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On homeomorphisms in the plane

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Let D be an open set in the plane Z of complex numbers and h a homeomorphism of the set D onto itself. Let h^n denote the nth iteration of the homeomorphism h and $h(U)$ the image of the set U by the homeomorphism h. For $U \subset D$ the set

$$U_h = \bigcup_{-\infty}^{\infty} h^n(U)$$

is called the *h-trajectory* of the set U. A set W is called *interior* in D if there exists a compact set V such that $W \subset V \subset D$.

We shall prove the following

THEOREM. *Let D be an open simple connected subset of the plane Z and h a homeomorphism of the set D onto itself. If for every set interior in D its h-trajectory is also interior in D, then the homeomorphism h has in D a fixed point, i.e. there exists $z \in D$ such that $h(z) = z$.*

Proof. Let $z_0 \in D$. Denote by A an open connected and interior in D subset of the set D including the h-trajectory of the point z_0. It is clear that such a set exists. Since $h^n(z_0) \in h^n(A) \cap A$, the trajectory A_h of the set A is a connected set. Moreover A_h is open, interior in D and h-invariant, i.e. $h(A_h) = A_h$. Add to the set $\bar{A}_h$ (1) all bounded components of its complement (if such components exist, i.e. if $\bar{A}_h$ disjoins the plane) and the obtained simple connected set denote by B. Evidently $h(B) = B$. The set B is bounded. In fact, the set $\bar{A}_h$ as the closure of the h-trajectory of the set A interior in D, is contained in a compact set W. Joining eventually all bounded components of its complement we may assume that the set W is simple connected. Then $B \subset W$. From the construction of the set B it follows that it is closure of its interior. Thus the set B is homeomorphic with an two-dimensional simplex and by the theorem of Brouwer there exists a $z \in B$ such that $h(z) = z$ which ends the proof.

(1) $\bar{U}$ denotes the closure of the set U.

Remark. If the set D is the whole plane, our theorem may be formulated in a somewhat different manner. Denote by $d(z)$ the diameter of the trajectory of a point z, i.e.

$$d(z) = \sup_{-\infty<n,m<\infty} |h^n(z) - h^m(z)| .$$

Then, if the function $d(z)$ is locally bounded ([2]), the homeomorphism h has a fixed point.

([2]) A function is called *locally bounded* if it is bounded in every bounded set.

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