediately from the Weierstrass theorem that $\lim_{h\to+\infty} P_n(M_h) = n$, $n = 2, 3, \dots$, for any sequence $(M_h)_{h=1,2,\dots}$ fulfilling conditions of Lemma 1.

LEMMA 2. *Let $(M_h)_{h=1,2,\dots}$ be an arbitrary sequence of real numbers, $M_h > 1$, $h = 1, 2, \dots$, such that $\lim_{h\to+\infty} M_h = +\infty$. From each of the families $\mathscr{F}_{M_h}$, $h = 1, 2, \dots$, let us choose arbitrarily one function $F_h(z) = z+\sum_{n=2}^{\infty} A_{nh}z^n$ and consider the sequence $(F_h)_{h=1,2,\dots}$. Then, for every number $\varepsilon > 0$ and any closed set $\Delta \subset K$, there exists an $h_0$ such that for all $h > h_0$ and $z \in \Delta$ we have*

$$|F_h(z)-F_0(z)| < \varepsilon.$$

The proof is a consequence of the inequality $P_6(M_h) \leqslant A_{6h} \leqslant 6$, $h = 1, 2, \dots$, of Remark 1, of the fact that $F_0$ is the only function in the family $S_K$ for which $A_{6F} = 6$, and of the Vitali–Osgood theorem.

Remark 2. It follows immediately from the Weierstrass theorem that for every sequence of extremal functions $(F_h)_{h=1,2,\dots}$, fulfilling conditions of Lemma 2, and for any $n$, $n = 2, 3, \dots$, $\lim_{h\to+\infty} A_{nh} = n$.

Since the sequence $(M_h)_{h=1,2,...}$ and the sequence of extremal functions $(F_h)_{h=1,2,...}$, corresponding to it have been chosen quite arbitrarily, we obtain immediately

COROLLARY 1. *If, for every* $M \in (1, +\infty)$, *any function* $F$ *of form* (1) $(A_{nF} = A_{nF}(M),\ n = 2, 3, \ldots)$ *belongs to the family* $\mathscr{F}_M$, *then, for every* $n$, $n = 2, 3, \ldots$, *and any* $\varepsilon > 0$, *there exists an* $M_n$ *such that for all* $M > M_n$ *and every function* $F \in \mathscr{F}_M$ *we have*

$$|A_{nF} - n| < \varepsilon.$$

LEMMA 3. *Let* $(M_h)_{h=1,2,...}$ *be an arbitrary sequence of real numbers,* $M_h > 1$, $h = 1, 2, \ldots$, *such that* $\lim_{h\to+\infty} M_h = +\infty$. *From each of the families* $\mathscr{F}_{M_h}$, $h = 1, 2, \ldots$, *let us choose arbitrarily a function* $F_h(z) = z + \sum_{n=2}^{\infty} A_{nh} z^n$ *and consider the sequence* $(F_h^m)_{h=1,2,...}$, *where* $m$ *is a fixed natural number,* $m \geqslant 2$. *Then, for every number* $\varepsilon > 0$ *and any closed set* $\Delta \subset K$, *there exists an* $h_0$ *such that for all* $h > h_0$ *and* $z \in \Delta$ *we have*

$$|F_h^m(z) - F_0^m(z)| < \varepsilon.$$

The proof is a consequence of the common boundednes of the sequence $(F_h^m)_{h=1,2,...}$ in every closed set contained in the disc $K$, of Lemma 2, and of the Vitali–Osgood theorem.

Remark 3. Let $(F_h^m)_{h=1,2,...}$ be an arbitrary sequence fulfilling conditions of Lemma 3. Denote

$$F_h^m(z) = \sum_{n=m}^{\infty} A_{nh}^{(m)} z^n, \qquad h = 1, 2, \ldots, \ m \geqslant 2,$$

$$F_0^m(z) = \sum_{n=m}^{\infty} A_{n0}^{(m)} z^n, \qquad m \geqslant 2.$$

It follows immediately from the Weierstrass theorem and from Lemma 3 that

$$\lim_{h\to+\infty} A_{nh}^{(m)} = A_{n0}^{(m)}, \qquad n = m,\ m+1, \ldots,\ m \geqslant 2.$$

From this remark, in view of the fact that the choice of the sequences $(M_h)_{h=1,2,...}$ and $(F_h^m)_{h=1,2,...}$ in Lemma 3 has been arbitrary, we obtain, in turn

COROLLARY 2. *Let* $m$, $m \geqslant 2$, *be an arbitrary fixed natural number. For every* $M \in (1, +\infty)$ *and any function* $F \in \mathscr{F}_M$ *let us denote*

$$(14) \qquad F^m(z) = \sum_{n=m}^{\infty} A_{nF}^{(m)} z^n, \qquad z \in K\ (A_{nF}^{(m)} = A_{nF}^{(m)}(M),\ n = m, m+1, \ldots).$$

*Then, for every* $n$, $n = m$, $m+1, \ldots$, *and every number* $\varepsilon > 0$, *there exists* *an* $M_n$ *such that for all* $M > M_n$ *and every function* $F \in \mathscr{F}_M$ *we have*

$$|A_{nF}^{(m)} - A_{n0}^{(m)}| < \varepsilon,$$

*where the numbers* $A_{n0}^{(m)}$, $n = m, m+1, \ldots,$ *denote the coefficients of the function* $F_0^m(z)$, *defined in Remark* 3.

LEMMA 4. *Suppose that*

1° $n$ *is an arbitrary fixed natural number;*

2° $(C_{0t}, C_{1t}, \ldots, C_{nt})_{t \in (1, +\infty)}$ *is a given family of* $(n+1)$*-element sequences of sets of real numbers*;

3° $\mathscr{C}_t = \{(c_0(t), c_1(t), \ldots, c_n(t)) \colon c_0(t) \in C_{0t}, c_1(t) \in C_{1t}, \ldots, c_n(t) \in C_{nt}\}$, $t \in (1, +\infty)$;

4° *there exists a sequence of numbers* $(c_0, c_1, \ldots, c_n)$ *which satisfies the condition: for every* $\eta > 0$ *there exists a* $t_0$ *such that for all* $t > t_0$ *the condition* $(c_0(t), c_1(t), \ldots, c_n(t)) \in \mathscr{C}_t$ *implies* $\max_{0 \leq k \leq n} |c_k(t) - c_k| < \eta$;

5° $W_0(z) = c_0 z^n + c_1 z^{n-1} + \ldots + c_n$, $\mathscr{W}_t = \{W(z, t) \colon W(z, t) = c_0(t) z^n + c_1(t) z^{n-1} + \ldots + c_n(t), (c_0(t), c_1(t), \ldots, c_n(t)) \in \mathscr{C}_t\}$, $t \in (1, +\infty)$.

*Then, for any number* $\varepsilon > 0$ *and every closed and bounded set* $\Delta$, *there exists a* $t'$ *such that for all* $t > t'$ *and every* $W(z, t) \in \mathscr{W}_t$ *we have*

$$|W(z, t) - W_0(z)| < \varepsilon$$

*for* $z \in \Delta$.

Proof. Let us take any number $\varepsilon > 0$ and a closed bounded set $\Delta$. Denote

$$L_\Delta = \max\{\sup_{z \in \Delta} |z|, \sup_{z \in \Delta} |z|^n, 1\}.$$

By 4°, there exists a $t'$ such that if $t > t'$ and $W(z, t) \in \mathscr{W}_t$, then, for every $k$, $k = 0, 1, \ldots, n$, we have $|c_k(t) - c_k| < \eta$, where $\eta = \varepsilon/(n+1) L_\Delta$. Then

$$\begin{aligned} &|W(z, t) - W_0(z)| \\ &\quad \leq |c_0(t) - c_0| |z|^n + |c_1(t) - c_1| |z|^{n-1} + \ldots + |c_n(t) - c_n| < \eta (n+1) L_\Delta = \varepsilon, \end{aligned}$$

which concludes the proof.

Remark 4. If assumptions 1°–5° of Lemma 4 are satisfied and if

6° $W_0(z) \not\equiv 0$,

7° $W_0(z)$ has a $k$-tuple zero at some point $z_0$,

then it follows from the Hurwitz theorem that there exists a $t'$ such that for all $t > t'$ each of the polynomials $W(z, t) \in \mathscr{W}_t$ has exactly $k$ zeros in every sufficiently small neighbourhood of the point $z_0$.

LEMMA 5. *If*

$$L_0(z) = z^8 + 2z^7 + 4z^6 + 6z^5 + 9z^4 + 6z^3 + 4z^2 + 2z + 1, \tag{15}$$

*then* $L_0(z)$ *can be factorized in the following way*:

(16) $$L_0(z) = \prod_{k=1}^{2} (z-z_k)(z-\bar{z}_k)(z-1/z_k)(z-1/\bar{z}_k),$$

*where* $|z_k| < 1$, $\operatorname{im} z_k \neq 0$, $k = 1, 2$, *and* $z_1 \neq z_2$.

Proof. Let us first suppose that polynomial (15) has in the disc $K$ one double root $z_0$ such that $\operatorname{im} z_0 \neq 0$. Then it follows from the symmetry of the real coefficients of the polynomial $L_0(z)$ that

(17) $$L_0(z) = (z-z_0)^2(z-\bar{z}_0)^2(z-1/z_0)^2(z-1/\bar{z}_0)^2.$$

The comparison of the coefficients of polynomials (15) and (17) gives an inconsistent system of equations, so that factorization (17) is impossible.

Next, suppose that $L_0(z)$ has in the disc $K$ a real root $r$. Of course, $r \neq 0$, $-1 < r < 1$. Applying the substitution $r+1/r = R$, we reduce the equation $L_0(r) = 0$ to the following inconsistent equation:

$$R^4+2R^3+3 = 0, \quad |R| > 2.$$

Consequently, this supposition is false, too.

In an analogous way we prove that the polynomial $L_0(z)$ has no roots on the circle $|z| = 1$. Thus the only possible factorization of the polynomial $L_0(z)$ is (16).

**4. The basic theorem.**

THEOREM. *Let* $S_R(M)$, $M > 1$, *be the families of functions* $F(z) = z + \sum_{n=2}^{\infty} A_{nF} z^n$ *holomorphic and univalent in the disc* $|z| < 1$, *having real coefficients and such that* $F \in S_R(M)$ *implies* $|F(z)| \leqslant M$. *Then there exists a constant* $M_6 > 1$ *such that for all* $M > M_6$ *and every function* $F \in S_R(M)$ *the estimation*

(18) $$A_{6F} \leqslant P_6(M) = 2\left(3-\frac{35}{M}+\frac{140}{M^2}-\frac{252}{M^3}+\frac{210}{M^4}-\frac{66}{M^5}\right)$$

*is true. The only function for which with a given* $M$, $M > M_6$, *equality holds in estimation* (18) *is the Pick function* $w = P(z, M)$, $P(0, M) = 0$, *given by the equation*

$$\frac{w}{(1-w/M)^2} = \frac{z}{(1-z)^2}.$$

The proof will consist of three parts.

1° Factorization of the right-hand side of the differential-functional equation (10) for $M$ sufficiently large. Let $F \in \mathscr{F}_M$, $w = f(z) = (1/M)F(z)$. Multiply both sides of equation (10) by $M$; then the function $f(z)$ satisfies the following differential-functional equation:

$$(zw'/w)^2 M(w) = N(z), \quad 0 < |z| < 1,$$

where

$$M(w) = M\cdot \mathfrak{M}(w), \quad N(z) = M\cdot \mathfrak{N}(z),$$

with $\mathfrak{M}(w)$ and $\mathfrak{N}(z)$ defined by formula (11). Let us recall the notation (cf. (14) and (11)):

$$F^m(z) = \sum_{n=m}^{\infty} A_{nF}^{(m)} z^n, \quad m = 2, 3, \ldots,$$

$$f^m(z) = \sum_{n=m}^{\infty} a_{nf}^{(m)} z^n, \quad m = 2, 3, \ldots$$

From the relationship

$$F(z) = M\cdot f(z)$$

it follows at once that

$$A_{nF}^{(m)} = M^m a_{nf}^{(m)}, \quad m = 2, 3, \ldots, \; n = m, m+1, \ldots$$

Thus

(19) $$N(z) = N_F(z) = 5A_{6F}+5A_{5F}(z+1/z)+4A_{4F}(z^2+1/z^2)+ +3A_{3F}(z^3+1/z^3)+2A_{2F}(z^4+1/z^4)+A_{1F}(z^5+1/z^5)-P,$$

where

(20) $$P = P_F = 2\min_{0\leqslant x\leqslant 2\pi}\left[\frac{A_{6F}^{(2)}}{M}\cos x+\frac{A_{6F}^{(3)}}{M^2}\cos 2x+\frac{A_{6F}^{(4)}}{M^3}\cos 3x+ +\frac{A_{6F}^{(5)}}{M^4}\cos 4x+\frac{A_{6F}^{(6)}}{M^5}\cos 5x\right].$$

It follows from Corollary 2 that for every $\eta_1 > 0$ there exists a constant $\hat{M}'$ such that for all $M > \hat{M}'$ we have $\max_{2\leqslant j\leqslant 6} |A_{6F}^{(j)}/M^{j-1}| < \eta_1$, and, in consequence, for every $\varepsilon_1 > 0$ there exists an $M' > \hat{M}'$ such that for all $M > M'$ we have $|P| < \varepsilon_1$, where $P$ is given by formula (20). In turn, from Corollary 1 it follows that for any $\eta_2 > 0$ there exists an $M''$ such that, for every $M > M''$, $\max_{1\leqslant j\leqslant 6} |A_{jF}-j| < \eta_2$ holds. From the above and from Lemma 4 we obtain that for any $\varepsilon_2 > 0$ and any compact set $\Delta$, there exists an $M''' \geqslant \max\{M', M''\}$ such that for $M > M'''$ and for every $z\in\Delta$ the estimate

$$|z^5(N(z)-N_0(z))| < \varepsilon_2$$

is satisfied, where $N(z)$ is given by formula (19) and

$$N_0(z) = 30+25(z+1/z)+16(z^2+1/z^2)+9(z^3+1/z^3)+4(z^4+1/z^4)+(z^5+1/z^5)$$
$$= \frac{1}{z^5}(z^{10}+4z^9+9z^8+16z^7+25z^6+30z^5+25z^4+16z^3+9z^2+4z+1).$$

Note that

$$N_0(z) = \frac{(z+1)^2}{z^5} L_0(z),$$

where $L_0(z)$ is given by formula (15). Then by Lemma 5, the function $N_0(z)$ has on the circle $|z| = 1$ exactly one double root $z = -1$ and four single complex roots $z_1$, $\bar{z}_1$, $z_2$, $\bar{z}_2$ inside the circle. Let us surround all the zeros of the function $N_0(z)$ with sufficiently small, disjoint discs. From Remark 4 we infer that there exists an $M_6 > M'''$ such that for all $M > M_6$ the zeros of the function $N(z)$ lie, respectively, in the chosen neighbourhood of the zeros of the function $N_0(z)$; in each of these neighbourhoods the number of zeros of the two functions (considering multiplications) is the same.

Let $\tilde{z}$, $|\tilde{z}| < 1$, be a root of the function $N(z)$. Then, for $M > M_6$, it lies in one of the neighbourhoods of the points $z_1$, $z_2$, $\bar{z}_1$, $\bar{z}_2$ and is a single complex root or it lies in the neighbourhood of the point $z = -1$ and is a single real root (for if it were a double complex or real root, there would exist in this neighbourhood four single roots $\tilde{z}$, $\bar{\tilde{z}}$, $1/\tilde{z}$, $1/\bar{\tilde{z}}$ or two double real roots $\tilde{z}$, $1/\tilde{z}$, respectively).

Next, let $\tilde{z} \neq -1$ be a root of the function $N(z)$, lying on the circle $|z| = 1$. Then it lies in the neighbourhood of the double root $z = -1$ of the function $N_0(z)$. Since $N(z)$ is non-negative on the circle $|z| = 1$ [4], the multiplicity of such a root is at least 2; besides, in the same neighbourhood there must lie root $\bar{\tilde{z}}$ of multiplicity at least 2, which contradicts the fact that the function $N(z)$ must there have exactly two roots (considering multiplicities). Consequently, if there exists any root of the function $N(z)$ on the circle $|z| = 1$, it must be equal to $-1$. It follows from the property of the function $N(z)$ [4] that such a root must exist, which allows to exlude the case where, in the neighbourhood of the point $-1$, a single real root lies inside the disc $|z| < 1$.

Finally, for $M > M_6$, the function $N(z)$ has on the circle $|z| = 1$ exactly one double root $z = -1$ and two pairs of conjugate single roots inside this circle, that is,

$$N(z) = \frac{1}{z^5}(z+1)^2 \prod_{k=1}^{2} (z-\tilde{z}_k)(z-\bar{\tilde{z}}_k)(z-1/\tilde{z}_k)(z-1/\bar{\tilde{z}}_k), \tag{21}$$

where

$$|\tilde{z}_k| < 1, \quad \tilde{z}_k \neq \bar{\tilde{z}}_k, \quad k = 1,2 \quad \text{and} \quad \tilde{z}_1 \neq \tilde{z}_2.$$

2° Discussion of the differential-functional equation (10). It follows from the above considerations that, for $M > M_6$, every function $w = f(z) = (1/M)F(z)$, $F \in \mathscr{F}_M$, satisfies the differential-functional equation

$$(22) \qquad (zw'/w)^2 M(w) = N(z), \quad 0 < |z| < 1,$$

where

$$(23) \quad M(w) = M_F(w) = \frac{1}{w^5}\left[\frac{A_{6F}^{(6)}}{M^5} w^{10} + \frac{A_{6F}^{(5)}}{M^4} w^9 + \frac{A_{6F}^{(4)}}{M^3} w^8 + \frac{A_{6F}^{(3)}}{M^2} w^7 + \right.$$
$$\left. + \frac{A_{6F}^{(2)}}{M} w^6 - Pw^5 + \frac{A_{6F}^{(2)}}{M} w^4 + \frac{A_{6F}^{(3)}}{M^2} w^3 + \frac{A_{6F}^{(4)}}{M^3} w^2 + \frac{A_{6F}^{(5)}}{M^4} w + \frac{A_{6F}^{(6)}}{M^5}\right],$$

and $N(z)$ and $P$ are given by formulae (21) and (20), respectively. Since the function $M(w)$ is symmetrical, with real coefficients, we see that if $M(w_0) = 0$, then also $M(\bar{w}_0) = 0$, $M(1/w_0) = 0$ and $M(1/\bar{w}_0) = 0$.

We infer from equation (22) that the images $\tilde{w}_k = f(\tilde{z}_k)$, $k = 1,2$, of the roots $\tilde{z}_k$ of the function $N(z)$ are roots of the function $M(w)$, since $f'(\tilde{z}_k) \neq 0$, $k = 1,2$. Besides, from the univalence of the function $f$ it follows that $\tilde{w}_1 \neq \tilde{w}_2$ and $\tilde{w}_k \neq \bar{\tilde{w}}_k$, $k = 1,2$. It is also known [4] that the function $M(w)$ has on the circle $|w| = 1$ at least one double root $\tilde{w}_0$. On account of (23) and the above properties of the function $M(w)$ we have $\tilde{w}_0 = 1$ or $\tilde{w}_0 = -1$. Thus the function $M(w)$ has the following factorization:

$$(24) \qquad M(w) = \frac{1}{w^5}(w-\tilde{w}_0)^2 \prod_{k=1}^{2} (w-\tilde{w}_k)(w-\bar{\tilde{w}}_k)(w-1/\tilde{w}_k)(w-1/\bar{\tilde{w}}_k),$$

where

$$|\tilde{w}_k| < 1, \quad \tilde{w}_k \neq \bar{\tilde{w}}_k, \quad k = 1,2, \quad \text{and} \quad \tilde{w}_1 \neq \tilde{w}_2 \quad \text{and} \quad \tilde{w}_0 = \pm 1.$$

It is known (cf. [19], [6]) that every function $f(z) = (1/M)F(z)$, $F \in \mathscr{F}_M$, maps the disc $|z| < 1$ onto the disc $|w| < 1$ with slits along a finite number of analytic arcs. If $\hat{w}$ is the initial point of such an arc, lying on the circle $|w| = 1$, then, by (22): $M(\hat{w}) = 0$, since at the pre-image $\hat{z}$ of this point the finite derivative $\hat{w}' = f'(\hat{z})$ does not exist and the right-hand side of equation (22) has a finite value. According to (24), for $M > M_6$ the function $M(w)$ has exactly one root $\tilde{w}_0$ on the circle $|w| = 1$, and so, it must be the initial point of the only slit in the image of the disc $K$ under the mapping $w = f(z)$; then, from the properties of the classes $S_R(M)$ considered it follows that slit is rectilinear and lies on the real axis.

At the same time, the point $z_0$, $|z_0| = 1$, which is the pre-image of the opposit end point $w_0$ of such a slit, must be, again in virtue of (22), a root of the function $N(z)$, since in the neighbourhood of such a point the function $f$ is twofold, and so $w_0' = f'(z_0) = 0$. In view of (21) we get $z_0 = -1$.

Note that also $\tilde{w}_0 = -1$. Indeed, taking account of (21) and (24), we may write equation (22) in the form:

$$\left(\frac{zw'}{w}\right)^2 \frac{1}{w^5}(w-\tilde{w}_0)^2 \prod_{k=1}^{2}(w-\tilde{w}_k)(w-\bar{\tilde{w}}_k)\left(w-\frac{1}{\tilde{w}_k}\right)\left(w-\frac{1}{\bar{\tilde{w}}_k}\right)$$
$$= \frac{1}{z^5}(z+1)^2 \prod_{k=1}^{2}(z-\tilde{z}_k)(z-\bar{\tilde{z}}_k)\left(z-\frac{1}{\tilde{z}_k}\right)\left(z-\frac{1}{\bar{\tilde{z}}_k}\right), \quad 0<|z|<1,$$

from which it is easily seen that the function $w = f(z)$ transforms the negative real half-axis onto the negative real half-axis. Consequently, the image of the point $z_0 = -1$, being the end point of the slit in the image of the disc $K$ under the mapping $w = f(z)$, lies on the negative real half-axis; and so, the point $w = +1$ cannot be the initial point of this slit, since $w = 0$ belongs to the image under consideration.

Reconsidering, for $M > M_6$ every function $w = f(z) = (1/M)F(z)$, $F \in \mathcal{F}_M$, maps the disc $|z| < 1$ onto the disc $|w| < 1$ with the only slit along the negative real half-axis, with the initial point $\tilde{w}_0 = -1$ and the terminal point $w_0 = \dot{w}_0$, $w_0 < 0$.

3° Construction of a function maximal with respect to functional (9) in the class $S_K(M)$ for $M > M_6$. It is known that the functions $p(z, M) = (1/M)P(z, M)$, $M > 1$, where $P(z, M) \in \mathcal{P}$, map the disc $K$ onto a disc with a slit along the negative real half-axis, with the initial point $w = -1$ and the terminal point $r_M = 1-2M+2\sqrt{M(M-1)}$ (cf. [17]).

For a fixed $M > M_6$, let us choose an $M_0$ so that the image of the disc $|z| < 1$ under the mapping $w = p_0(z) = p(z, M_0)$ should be identical with the image of the disc $K$ under the mapping $w = f(z) = (1/M)F(z)$, $F \in \mathcal{F}_M$, obtained in the second part of the proof; in other words, we choose the $M_0$ so that $r_{M_0} = w_0$. From Schwarz's lemma we obtain $M = M_0$ and $p_0^{-1}(f(z)) = z$. Thus finally, $f(z) = p_0(z)$, $z \in K$.

From the above reasoning it follows that, for a given $M$, $M > M_6$, the only function maximal with respect to functional (9) in the class $S_K(M)$ is the function $P(z, M) = M \cdot p_0(z) = \sum_{n=1}^{\infty} P_n(M)z^n$. We calculate the sixth coefficient of this function by making use of (7). We obtain

$$P_6(M) = 2\left(3-\frac{35}{M}+\frac{140}{M^2}-\frac{252}{M^3}+\frac{210}{M^4}-\frac{66}{M^5}\right),$$

and so, estimation (18) holds, which concludes the proof of the theorem.

**5. Final remarks.** It is evident that if $F \in S_K(M)$, $M > 1$, the function $G$ defined by the formula: $G(z) = -F(-z)$, $z \in K$, belongs to $S_K(M)$, too. Thus the theorem just obtained implies

COROLLARY 3. *For all $M > M_6$ and any function $F \in S_R(M)$, the estimation*

$$A_{6F} \geqslant -P_6(M)$$

*takes place. The only function for which equality holds is the function* $\tilde{P}(z) = -P(-z, M)$.

The problem of determining a minimal $M_6$, such that for all $M > M_6$ the Pick function is an extremal function, remains open.

At last, let us notice that it was possible to carry out the proof of the theorem without complicated integration of differential-functional equation (10) (cf. [10]). Instead, its properties and the result of Dieudonné had to be employed.

### References

[1] L. Bieberbach, *Über die Koefficienten derjenigen Potenzreihen, welche schlichte Abbildung des Einheitskreises vermitteln*, Sitzungsber. der Preuss. Akad. der Wiss. 38, Berlin 1916.

[2] Z. Charzyński and M. Schiffer, *A new proof of the Bieberbach conjecture for the fourth coefficient*, Archive for Rational Mechanics and Analysis, Vol. 5, No 3 (1960).

[3] J. Dieudonné, *Sur les fonctions univalentes*, C. R. 192 (1931), p. 1148–1150.

[4] I. Dziubiński, *L'équation des fonctions extrémales dans la famille des fonctions univalentes symétriques et bornées*, Łódzkie Towarzystwo Naukowe, Societas Scientarum Lodziensis, Sec. III, No 65 (1960).

[5] P. R. Garabedian and M. Schiffer, *A proof of the Bieberbach conjecture for the fourth coefficient*, J. Ration. Mech. Analys. 4 (1955), p. 427–465.

[6] В.В. Голубев, *Лекции по аналитической теории дифференциальных уравнений*, (V. Golubev, *Lectures on the analytical theory of differential equations*). Gos. Izd. Teh. Lit., Moscow 1950.

[7] Z. J. Jakubowski, *O oszacowaniu współczynnika $B_4$ w rodzinie funkcji jednolistnych, ograniczonych, o współczynnikach rzeczywistych*, Zesz. Nauk. P. Ł., Nr 33, Chemia Z. 9 (1961), p. 55–71.

[8] — *Maksimum funkcjonału $A_3 + \alpha A_2$ w rodzinie funkcji jednolistnych o współczynnikach rzeczywistych*, Zesz. Nauk. U. Ł., Nauki Mat.-Przyr., Ser. II, Zesz. 20, Łódź 1966, p. 43–61.

[9] — *Sur les coefficients des fonctions univalentes et symétriques dans un cercle unitaire*, Bull. Acad. Polon. Sci., Sér. Math. Astr. Phys. 14 (1966), p. 643–646.

[10] — *Sur les coefficients des fonctions univalentes dans le cercle unité*, Ann. Polon. Math. 19 (1967), p. 207–233.

[11] W. Janowski, *Le maximum des coefficients $A_2$ et $A_3$ des fonctions univalentes bornées*, ibidem 2 (1955), p. 145–160.

[12] K. Löwner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises*, Math. Ann. 89 (1923), p. 103–121.

[13] L. Mikołajczyk, *The variability regions of the coefficients $A_2$ and $A_3$ of univalent, symmetrical and bounded functions*, Colloq. Math. 11 (1964), p. 289–290.

[14] M. Ozawa, *On the Bieberbach conjecture for the sixth coefficient*, Kódai Mathematical Seminar Reports 21 (1969), p. 97–128.

[15] R. N. Pederson, *A proof of the Bieberbach conjecture for the sixth coefficients*, Arch. Rat. Mech. Anal. 31 (1968), p. 331–351.

[16] — and M. Schiffer, *A proof of the Bieberbach conjecture for the fifth coefficient*, ibidem 45 (1972), p. 161–193.

[17] G. Pick, *Über die konforme Abbildung eines Kreises auf eim schlichtes und zugleich beschränktes Gebiet*, Sitzungsber. Akad. Wiss. Wien Abt. II a (1917), p. 247–263.

[18] A. C. Schaeffer and D. C. Spencer, *The coefficients of schlicht functions*, Duke Math. J. 12 (1945), p. 107–125.

[19] M. Schiffer and O. Tammi, *The fourth coefficients of a bounded real univalent function*, Ann. Acad. Sci. Fennicae, Ser. AI, No. 354 (1965), p. 1–34.

[20] — — *On the fourth coefficient of bounded univalent functions*, Trans. Amer. Math. Soc. 119 (1965), p. 67–78.

[21] — — *On bounded univalent functions which are close to identity*, Ann. Acad. Sci. Fennicae Ser. AI, Math. (1968), p. 3–26.

[22] L. Siewierski, *The local solution of coefficient problem for bounded schlicht functions*, Soc. Sci. Lodziensis, Sec. III (1960), p. 7 13.

[23] — *Sharp estimation of the coefficients of bounded univalent functions near the identity*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astr. et Phys. 16 (7) (1968), p. 575–576.

[24] — *Sharp estimation of the coefficients of bounded univalent functions close to identity*, Dissert. Math. 86 (1971), p. 1–153.

[25] O. Tammi, *On the maximalization of the coefficient $A_3$ of bounded schlicht functions*, Ann. Acad. Sci. Fennicae, Ser. AI, 140 (1953).

[26] — *On optimizing parameters of the power inequality for $a_4$ in the class of bounded univalent functions*, ibidem Ser. A, Math. (1973).

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