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TRIGONOMETRIC INTERPOLATION, V

BY

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1. Preliminaries. Given a function $f(s)$ Riemann-integrable over any finite interval, and $l > 0$, let

$$I_n^l(x;\, f) = \frac{1}{2}a_0 + \sum_{k=1}^{n}\left(a_k\cos\frac{k\pi x}{l} + b_k\sin\frac{k\pi x}{l}\right)$$

be the n-th interpolating polynomial with nodes

$$s_j = 2lj/(2n+1) \quad (j = 0, \pm 1, \pm 2, \ldots).$$

Denote by $\tilde{I}_n^l(x;\, f)$ the polynomial conjugate to $I_n^l(x;\, f)$, that is

$$\tilde{I}_n^l(x;\, f) = \sum_{k=1}^{n}\left(a_k\sin\frac{k\pi x}{l} - b_k\cos\frac{k\pi x}{l}\right).$$

Write, for $\nu \leqslant n$,

$$I_{n,\nu}^l(x;\, f) = \frac{1}{2}a_0 + \sum_{k=1}^{\nu}\left(a_k\cos\frac{k\pi x}{l} + b_k\sin\frac{k\pi x}{l}\right),$$

$$\tilde{I}_{n,\nu}^l(x;f) = \sum_{k=1}^{\nu}\left(a_k\sin\frac{k\pi x}{l} - b_k\cos\frac{k\pi x}{l}\right),$$

$$\sigma_{n,\nu}^l(x;\, f) = \frac{1}{\nu}\sum_{m=0}^{\nu-1} I_{n,m}^l(x;\, f), \qquad \tilde{\sigma}_{n,\nu}^l(x;\, f) = \frac{1}{\nu}\sum_{m=1}^{\nu-1} \tilde{I}_{n,m}^l(x;\, f).$$

Using the integral notation as in Section 1 of [1], we easily get the fundamental formulae

$$I_{n,\nu}^l(x;\, f) = \frac{1}{l}\int_{-l}^{l} f(s)D_\nu^l(s-x)\,d\omega_n^l(s),$$

$$\tilde{I}^l_{n,\nu}(x;\,f) = -\frac{1}{l}\int_{-l}^{l} f(s)\tilde{D}^l_\nu(s-x)\,d\omega^l_n(s),$$

$$\sigma^l_{n,\nu}(x;\,f) = \frac{1}{l}\int_{-l}^{l} f(s)K^l_\nu(s-x)\,d\omega^l_n(s),$$

$$\tilde{\sigma}^l_{n,\nu}(x;\,f) = -\frac{1}{l}\int_{-l}^{l} f(s)\tilde{K}^l_\nu(s-x)\,d\omega^l_n(s),$$

where

$$D^l_\nu(t) = \frac{1}{2}+\sum_{k=1}^{\nu}\cos\frac{k\pi t}{l} = \frac{\sin(2\nu+1)\dfrac{\pi t}{2l}}{2\sin\dfrac{\pi t}{2l}},$$

$$\tilde{D}^l_\nu(t) = \sum_{k=1}^{\nu}\sin\frac{k\pi t}{l} = \frac{1}{2}\cot\frac{\pi t}{2l} - \frac{\cos(2\nu+1)\dfrac{\pi t}{2l}}{2\sin\dfrac{\pi t}{2l}},$$

$$K^l_\nu(t) = \frac{1}{\nu}\sum_{m=0}^{\nu-1}D^l_m(t) = \frac{1}{2\nu}\left(\frac{\sin\dfrac{\nu\pi t}{2l}}{\sin\dfrac{\pi t}{2l}}\right)^2,$$

$$\tilde{K}^l_\nu(t) = \frac{1}{\nu}\sum_{m=0}^{\nu-1}\tilde{D}^l_m(t) = \frac{1}{2}\cot\frac{\pi t}{2l} - \frac{\sin\dfrac{\nu\pi t}{l}}{\nu\left(2\sin\dfrac{\pi t}{2l}\right)^2}$$

(cf. [2], p. 4, 5, 8, 21, 22, 48 and 54).

As in the 2π-periodic case,

$$\frac{1}{l}\int_{-l}^{l} D^l_\nu(s-x)\,d\omega^l_n(s) = \frac{1}{l}\int_{-l}^{l} K^l_\nu(s-x)\,d\omega^l_n(s) = 1$$

and

$$\int_{-l}^{l}\tilde{D}^l_\nu(s-x)\,d\omega^l_n(s) = \int_{-l}^{l}\tilde{K}^l_\nu(s-x)\,d\omega^l_n(s) = 0.$$

Therefore, for example,

$$\sigma^l_{n,\nu}(x;\, f)-f(x) = \frac{1}{l}\int\limits_{-l}^{l}\{f(s)-f(x)\}K^l_\nu(s-x)\,d\omega^l_n(s), \tag{1}$$

$$\tilde{I}^l_{n,\nu}(x;\, f) = -\frac{1}{l}\int\limits_{-l}^{l}\{f(s)-f(x)\}\tilde{D}^l_\nu(s-x)\,d\omega^l_n(s). \tag{2}$$

In [1] we have presented some theorems concerning convergence of $I^l_n(x;\, f)$ as $l\to\infty$, $(l/n)\to 0$. Here the behaviour of $\sigma^l_{n,\nu}(x;\, f)$, $\tilde{I}^l_{n,\nu}(x;\, f)$ and $\tilde{\sigma}^l_{n,\nu}(x;\, f)$ will be examined. Considering the penultimate sum we introduce the following expressions:

$$\tilde{f}^l_{n,\nu}(x) = -\frac{1}{2l}\left(\int\limits_{x-l}^{x-l/\nu}+\int\limits_{x+l/\nu}^{x+l}\right)\{f(s)-f(x)\}\cot\frac{\pi(s-x)}{2l}\,d\omega^l_n(s),$$

$$\hat{f}^l_{n,\nu}(x) = -\frac{1}{\pi}\left(\int\limits_{x-l}^{x-l/\nu}+\int\limits_{x+l/\nu}^{x+l}\right)\frac{f(s)-f(x)}{s-x}\,d\omega^l_n(s),$$

$$\psi_x(t) = f(x+t)-f(x-t).$$

Our investigations extend the corresponding results given in Chapter X of [2].

2. Convergence of $\sigma^l_{n,\nu}(x;\, f)$. In this section we examine the pointwise and mean convergence of the indicated operator, assuming that $l\to\infty$, $(l/\nu)\to 0$.

THEOREM 1. *Let $f(s)$ be a function Riemann-integrable over every finite interval, continuous at a fixed $x\in(-\infty,\infty)$. Suppose that there exist a positive number c and a non-negative function $\lambda(s)$, non-increasing in the interval $\langle c,\infty)$, such that*

$$|s^{-2}f(s)|\leqslant\lambda(|s|) \quad \textit{when } |s|\geqslant c,$$

and

$$\int\limits_{c}^{\infty}\lambda(s)\,ds<\infty.$$

Then

$$\lim_{l/\nu\to 0}\sigma^l_{n,\nu}(x;\, f) = f(x) \qquad (l\to\infty).$$

Proof. Without loss of generality, we may suppose that $x\geqslant 0$, $l>2x$.

In view of (1),

$$\sigma_{n,\nu}^{l}(x;\,f)-f(x) = \frac{1}{l}\int\limits_{x-l}^{x+l}\{f(s)-f(x)\}K_{\nu}^{l}(s-x)\,d\omega_{n}^{l}(s)+$$

$$+\frac{1}{l}\Big(\int\limits_{-l}^{x-l}-\int\limits_{l}^{x+l}\Big)\{f(s)-f(x)\}K_{\nu}^{l}(s-x)\,d\omega_{n}^{l}(s) = P_{n,\nu}^{l}(x)+\big(Q_{n,\nu}^{l}(x)-R_{n,\nu}^{l}(x)\big).$$

Further,

$$|Q_{n,\nu}^{l}(x)-R_{n,\nu}^{l}(x)|\leqslant\frac{1}{l}\Big(\int\limits_{-l}^{x-l}+\int\limits_{l}^{x+l}\Big)\frac{|f(s)-f(x)|}{2\nu\left|\sin\dfrac{\pi(s-x)}{2l}\right|^{2}}\,d\omega_{n}^{l}(s)$$

$$\leqslant\frac{9l}{4\nu}\int\limits_{-l}^{x-l}\frac{|f(s)|+|f(x)|}{(x-s)^{2}}\,d\omega_{n}^{l}(s)+\frac{l}{2\nu}\int\limits_{l}^{x+l}\frac{|f(s)|+|f(x)|}{(s-x)^{2}}\,d\omega_{n}^{l}(s)$$

and $|f(s)|\leqslant s^{2}\lambda(s)$. Hence

(3) $$\sigma_{n,\nu}^{l}(x;\,f)-f(x) = P_{n,\nu}^{l}(x)+o(1)\quad \text{as } l\to\infty,\ l/\nu\leqslant 1.$$

Choose, for a given $\varepsilon>0$, a positive $\delta=\delta(\varepsilon)$ such that

$$|f(s)-f(x)|<\varepsilon\quad \text{if } |s-x|\leqslant\delta.$$

Then

$$P_{n,\nu}^{l}(x) = \frac{1}{l}\int\limits_{x-\delta}^{x+\delta}\{f(s)-f(x)\}K_{\nu}^{l}(s-x)\,d\omega_{n}^{l}(s)+$$

$$+\frac{1}{l}\Big(\int\limits_{x-l}^{x-\delta}+\int\limits_{x+\delta}^{x+l}\Big)\{f(s)-f(x)\}K_{\nu}^{l}(s-x)\,d\omega_{n}^{l}(s) = A+B.$$

In the case $l\geqslant\delta$,

$$|A|\leqslant\frac{\varepsilon}{l}\int\limits_{x-\delta}^{x+\delta}K_{\nu}^{l}(s-x)\,d\omega_{n}^{l}(s)\leqslant\varepsilon$$

and

$$|B|\leqslant\frac{1}{l}\Big(\int\limits_{x-l}^{x-\delta}+\int\limits_{x+\delta}^{x+l}\Big)\frac{|f(s)|+|f(x)|}{2\nu\left|\sin\dfrac{\pi(s-x)}{2l}\right|^{2}}\,d\omega_{n}^{l}(s).$$

Proof. Retaining the symbols $P_{n,\nu}^{l}(x)$, $Q_{n,\nu}^{l}(x)$ and $R_{n,\nu}^{l}(x)$ used above, we have

$$\Big\{\int_{-l}^{l}|\sigma_{n,\nu}^{l}(x;f)-f(x)|^{r}dx\Big\}^{1/r}$$
$$\leqslant\Big\{\int_{-l}^{l}|P_{n,\nu}^{l}(x)|^{r}dx\Big\}^{1/r}+\Big\{\int_{-l}^{l}|Q_{n,\nu}^{l}(x)|^{r}dx\Big\}^{1/r}+\Big\{\int_{-l}^{l}|R_{n,\nu}^{l}(x)|^{r}dx\Big\}^{1/r}$$
$$=W_1+W_2+W_3.$$

Choose, for an arbitrary $\varepsilon>0$, a positive $\delta=\delta(\varepsilon)$ such that

$$\vartheta(t)\leqslant\varepsilon\quad\text{when } t\leqslant\delta.$$

Evidently, if $l>\delta$,

$$W_1^r\leqslant\int_{-l}^{l}\Big|\frac{\varepsilon}{l}\int_{x-\delta}^{x+\delta}K_{\nu}^{l}(s-x)\,d\omega_n^l(s)\Big|^r dx+$$
$$+\int_{-l}^{l}\Big|\frac{1}{l}\Big(\int_{x-l}^{x-\delta}+\int_{x+\delta}^{x+l}\Big)\vartheta(|s-x|)K_{\nu}^{l}(s-x)\,d\omega_n^l(s)\Big|^r dx$$
$$\leqslant 2l\varepsilon^r+\int_{-l}^{l}\Bigg|\frac{1}{l}\Big(\int_{x-l}^{x-\delta}+\int_{x+\delta}^{x+l}\Big)\frac{\vartheta(|s-x|)}{2\nu\Big|\sin\dfrac{\pi(s-x)}{2l}\Big|^2}\,d\omega_n^l(s)\Bigg|^r dx.$$

Since $\vartheta(t)\leqslant Ct^{1/r}$ for $t\geqslant 0$, where C is a positive constant, we obtain

$$W_1\leqslant\Big\{2l\varepsilon^r+\int_{-l}^{l}\Big|\frac{Cl}{2\nu}\Big(\int_{x-l}^{x-\delta/2}+\int_{x+\delta/2}^{x+l}\Big)|s-x|^{1/r-2}ds\Big|^r dx\Big\}^{1/r};$$

whence $l^{-1/r}W_1\leqslant 4^{1/r}\varepsilon$ for sufficiently small l/ν.

Taking $\nu>4$, we have

$$W_3^r=\Big(\int_{-l}^{-l+2l/\nu}+\int_{-l+2l/\nu}^{-l/2}+\int_{-l/2}^{0}\Big)\Big|\frac{1}{l}\int_{x+l}^{l}\{f(s)-f(x)\}K_{\nu}^{l}(s-x)\,d\omega_n^l(s)\Big|^r dx+$$
$$+\Big(\int_{0}^{l-2l/\nu}+\int_{l-2l/\nu}^{l}\Big)\Big|\frac{1}{l}\int_{l}^{x+l}\{f(s)-f(x)\}K_{\nu}^{l}(s-x)\,d\omega_n^l(s)\Big|^r dx$$
$$=(A_1+A_2+A_3)+(B_1+B_2).$$

It is easily seen that

$$A_1 \leqslant \int_{-l}^{-l+2l/\nu} \left| \frac{1}{l} \int_{x+l}^{l} \vartheta(s-x) K_\nu^l(s-x)\, d\omega_n^l(s) \right|^r dx$$

$$\leqslant \int_{-l}^{-l+2l/\nu} \left| \vartheta(2l) \frac{1}{l} \int_{x+l}^{l} K_\nu^l(s-x)\, d\omega_n^l(s) \right|^r dx$$

$$\leqslant \{\vartheta(2l)\}^r \frac{2l}{\nu} \leqslant 4C^r l \frac{l}{\nu}.$$

Further,

$$A_2 \leqslant \int_{-l+2l/\nu}^{-l/2} \left| \frac{1}{l} \int_{x+l}^{l} \frac{|f(s)-f(x)|}{2\nu \left| \sin\left(\pi - \dfrac{\pi(s-x)}{2l}\right) \right|^2} d\omega_n^l(s) \right|^r dx$$

$$\leqslant \int_{-l+2l/\nu}^{-l/2} \left| \frac{l}{2\nu} \int_{x+l}^{l} \frac{\vartheta(s-x)}{(2l+x-s)^2} d\omega_n^l(s) \right|^r dx$$

$$\leqslant \left\{ \frac{l}{2\nu} \vartheta(2l) \right\}^r \int_{-l+2l/\nu}^{-l/2} \frac{dx}{(l+x-l/\nu)^r} \leqslant \frac{2}{r-1} \left(\frac{C}{2} \right)^r l \frac{l}{\nu},$$

$$A_3 \leqslant \int_{-l/2}^{0} \left| \frac{1}{l} \int_{x+l}^{l} \frac{\vartheta(s-x)}{2\nu \left| \sin \dfrac{\pi(s-x)}{2l} \right|^2} d\omega_n^l(s) \right|^r dx$$

$$\leqslant \int_{-l/2}^{0} \left| \frac{9Cl}{4\nu} \int_{x+l-l/\nu}^{l} (s-x)^{1/r-2} ds \right|^r dx \leqslant \frac{3}{8} \left\{ \frac{3Cr}{r-1} \right\}^r l \frac{l}{\nu}.$$

Analogously,

$$B_1 \leqslant \int_{0}^{l-2l/\nu} \left| \frac{l}{2\nu} \int_{l}^{x+l} \frac{\vartheta(s-x)}{(s-x)^2} d\omega_n^l(s) \right|^r dx$$

$$\leqslant \int_{0}^{l-2l/\nu} \left| \frac{Cl}{2\nu} \int_{l-l/\nu}^{x+l} (s-x)^{1/r-2} ds \right|^r dx \leqslant \left\{ \frac{Cr}{2(r-1)} \right\}^r l \frac{l}{\nu},$$

$$B_2 \leqslant \int\limits_{l-2l/\nu}^{l} \left| \frac{\nu}{2l} \int\limits_{l}^{l+2l/\nu} \vartheta(s-x)\, d\omega_n^l(s) \right|^r dx + \int\limits_{l-2l/\nu}^{l} \left| \frac{l}{2\nu} \int\limits_{l+2l/\nu}^{x+l} \frac{\vartheta(s-x)}{(s-x)^2}\, d\omega_n^l(s) \right|^r dx$$

$$\leqslant \int\limits_{l-2l/\nu}^{l} \frac{C\nu}{2l} \int\limits_{l}^{l+3l/\nu} (s-x)^{1/r} ds \Bigg|^r dx + \int\limits_{l-2l/\nu}^{l} \left| \frac{Cl}{2\nu} \int\limits_{l+l/\nu}^{x+l} (s-x)^{1/r-2} ds \right|^r dx$$

$$\leqslant \int\limits_{l-2l/\nu}^{l} \left| \frac{C\nu r}{2l(r+1)} \left(l+\frac{3l}{\nu}-x\right)^{1/r+1} \right|^r dx + \int\limits_{l-2l/\nu}^{l} \left| \frac{Clr}{2\nu(r-1)} \left(l+\frac{l}{\nu}-x\right)^{1/r-1} \right|^r dx$$

$$\leqslant \frac{5}{2} \left\{ \frac{5Cr}{2(r+1)} \right\}^r l \frac{l}{\nu} + \frac{1}{2} \left\{ \frac{Cr}{2(r-1)} \right\}^r l \frac{l}{\nu}.$$

Thus,

$$\lim_{l/\nu \to 0} \frac{1}{l^{1/r}} W_3 = 0 \qquad (l \to \infty)$$

and, by symmetry, the quantity W_3 can be replaced here by W_2. Hence the theorem follows.

In the case $r = 1$, relation (4) is true when $\vartheta(t) \leqslant Ct$ for $t \geqslant 0$ and $(l/\nu)\log \nu \to 0$.

3. Convergence of $\tilde{I}_{n,\nu}^l(x;f)$ and $\tilde{\sigma}_{n,\nu}^l(x;f)$. Now we shall prove the following

THEOREM 3. *Consider a function $f(s)$ Riemann-integrable over any finite interval, continuous at a certain $x \in (-\infty, \infty)$. Suppose that there exist two positive numbers c, γ and non-negative functions $\lambda(s)$ and $\mu(s)$ monotonic in $\langle c, \infty)$ and $\langle 0, \gamma \rangle$, respectively, such that*

$$|s^{-1}f(s)| \leqslant \lambda(|s|) \quad \text{if } |s| \geqslant c,$$

$$|f(x+t)-f(x)| \leqslant \mu(|t|) \quad \text{if } |t| \leqslant \gamma.$$

Moreover,

$$\int\limits_{c}^{\infty} \lambda(s)\, ds < \infty \quad \text{and} \quad \int\limits_{0}^{\gamma} t^{-1}\mu(t)\, dt < \infty.$$

Then

$$\lim_{l/\nu \to 0} \tilde{I}_{n,\nu}^l(x;\, f) = -\frac{1}{\pi} \int\limits_{0}^{\infty} \frac{\psi_x(t)}{t}\, dt \qquad (l \to \infty).$$

(The last two integrals are taken in the sense of Lebesgue.)

Proof. By the assumption, $\lambda(s)$ is non-increasing in $\langle c, \infty)$, $\lambda(s) \to 0$ as $s \to \infty$, and $\mu(s)$ in non-decreasing in $\langle 0, \gamma \rangle$, $\mu(0) = 0$. As in the proof of Theorem 1, let x be a non-negative number and $l > 2x$.

In view of (2),

$$\tilde{I}^l_{n,\nu}(x;\, f) = -\frac{1}{l}\int\limits_{x-l}^{x+l} \{f(s)-f(x)\}\tilde{D}^l_\nu(s-x)\,d\omega^l_n(s) +$$

$$+\frac{1}{l}\Big(\int\limits_{l}^{x+l} - \int\limits_{-l}^{x-l}\Big)\{f(s)-f(x)\}\tilde{D}^l_\nu(s-x)\,d\omega^l_n(s) = F^l_{n,\nu}(x)+G.$$

Since

$$|G| \leqslant \frac{1}{l}\Big(\int\limits_{l}^{x+l} + \int\limits_{-l}^{x-l}\Big)\frac{|f(s)-f(x)|}{\left|\sin\dfrac{\pi(s-x)}{2l}\right|}\,d\omega^l_n(s)$$

$$\leqslant \int\limits_{l}^{x+l} \frac{|f(s)|+|f(x)|}{s-x}\,d\omega^l_n(s) + \frac{3}{\sqrt{2}}\int\limits_{-l}^{x-l}\frac{|f(s)|+|f(x)|}{x-s}\,d\omega^l_n(s)$$

and $|f(s)| \leqslant |s|\lambda(|s|)$, we obtain

(5) $$\tilde{I}^l_{n,\nu}(x;\, f) = F^l_{n,\nu}(x)+o(1) \quad \text{as } l\to\infty,\ l/n \leqslant 1.$$

Given any $\varepsilon > 0$, let us choose a positive $\delta = \delta(\varepsilon)$ such that

$$|f(s)-f(x)| < \varepsilon \quad \text{if } |s-x| < \delta,$$

and

$$\int\limits_0^{3\delta} \frac{\mu(t)}{t}\,dt < \varepsilon.$$

Then, putting

$$H^l_\nu(t) = \frac{\cos(2\nu+1)\dfrac{\pi t}{2l}}{2\sin\dfrac{\pi t}{2l}},$$

we have

$$F^l_{n,\nu}(x) - \tilde{f}^l_{n,\nu}(x) = -\frac{1}{l}\int\limits_{x-l/\nu}^{x+l/\nu}\{f(s)-f(x)\}\tilde{D}^l_\nu(s-x)\,d\omega^l_n(s) +$$

$$+\frac{1}{l}\Big(\int\limits_{x-l}^{x-l/\nu} + \int\limits_{x+l/\nu}^{x+l}\Big)\{f(s)-f(x)\}H^l_\nu(s-x)\,d\omega^l_n(s) = A+B.$$

In the case $l/\nu < \delta$ $(1 \leqslant \nu \leqslant n)$,

$$|A| \leqslant \frac{\varepsilon}{l}\int\limits_{x-l/\nu}^{x+l/\nu}|\tilde{D}^l_\nu(s-x)|\,d\omega^l_n(s) \leqslant \frac{\varepsilon\nu}{l}\left(\frac{2l}{\nu}+\frac{2l}{2n+1}\right) < 3\varepsilon.$$

If $\delta < l < \nu\delta$,

$$\frac{1}{l}\left|\left(\int_{x-\delta}^{x-l/\nu} + \int_{x+l/\nu}^{x+\delta}\right)\{f(s)-f(x)\}H_\nu^l(s-x)\,d\omega_n^l(s)\right|$$

$$\leqslant \frac{1}{l}\left(\int_{x-\delta}^{x-l/\nu} + \int_{x+l/\nu}^{x+\delta}\right)\frac{\mu(|s-x|)}{2\left|\sin\dfrac{\pi(s-x)}{2l}\right|}\,d\omega_n^l(s)$$

$$\leqslant \int_{x-\delta}^{x-l/\nu}\frac{\mu(x-s)}{2(x-s)}\,d\omega_n^l(s) + \int_{x+l/\nu}^{x+\delta}\frac{\mu(s-x)}{2(s-x)}\,d\omega_n^l(s)$$

$$\leqslant 3\int_0^{3\delta} t^{-1}\mu(t)\,dt < 3\varepsilon$$

(see [2], p. 17, 18 and 50).

Therefore, for all these l and ν,

$$|B| \leqslant 3\varepsilon + \frac{1}{l}\left|\left(\int_{x-l}^{x-\delta} + \int_{x+\delta}^{x+l}\right)\{f(s)-f(x)\}H_\nu^l(s-x)\,d\omega_n^l(s)\right|.$$

Reasoning as in Section 2 of [1], it can easily be observed that two last integrals tend to zero as $l/\nu \to 0$. Hence

$$|F_{n,\nu}^l(x) - \tilde{f}_{n,\nu}^l(x)| < 7\varepsilon,$$

whenever l/ν and $1/l$ are small enough.

Under the assumption $\delta < l < \nu\delta$, consider now the difference

$$Z_{n,\nu}^l(x) \equiv \tilde{f}_{n,\nu}^l(x) - \hat{f}_{n,\nu}^l(x)$$

$$= \frac{1}{\pi}\left(\int_{x-l}^{x-l/\nu} + \int_{x+l/\nu}^{x+l}\right)\frac{f(s)-f(x)}{s-x}\left\{1 - \frac{\pi(s-x)}{2l}\cot\frac{\pi(s-x)}{2l}\right\}d\omega_n^l(s).$$

Since

$$\left(\int_{x-\delta}^{x-l/\nu} + \int_{x+l/\nu}^{x+\delta}\right)\frac{\mu(|s-x|)}{|s-x|}\left|1 - \frac{\pi(s-x)}{2l}\cot\frac{\pi(s-x)}{2l}\right|d\omega_n^l(s) \leqslant 6\varepsilon$$

and

$$\left|\left(\int_{x-l}^{x-\delta} + \int_{x+\delta}^{x+l}\right)\frac{f(x)}{s-x}\left\{1 - \frac{\pi(s-x)}{2l}\cot\frac{\pi(s-x)}{2l}\right\}d\omega_n^l(s)\right| \leqslant \frac{4l\,|f(x)|}{(2n+1)\delta},$$

we obtain

$$|Z_{n,\nu}^l(x)| \leqslant 3\varepsilon + \frac{1}{\pi}\left|\left(\int_{x-l}^{x-\delta} + \int_{x+\delta}^{x+l}\right)\frac{f(s)}{s-x}\left\{1-\frac{\pi(s-x)}{2l}\cot\frac{\pi(s-x)}{2l}\right\}d\omega_n^l(s)\right|$$

for sufficiently small l/ν.

By our hypothesis, there is a $\Delta \geqslant \max(\delta, 1+c+x)$ such that

$$\left(\int_{-\infty}^{x-\Delta+1} + \int_{x+\Delta-1}^{\infty}\right)\lambda(|t|)dt < \varepsilon.$$

Consequently,

$$\left|\left(\int_{x-l}^{x-\Delta} + \int_{x+\Delta}^{x+l}\right)\frac{f(s)}{s-x}\left\{1-\frac{\pi(s-x)}{2l}\cot\frac{\pi(s-x)}{2l}\right\}d\omega_n^l(s)\right|$$

$$\leqslant \left(\int_{x-l}^{x-\Delta} + \int_{x+\Delta}^{x+l}\right)\frac{|f(s)|}{|s-x|}\, d\omega_n^l(s) \leqslant 2\left(\int_{x-l}^{x-\Delta} + \int_{x+\Delta}^{x+l}\right)\lambda(|s|)d\omega_n^l(s)$$

$$\leqslant 2\left(\int_{-\infty}^{x-\Delta+1} + \int_{x+\Delta-1}^{\infty}\right)\lambda(|s|)ds < 2\varepsilon$$

when $l > \Delta$, $l/n \leqslant 1$. Hence

$$|Z_{n,\nu}^l(x)| \leqslant 4\varepsilon + \frac{1}{\pi}\left|\left(\int_{x-\Delta}^{x-\delta} + \int_{x+\delta}^{x+\Delta}\right)\frac{f(s)}{s-x}\left\{1-\frac{\pi(s-x)}{2l}\cot\frac{\pi(s-x)}{2l}\right\}d\omega_n^l(s)\right|$$

if $l > \Delta$ and l/ν are small enough. The last term does not exceed

$$\left(1-\frac{\pi\Delta}{2l}\cot\frac{\pi\Delta}{2l}\right)\frac{1}{\pi\delta}\left(\int_{x-\Delta}^{x-\delta} + \int_{x+\delta}^{x+\Delta}\right)|f(s)|\,d\omega_n^l(s) < \varepsilon$$

for sufficiently large l and small l/n. Thus we have

$$|Z_{n,\nu}^l(x)| \leqslant 5\varepsilon,$$

whence

$$|F_{n,\nu}^l(x) - \hat{f}_{n,\nu}^l(x)| < 12\varepsilon, \tag{6}$$

whenever l/ν together with $1/l$ are small enough.

Finally, we prove that

$$\lim_{l/\nu\to 0}\hat{f}_{n,\nu}^l(x) = -\frac{1}{\pi}\int_0^\infty \frac{\psi_x'(t)}{t}\,dt \qquad (l\to\infty). \tag{7}$$

By our assumptions, the existence of this integral is evident.

As before, for any positive $\eta \leqslant \delta$,

$$\left|\left(\int\limits_{x-\eta}^{x-l/\nu} + \int\limits_{x+l/\nu}^{x+\eta}\right) \frac{\mu(|s-x|)}{|s-x|}\, d\omega_n^l(s)\right| \leqslant 6\varepsilon \quad \text{if } \frac{l}{\nu} < \eta$$

and

$$\left(\int\limits_{x-l}^{x-\eta} + \int\limits_{x+\eta}^{x+l}\right) \frac{f(x)}{s-x}\, d\omega_n^l(s) \to 0 \quad \text{if } \frac{l}{n} \to 0.$$

Therefore,

$$\left|\hat{f}_{n,\nu}^l(x) + \frac{1}{\pi}\left(\int\limits_{x-l}^{x-\eta} + \int\limits_{x+\eta}^{x+l}\right) \frac{f(s)}{s-x}\, d\omega_n^l(s)\right| \leqslant 3\varepsilon$$

for l/ν small enough. Taking an arbitrary $\Lambda \geqslant \Delta$ and $l > \Lambda$, $l/n \leqslant 1$, we obtain

$$\left(\int\limits_{x-l}^{x-\Lambda} + \int\limits_{x+\Lambda}^{x+l}\right) \left|\frac{f(s)}{s-x}\right| d\omega_n^l(s) \leqslant 2\varepsilon.$$

Hence

$$\left|\hat{f}_{n,\nu}^l(x) + \frac{1}{\pi}\left(\int\limits_{x-\Lambda}^{x-\eta} + \int\limits_{x+\eta}^{x+\Lambda}\right) \frac{f(s)}{s-x}\, ds\right| < 4\varepsilon$$

if l/ν and $1/l$ are small enough.

But

$$\int\limits_0^\infty \frac{\psi_x(t)}{t}\, dt = \lim_{\substack{h\to 0+\\ H\to\infty}} \int\limits_h^H \frac{\psi_x(t)}{t}\, dt = \lim_{\substack{h\to 0+\\ H\to\infty}} \left(\int\limits_{x-H}^{x-h} + \int\limits_{x+h}^{x+H}\right) \frac{f(s)}{s-x}\, ds.$$

Thus, relation (7) is established and, by (5) and (6), the result follows.

In Theorem 3 the sums $\tilde{I}_{n,\nu}^l(x; f)$ can be replaced by their arithmetic means $\tilde{\sigma}_{n,\nu}^l(x; f)$.

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