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On monotonic solutions of some functional equations

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INTRODUCTION

The purpose of the present paper is to investigate the monotonic solutions of the functional equation of the first order.

Chapter I has a preliminary character. We give definitions of concepts that are needed in the following chapters and prove their main properties. Besides new concepts (the almost dense set, the almost continuous space, the compactly ordered space) we define also some already known ones. This is caused by the fact that various authors use different terminology.

In Chapter II we investigate increasing solutions of nonlinear equations of the first order

$$\varphi[f(x)] = H(x, \varphi(x)) \quad \text{and} \quad \varphi(x) = F(x, \varphi[f(x)]),$$

where f, H, F are given functions. The function f is defined in a subset A of a linearly ordered space X and takes the values in X and the functions H and F are defined in $A \times Y$ and have their values in Y, where Y is a linearly ordered space. We seek the function φ defined in A and having its values in Y.

The purpose of Chapter III is to study the existence and uniqueness of monotonic solutions of the equations

$$\varphi[f(x)] = \varphi(x) + h(x), \quad \varphi[f(x)] = g(x)\varphi(x),$$

where g, h are given functions defined in $A \subset X$ except for points of a set $S \in \Omega$. Here Ω is a σ-ideal of sets such that $X \notin \Omega$. Values of these functions lie in R and $g(x) > 0$. (Here R denotes the set of all real numbers.) The function f is such as in Chapter II.

Similar problems as in Chapter III, in the case where all functions are defined in an interval contained in R and have values in R, were studied by: J. Burek and M. Kuczma [2], M. Kuczma [8], [11], [12], and A. Smajdor [18]; and for $f(x) = x+1$ by F. John [6] and M. Kuczma [9].

The methods of proofs of theorems 3.1, 3.2, 3.9 and 3.16 in this chapter originate from papers [6], [12] and [18].

Chapter IV contains investigations of the existence and uniqueness of monotonic solutions of the general linear functional equation

$$\varphi[f(x)] = g(x)\varphi(x) + h(x),$$

where $g(x) > 0$. For $g(x) < 0$ analogous questions have been dealt with by M. Kuczma [10] ($g(x) = -1$), D. Brydak and J. Kordylewski [1] ($g(x) \equiv \text{const} < 0$) as well as A. Smajdor in [17].

The proofs of the theorems given in Chapter IV are based on the results of the preceding chapters.

CHAPTER I

DEFINITIONS AND AUXILIARY THEOREMS

§ 1. We say that a set X is *ordered by the relation* $\leqslant$ iff for some pairs $(x, y) \in X \times X$ the relation $x \leqslant y$ is defined and

(a) for every $x \in X$ we have $x \leqslant x$;

(b) if $x \leqslant y$ and $y \leqslant x$, then $x = y$;

(c) if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$.

A set X, ordered by a relation $\leqslant$, will be called an *ordered space*.

We say that x *precedes* y, and write $x < y$, when $x \leqslant y$ and $x \neq y$.

An element z of a subset S of an ordered space X is called the *greatest* [*least*] element of the set S iff each element $z' \in S$ fulfils the inequality $z' \leqslant z$ [$z \leqslant z'$].

Let X be an ordered space and let $S \subset X$. An element $z \in X$ is an *upper bound* [*lower bound*] of S iff for every $x \in S$ the inequality $x \leqslant z$ [$z \leqslant x$] holds. We say that a set is *bounded* [*upper bounded, lower bounded*] iff it possesses an upper bound and a lower bound [an upper bound, a lower bound].

The least element (provided it exists) of the set of upper bounds of S will be called the *supremum* of the set S and will be denoted shortly $\sup S$. The greatest element (provided it exists) of the set of lower bounds of S will be called the *infimum* of S and denoted $\inf S$.

A function $T(x)$ defined in a subset S of an ordered space X and having its values in an ordered space Y is called *increasing* [*decreasing*] iff for every pair $(x, y) \in S \times S$ such that $x \leqslant y$ we have the inequality $T(x) \leqslant T(y)$ [$T(y) \leqslant T(x)$]. A function T defined in $S \subset X$ and having its values in Y is called *strictly increasing* [*strictly decreasing*] iff for every pair $(x, y) \in S \times S$ such that $x < y$ we have the inequality $T(x) < T(y)$ [$T(y) < T(x)$].

LEMMA 1.1 (A. Pelczar [14], [15]). *Let X be an ordered space and let $T\colon X \to X$ be an increasing function. If the set*

$$\{x \in X\colon\ x \leqslant T(x)\}$$

is not empty and possesses the supremum in X, then the set of solutions of the equation

$$T(x) = x$$

has the greatest element.

We say that the ordering relation $\leqslant$ *orders* the set X linearly iff for every $x, y \in X$ one of the two inequalites

$$x \leqslant y \quad \text{or} \quad y \leqslant x.$$

holds.

Let X be a linearly ordered space. By the *open* [*closed*] *interval* with the ends $a, b \in X$ we understand the set

$$(a, b) \overset{\text{df}}{=} \{x \in X\colon\ a < x < b\} \qquad [[a, b] \overset{\text{df}}{=} \{x \in X\colon\ a \leqslant x \leqslant b\}].$$

The set

$$(a, b] \overset{\text{df}}{=} \{x \in X\colon\ a < x \leqslant b\} \qquad [[a, b) \overset{\text{df}}{=} \{x \in X\colon\ a \leqslant x < b\}]$$

is called the *left-open* [*right-open*] interval with the ends $a, b \in X$. Besides, we put

$$(-\infty, a) \overset{\text{df}}{=} \{x \in X\colon\ x < a\}, \qquad (-\infty, a] \overset{\text{df}}{=} \{x \in X\colon\ x \leqslant a\},$$
$$(a, +\infty) \overset{\text{df}}{=} \{x \in X\colon\ a < x\} \qquad [a, +\infty) \overset{\text{df}}{=} \{x \in X\colon\ a \leqslant x\}.$$

An arbitrary open interval containing $x_0 \in X \setminus \{\inf X, \sup X\}$ is called a *neighbourhood* of x_0. If $x_0 = \inf X$ $[x_0 = \sup X]$, then by a neighbourhood of x_0 we understand any interval of the form $[x_0, a)$ $[(a, x_0]]$. The family of neighbourhoods so defined determines a topology in X (cf. [5], p. 52). This topology is called the *order compatible topology*. In the sequel the ordered space will always be treated as a topological space with the order compatible topology. All topological concepts (closed set, closure of a set, etc.) appearing in this paper are connected with the order compatible topology.

LEMMA 1.2. *The infimum* [*supremum*] (*provided is exists*) *of a subset of the linearly ordered space belongs to the closure of this subset.*

Proof. Let $S \subset X$ and $q = \inf S$. If $q = \sup X$, then $S = \{q\}$ and q belongs to the closure of S. Suppose that $q \neq \sup X$. If there existed a $d \in X$, $d > q$, such that $[q, d) \cap S = \emptyset$, then d would be a lower bound of S and, since $d > q$, we obtain a contradiction with $q = \inf S$. Thus for every $d \in X$ and $d > q$ the set $[q, d) \cap S$ is not empty, hence for every neighbourhood $U_q \subset X$ of q the set $U_q \cap S$ is not empty. Consequently $q \in \bar{S}$.

DEFINITION. We say that a subset Z of a linearly ordered space X is *dense* [*almost dense*] iff for every $x, y \in X$ such that $x < y$ there exists a $z \in Z$ such that $x < z < y$ $[x \leqslant z < y]$.

A dense set is dense in X in the topological sense.

We say that two linearly ordered spaces X and Y are *similar* iff there exists a strictly increasing mapping $\sigma\colon X \to Y$ such that $\sigma(X) = Y$. Such mapping is called a similarity. Every similarity is a homeomorphism.

LEMMA 1.3 (cf. [13]). *Every linearly ordered space X is similar to the family of intervals $P_x = (-\infty, x)$ linearly ordered by the inclusion.*

A pair (A, A') of subsets of a linearly ordered space X is called a *section* iff the following conditions are fulfilled:

(i) $A \cup A' = X$,

(ii) if $a \in A$ and $b \in A'$, then $a < b$.

LEMMA 1.4 (cf. [13]). *If pairs (A, A') and (B, B') are sections of a linearly ordered space, then $A \subset B$ or $B \subset A$.*

Let $a = (A, A')$, $b = (B, B')$ be sections of a linearly ordered space X. We say that a and b satisfy the relation $a \leqslant b$ iff $A \subset B$. According to Lemma 1.4 the relation $\leqslant$ linearly orders the set of all section of the linearly ordered space X.

For every section (A, A') of the space X there are the following cases possible:

(a) the set A has the greatest element and the set A' has the least element;

(b) the set A has the greatest element but the set A' has not the least element;

(c) the set A has not the greatest element but the set A' has the least element;

(d) the set A has not the greatest element and the set A' has not the least element.

DEFINITION. A linearly ordered space is said to be: *dense* iff any section does not fulfil condition (a); *continuous* iff every section fulfils condition (b) or (c); *almost continuous* iff any section does not fulfil condition (d).

Every continuous linearly ordered space is almost continuous.

DEFINITION. A linearly ordered space X which is almost continuous and bounded is called a *compactly ordered space.*

Since the boundedness is meant as the boundedness in the space X, it follows directly from the above definition that every compactly ordered space contains the greatest and the least element.

We denote by Z the set of all sections of the space which are of the form $a_x = (P_x, X \setminus P_x)$, and by X^* the set of all sections possessing property (d) or belonging to Z. The space X^* is linearly ordered (see Lemma 1.4).

LEMMA 1.5. *The space X^* is compactly ordered. If X is dense, then X^* is continuous.*

Proof. At first we show that the set Z is almost dense in X^*. Let $a, b \in X^*$ and $a < b$. If $a = (A, A')$, $b = (B, B')$ then $A \subset B$ and $A \neq B$. So there exists

$$x_0 \in B \setminus A. \tag{1.1}$$

If $x \in A$ then, since $x_0 \in A'$, we have the inequality $x < x_0$, that is $x \in P_{x_0}$. Thus we have shown that

$$A \subset P_{x_0}. \tag{1.2}$$

Let $x \in P_{x_0}$. If $x \in B'$, then since $x_0 \in B$, we get the inequality $x_0 < x$ which contradicts the condition $x \in P_{x_0}$. Thus we have the inclusion

$$P_{x_0} \subset B. \tag{1.3}$$

Conditions (1.1), (1.2) and (1.3) imply the inequality

$$a \leqslant a_{x_0} < b.$$

Hence the set Z is almost dense in X^*.

I. Suppose that there exists a section $c = (C, C')$ of the space X^* possessing property (d). Then the pair $(C \cap Z, C' \cap Z)$ is a section of the set Z. Let $d = (D, D')$ be the section of the space X defined in the following manner

$$D \overset{\mathrm{df}}{=} \bigcup_{(A, A') \in C \cap Z} A, \qquad D' \overset{\mathrm{df}}{=} X \setminus D.$$

If D had the greatest element a_0, then there would exist an A such that $(A, A') \in C \cap Z$, $a_0 \in A$, and a_0 is an upper bound for A, which is not possible. So D has not the greatest element and hence $d \in X^*$. If $a \in C$, then, since C has not the greatest element, there exists $a_1 \in C$, $a < a_1$. But Z is almost dense, so there exists $a_2 \in Z$ such that

$$a \leqslant a_2 < a_1. \tag{1.4}$$

If we had $a_2 \in C'$, then the relation $a_1 \in C$ would imply the inequality $a_1 < a_2$. Thus $a_2 \in C \cap Z$. On account of the definition of d we have $a_2 \leqslant d$. Hence and by (1.4)

$$a \leqslant d.$$

Similarly we can prove that for every $b \in C'$

$$d \leqslant b.$$

The last two inequalities show that d is the greatest element of C or the least element of C'. This is a contradiction with the hypothesis that c fulfils conditions (d). Thus we have shown that X^* is almost continuous.

The space X^* has the least and greatest element. The least element is $(\emptyset, X)$. If X has the greatest element $\hat{x}$, then the greatest element of X^*

is the section $a_{\hat{x}}$. If X has not the greatest element, then the greatest element of X^* is the section $(X, \emptyset)$. Consequently the space X^* is bounded. Thus we have proved that the space X^* is compactly ordered.

II. Now suppose that X is dense. Let $a, b \in X^*$ and $a < b$. If $a = (A, A')$, $b = (B, B')$, then $A \subset B$ and $B \setminus A \neq \emptyset$. Since X is dense and for every $x \in X$ the set $X \setminus P_x$ has the least element (viz. x), the set P_x cannot have the greatest element. Hence neither A nor B has the greatest element. There exists an $x_0 \in B \setminus A$ and, since B has not the greatest element, there exists an $x_1 > x_0$, $x_1 \in B \setminus A$. Then $A \subset P_{x_1} \subset B$ and $x_0 \in P_{x_1} \setminus A$, $x_1 \in B \setminus P_{x_1}$ that is

$$a < a_{x_1} < b.$$

We have shown that if X is dense, then X^* is dense, too. Moreover X^* is almost continuous, and thus it is continuous.

According to Lemma 1.3 the space X is similar to the set Z, so we have.

LEMMA 1.6. *Every linearly ordered space X is similar to a subspace of the space X^*.*

Now we shall prove

LEMMA 1.7. *In a compactly ordered space X each set possesses supremum and infimum.*

Proof. Let the set A be a subset of the space X and $b = (B, B')$ where $B' \overset{\mathrm{df}}{=} \{x \in X\colon\ y < x \text{ for every } y \in A\}$ and $B = X \setminus B'$. In a compactly ordered space each section has property (a) or (b) or (c). If b has property (a) or (b), then the greatest element of the set B is the supremum of A. If b has property (c), then the supremum of the set A is furnished by the least element of the set B'. Similary we show the existence of the infimum.

Taking into account that a compactly ordered space contains the least element, we infer from Lemma 1.7 (cf. [5], p. 144) that this space is compact.

LEMMA 1.8. *From every sequence in a linearly ordered space we can choose a monotonic subsequence.*

Proof. Let $\{x_n\}$ be an arbitrary sequence of elements of a linearly ordered space X. Suppose that there exists a subsequence $\{x_{n_\nu}\}$ of the sequence $\{x_n\}$ such that for every ν_0 there exists a ν fulfilling the condition

$$x_{n_{\nu_0}} < x_{n_\nu}. \tag{1.5}$$

Then for every ν_0 there exist infinitely many ν fulfilling (1.5). Let $x_{n_{\nu_1}} = x_{n_1}$ and let $x_{n_{\nu_2}}$ be an element of the sequence $\{x_{n_\nu}\}$ fulfilling (1.5) for $\nu_0 = \nu_1$ and such that $\nu_1 < \nu_2$. Similarly, $x_{n_{\nu_{\mu+1}}}$ is a term of the sequence $\{x_{n_\nu}\}$

fulfilling (1.5) for $\nu_0 = \nu_\mu$ such that $\nu_{\mu+1} > \nu_\mu$. The sequence $\{x_{n_{\nu_\mu}}\}$ is an increasing subsequence of $\{x_n\}$.

Now suppose that for every subsequence $\{x_{n_\nu}\}$ of $\{x_n\}$ there exists a ν_0 such that for every ν the inequality

$$x_{n_\nu} \leqslant x_{n_{\nu_0}}.$$

is fulfilled. There is an element x_{n_1} of the sequence $\{x_n\}$ such that

$$x_n \leqslant x_{n_1} \quad \text{for} \quad n \geqslant 1. \tag{1.6}$$

There is an element x_{n_2} in the sequence $x_{n_1+1}, x_{n_1+2}, \ldots$ such that

$$x_n \leqslant x_{n_2} \quad \text{for} \quad n \geqslant n_1+1. \tag{1.7}$$

Similarly, there exists $x_{n_{k+1}}$ in the sequence $x_{n_k+1}, x_{n_k+2}, \ldots$ such that

$$x_n \leqslant x_{n_{k+1}} \quad \text{for} \quad n \geqslant n_k+1. \tag{1.8}$$

It follows by (1.6), (1.7) and (1.8) that the sequence $\{x_{n_k}\}$ is decreasing. This completes the proof.

LEMMA 1.9. *In a compactly ordered space from every sequence $\{x_n\}$ one can choose a convergent subsequence.*

Proof. In virtue of Lemma 1.8 we can choose a monotonic, say increasing, subsequence $\{x_{n_k}\}$. In view of Lemma 1.7 the set of terms of this sequence has the supremum. We shall show that

$$\lim_{k\to\infty} x_{n_k} = \sup_k x_{n_k}.$$

Let $x = \sup_k x_{n_k}$ and let U be a neighbourhood of the point x. There exists an interval $(a, x] \subset U$. According to the definition of the supremum there exists $x_{n_{k_0}} \in (a, x]$. The sequence $\{x_{n_k}\}$ is increasing, thus for $k \geqslant k_0$ we have $x_{n_k} \in (a, x]$. Hence for $k > k_0$ we have $x_{n_k} \in U$, and consequently $\lim_{k\to\infty} x_{n_k} = x$. If $\{x_{n_k}\}$ is decreasing then $\lim_{k\to\infty} x_{n_k} = \inf_k x_{n_k}$.

As is known, a space X is called *connected* iff it does not admit a representation in the form of a union $X_1 \cup X_2$ of two closed, separated and non-empty subsets. A set $A \subset X$ is connected iff it is a connected subspace of X. We shall prove

LEMMA 1.10. *A linearly ordered and continuous space X is connected.*

Proof. Suppose that there exists closed, separated and non-empty sets X_1 and X_2 such that $X_1 \cup X_2 = X$. Let $a \in X_1$, $b \in X_2$. We assume that $a < b$. Let C' denote the set of all upper bounds of the set $X_1 \cap [a, b]$, $C \overset{\mathrm{df}}{=} X \setminus C'$. The set $X_1 \cap [a, b]$ is closed. The space X is continuous, so the section (C, C') has property (b) or (c). Suppose that C has the

greatest element c_0. Any element of the set C' does not belong to X_1, since C' has not the least element. Thus c_0 is also an upper bound of $X_1 \cap [a, b]$. This is not possible.

Now, we suppose that C has not the greatest element. In such a case C' has the least element c_1 and $c_1 = \sup X_1 \cap [a, b]$. It follows by Lemma 1.2 that c_1 is a point of the closure of the set $X_1 \cap [a, b]$. This set is closed, so

$$c_1 \in X_1 \cap [a, b]. \tag{1.9}$$

Since c_1 is the upper bound of the set $X_1 \cap [a, b]$, for every $d \in (c_1, d]$ we must have $(c_1, d) \cap X_1 = \emptyset$. On the other hand, since the space X is dense, we have $(c_1, d) \neq \emptyset$ for every d, $c_1 < d \leqslant b$. Thus for every $d \in (c_1, b]$ we have $(c_1, d) \cap X_2 \neq \emptyset$. Hence c_1 is a point of the closure of the set X_2. The set X_2 is closed, so $c_1 \in X_2$, and this is contradictory with (1.9). This completes the proof.

Every interval of a continuous space is a continuous subspace, so intervals in the continuous space are connected sets. On the other hand, every connected subset of a linearly ordered space is an interval. Thus we have

COROLLARY. *A subset A of a connected space is connected if and only if it is an interval.*

From the fact that a continuous image of a connected set is connected we obtain

LEMMA 1.11. *Suppose that X and Y are linearly ordered spaces and $A \subset X$ is a connected set. Then every continuous function $f\colon A \to Y$ has the Darboux property.*

LEMMA 1.12. *Let functions $f\colon X \to Y$, $g\colon X \to Y$, where X and Y are linearly ordered spaces, fulfil the inequality*

$$f(x) \leqslant g(x)$$

in a neighbourhood S_{x_0} of x_0, and suppose that f, g have limits at x_0. Then

$$\lim_{x \to x_0} f(x) \leqslant \lim_{x \to x_0} g(x).$$

Proof. Suppose the contrary, i.e.

$$a = \lim_{x \to x_0} f(x) > b = \lim_{x \to x_0} g(x).$$

If the interval (b, a) is empty, then there exists a neighbourhood $S^0_{x_0}$ of x_0 such that

$$f(S^0_{x_0}) \subset (b, +\infty) \subset [a, +\infty) \quad \text{and} \quad g(S^0_{x_0}) \subset (-\infty, a) \subset (-\infty, b).$$

Thus for $x \in S^0_{x_0} \cap S_{x_0}$ we have $f(x) \geqslant a$ and $g(x) \leqslant b$, whence

$$a \leqslant f(x) \leqslant g(x) \leqslant b,$$

in spite of the supposition $b < a$. If the interval (b, a) is not empty, then for $c \in (b, a)$ there exists a neighbourhood $S^1_{x_0}$ such that

$$f(S^1_{x_0}) \subset (c, +\infty) \quad \text{and} \quad g(S^1_{x_0}) \subset (-\infty, c).$$

Hence for $x \in S_{x_0} \cap S^1_{x_0}$ we have $c < f(x) \leqslant g(x) < c$, which is impossible. Thus $a \leqslant b$.

§ 2. Definition. We say that a class $\Omega \neq \emptyset$ of subsets of a set X is a *σ-ideal* iff it fulfils the conditions:

(I) If $S_i \in \Omega$, $i = 1, 2, \ldots$, then $\bigcup_{i=1}^{\infty} S_i \in \Omega$.

(II) If $S_1 \subset S_2$, $S_2 \in \Omega$, then $S_1 \in \Omega$.

In the sequel we shall assume also that

(III) $X \notin \Omega$.

The following are examples of σ-ideals which do not contain X:

(a) the class of at most countable sets in an uncountable space X;

(b) the class of sets of measure zero in the space R of real numbers;

(c) the class of sets of the first category in the space R of real numbers;

(d) the class which has the empty set as the only element.

In the sequel we consider a fixed σ-ideal Ω which does not contain X.

We shall say that a condition holds *Ω-almost everywhere* (Ω-a.e.) in X iff there exixts a set $S \in \Omega$ such that this condition holds in the set $X \setminus S$.

Let a real-valued function h be defined Ω-a.e. in a set A. We define the essential supremum (essential infimum) of the function h with respect to Ω as follows:

$$\sup_{x \in A} \operatorname{ess}_{(\Omega)} h(x) \overset{\text{df}}{=} \inf_{S \in \Omega} [\sup_{x \in A \setminus S} h(x)] \quad \left(\inf_{x \in A} \operatorname{ess}_{(\Omega)} h(x) \overset{\text{df}}{=} \sup_{S \in \Omega} [\inf_{x \in A \setminus S} h(x)]\right).$$

In particular, the function h can be a measurable function on a set A, and Ω can consist of sets of measure zero. Then a condition holds Ω-almost everywhere iff it holds almost everywhere in the usual sense and $\sup_{x \in A} \operatorname{ess}_{(\Omega)} h(x)$ is equal to the usual essential supremum of a measurable function (cf. [16], p. 13). If the class Ω consists of the empty set only, then a condition holds Ω-a.e. iff it holds everywhere, and the essential supremum is the supremum in A.

Lemma 1.13. *Let h be a function defined Ω-a.e. in a subset A of a linearly ordered space X, with values in the set of real numbers. Then there exists a set $S_0 \in \Omega$ such that*

$$\inf_{x \in A} \operatorname{ess}_{(\Omega)} h(x) = \inf_{x \in A \setminus S_0} h(x).$$

Proof. Let

$$a_0 = \operatorname*{infess}_{x\in A\ (\Omega)} h(x) = \sup_{S\in\Omega}[\inf_{x\in A\setminus S} h(x)].$$

First we suppose that $a_0 > -\infty$. Let

$$a_n = \begin{cases} n & \text{if} \quad a_0 = +\infty, \\ a_0 - \dfrac{1}{n} & \text{if} \quad a_0 < +\infty. \end{cases}$$

For every positive integer n there exists a set $S_n \in \Omega$ such that

$$\inf_{x\in A\setminus S_n} h(x) > a_n. \tag{1.10}$$

Let $S_0 = \bigcup_{n=1}^{\infty} S_n$. Then $S_0 \in \Omega$ and

$$a_0 = \sup_{S\in\Omega}[\inf_{x\in A\setminus S} h(x)] \geqslant \inf_{x\in A\setminus S_0} h(x). \tag{1.11}$$

On the other hand, for every positive integer n

$$\inf_{x\in A\setminus S_0} h(x) \geqslant \inf_{x\in A\setminus S_n} h(x),$$

since $A\setminus S_0 \subset A\setminus S_n$. Hence and by (1.10) we have

$$\inf_{x\in A\setminus S_0} h(x) > a_n,$$

and with $n \to \infty$ we obtain

$$a_0 \leqslant \inf_{x\in A\setminus S_0} h(x). \tag{1.12}$$

Inequalities (1.11) and (1.12) yield the assertion of the lemma.

If $a_0 = -\infty$, then for every $S \in \Omega$ we have $\inf_{x\in A\setminus S} h(x) = a_0$.

DEFINITION. Let h be a real-valued function defined Ω-a.e. in $A \subset X$, where X is a linearly ordered space. We say that h attains the *essential infimum* at a point $x_0 \in \bar{A}$ iff there exists a sequence $\{x_n\}$; $x_n \in A$, $n = 1, 2, \ldots$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} h(x_n) = \operatorname*{infess}_{x\in A\ (\Omega)} h(x)$.

LEMMA 1.14. *Let h be a real-valued function defined Ω-a.e. in $A \subset X$, where X is a compactly ordered space. Then there exists a point $x_0 \in \bar{A}$ such that the essential infimum of h in the set A is attained at this point.*

Proof. In virtue of Lemma 1.13 there exists the set $S_0 \in \Omega$ such that $a_0 = \operatorname*{infess}_{x\in A\ (\Omega)} h(x) = \inf_{A\setminus S_0} h(x)$. If $a_0 \in R$, then for every positive number

n there exists an element $x_n \in A \setminus S_0$ such that

$$a_0 \leqslant h(x_n) < a_0 + \frac{1}{n}. \tag{1.13}$$

According to Lemmas 1.8 and 1.9 there exists a monotonic and convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let

$$x_0 = \lim_{k\to\infty} x_{n_k}.$$

By (1.13) we get $\lim\limits_{k\to\infty} h(x_{n_k}) = a_0$.

If $a_0 = -\infty$, the proof is analogous. Then we replace (1.13) by the inequality $h(x_n) < -n$.

CHAPTER II

INCREASING SOLUTIONS OF A FUNCTIONAL EQUATION OF THE FIRST ORDER

In this chapter we shall study the increasing solutions of the equations

$$\varphi[f(x)] = H(x, \varphi(x)) \tag{2.1}$$

and

$$\varphi(x) = F(x, \varphi[f(x)]), \tag{2.2}$$

where x varies in a certain subset of a linearly ordered space X.

§ 1. We make the following hypotheses:

(H_1) X and Y are linearly ordered spaces, $A \subset X$.

(H_2) $f\colon A \to A$ is an increasing function satisfying the inequality $x < f(x)$ for every $x \in A$ and such that for every $x \in A$ the sequence $\{f^n(x)\}$ of the iterates of f has not an upper bound in A.

Suppose that hypotheses (H_1), (H_2) are fulfilled, $Q \subset R$, the function $H\colon A \times Q \to Q$ is increasing with respect to the first variable and strictly increasing with respect to the second variable. Let φ be an increasing solution of (2.1) and $x_1, x_2 \in A$, $x_1 < x_2$, $f(x_1) = f(x_2)$; then

$$H(x_1, \varphi(x_1)) = H(x_2, \varphi(x_2)). \tag{2.3}$$

Taking into account the fact that H is strictly increasing with respect to the second variable, the inequality $\varphi(x_1) < \varphi(x_2)$ is incompatible with (2.3). Therefore $\varphi(x_1) = \varphi(x_2)$. From (2.3) we obtain the equality

$$H(x_1, \varphi(x_1)) = H(x_2, \varphi(x_1)).$$

DEFINITION. We say that $g \in S_f(B)$, $B \subset A$, iff for every x_1, $x_2 \in B$ the equality $f(x_1) = f(x_2)$ implies $g(x_1) = g(x_2)$.

The above considerations justify the following hypothesis

(H_3) Q is a subset of the space Y, $H\colon A \times Q \to Q$ is an increasing

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function with respect to both variables and for every fixed $y \in Q$ we have $H(x, y) \in S_f(A)$. Moreover, for every $x \in A$ the set

$$Q_x = \{y \in Q\colon\ y \leqslant H(x, y)\}$$

is not empty.

We shall prove

THEOREM 2.1. *Let hypotheses* (H_1), (H_2) *and* (H_3) *be fulfilled. If* $a_0 = \inf A$ *exists,* $a_0 \in A$ *and* Q *is a continuous subspace of* Y, *then for every increasing function* $\varphi_0 \in S_f\big(A \cap [a_0, f(a_0)]\big)$ *defined on* $A \cap [a_0, f(a_0)]$, *having values in* Q *and such that*

$$\varphi_0[f(a_0)] = H\big(a_0, \varphi_0(a_0)\big) \tag{2.4}$$

there exists a (not necessarily unique) increasing function φ *which is an extension of* φ_0 *onto* A *and satisfies equation* (2.1) *in* A.

Proof. Let

$$a_n = f^n(a_0), \qquad n = 1, 2, \ldots \tag{2.5}$$

We choose a $y_0 \in Q_{a_0}$ and an arbitrary increasing function $\varphi_0 \in S_f(A \cap [a_0, a_1])$ defined on $A \cap [a_0, a_1]$, having values in Q and such that

$$\varphi_0(a_0) = y_0, \qquad \varphi_0(a_1) = H(a_0, y_0). \tag{2.6}$$

In the set $f(A \cap [a_0, a_1]) \subset [a_1, a_2] \cap A$ we define a function φ_1 by the formula

$$\varphi_1(x) = H\big(x', \varphi_0(x')\big) \tag{2.7}$$

where x' is an element of the set $f^{-1}(\{x)\}$. According to (H_3) the value $\varphi_1(x)$ is independent of the choice of x'. The functions φ_0 and φ_1 fulfil the equation

$$\varphi_1[f(x)] = H\big(x, \varphi_0(x)\big) \tag{2.8}$$

in $[a_0, a_1] \cap A$. Let $x_1, x_2 \in f([a_0, a_1] \cap A)$, $x_1 < x_2$, and $x_1' \in f^{-1}(\{x_1\})$, $x_2' \in f^{-1}(\{x_1\})$. If we had $x_1' \geqslant x_2'$, then, according to the monotonicity of f, $x_1 = f(x_1') \geqslant f(x_2') = x_2$. Hence $x_1' < x_2'$. The function H is increasing with respect to both variables, whence from (2.7) we obtain

$$\varphi_1(x_1) = H\big(x_1', \varphi_0(x_1')\big) \leqslant H\big(x_2', \varphi_0(x_2')\big) = \varphi_1(x_2).$$

Now, φ_1 is increasing on the set $f([a_0, a_1] \cap A)$. From (2.8) we obtain $\varphi_1(a_1) = \varphi_1[f(a_0)] = H\big(a_0, \varphi_0(a_0)\big)$, and from (2.6) we have

$$\varphi_1(a_1) = \varphi_0(a_1).$$

If the set $A \cap [a_1, a_2] \setminus f([a_0, a_1] \cap A)$ is not empty, then we define the function φ_1 arbitrarily on this set, but in such a manner that φ_1 is increasing on $[a_1, a_2] \cap A$.

Similarly we define a function φ_2 on the set $[a_2, a_3] \cap A$, φ_3 on $A \cap [a_3, a_4]$ and generally φ_n on $[a_n, a_{n+1}] \cap A$. These functions are increasing and they fulfil the conditions

$$\varphi_{n+1}[f(x)] = H\big(x, \varphi_n(x)\big) \tag{2.9}$$

for $x \in A \cap [a_n, a_{n+1}]$, $n = 0, 1, \ldots$, and

$$\varphi_{n+1}(a_{n+1}) = \varphi_n(a_{n+1}), \quad n = 0, 1, \ldots \tag{2.10}$$

The sequence $\{a_n\}$ is strictly increasing and has not an upper bound in A, whence

$$A = \bigcup_{n=0}^{\infty} [a_n, a_{n+1}] \cap A. \tag{2.11}$$

According to the monotonicity of φ_n and (2.10), (2.11) the function

$$\varphi(x) = \varphi_n(x) \quad \text{for} \quad x \in [a_n, a_{n+1}] \cap A, \ n = 0, 1, \ldots$$

is increasing on A. By (2.9) the function φ satisfies equation (2.1) in A.

DEFINITION. Let Φ be a family of increasing functions φ_0 defined on $[a_0, a_1] \cap A$ and fulfilling the condition

$$\varphi_0(a_1) = H\big(a_0, \varphi_0(a_0)\big).$$

We say that equation (2.1) has an *increasing solution depending on an arbitrary function* iff the family Φ is not finite-parameter (1) and for every $\varphi_0 \in \Phi$ there exists exactly one increasing function φ on A which is an extension of φ_0 onto A and satisfies equation (2.1) on A.

THEOREM 2.2. *Suppose that X and Y are continuous linearly ordered spaces, $A = \{x \in X\colon x \geqslant a_0\}$, hypotheses* ($\mathrm{H}_2$) *and* ($\mathrm{H}_3$) *are fulfilled, f is continuous and the set Q is an interval.*

If solutions of the inequality

$$y < H(a_0, y) \tag{2.12}$$

exist, then equation (2.1) *has in A an increasing solution depending on an arbitrary function.*

If the only solutions of the inequality

$$y \leqslant H(a_0, y)$$

are those fulfilling

$$y = H(a_0, y), \tag{2.13}$$

then each solution of equation (2.13) *determines exactly one increasing solution of* (2.1).

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(1) A family Φ of functions is k-parameter iff for every $x_1, \ldots, x_k$ and $l_1, \ldots, l_k$ there exists at most one function $\varphi \in \Phi$ such that $\varphi(x_1) = l_1, \ldots, \varphi(x_k) = l_k$.

Proof. According to Lemma 1.10 (cf. Corollary) the set A is connected. A continuous function on a connected set has the Darboux property (Lemma 1.11), whence

$$f([a_n, a_{n+1}]) = [a_{n+1}, a_{n+2}].$$

Hence, according to Theorem 2.1, for every increasing function $\varphi_0 \in S_f([a_0, a_1])$ defined on $[a_0, a_1]$, having values in Q and fulfilling (2.5), there exists exactly one increasing function φ which is an extension of φ_0 onto A and satisfies equation (2.1) in A. If a solution of the inequality (2.12) exists, then the family Φ of functions φ_0 is not finite-parameter. If there exists no solution of (2.12), then every increasing solution of (2.1) on A is constant on $[a_0, a_1]$. Then each solution of (2.13) determines exactly one increasing solution of (2.1) in A.

LEMMA 2.1. *Suppose that X is a compactly ordered space, $A \subset X$ and*

$$A = \bar{A} \setminus \{\inf A, \sup A\}. \tag{2.14}$$

If f fulfils hypothesis (H_2), *is continuous and for an $a \in A$ we have*

$$B_n \overset{\mathrm{df}}{=} f^n(A) \cap [a, f(a)] \neq \emptyset \quad \textit{for} \quad n = 0, 1, \ldots, \tag{2.15}$$

where f^n is n-th iterate of f, then there exists a sequence $\{x_n\}$ such that $x_n \in A$ $(n = 0, 1, \ldots)$ and

$$f(x_n) = x_{n-1} \quad \textit{for} \quad n = 1, 2, \ldots \tag{2.16}$$

Proof. We put

$$f_1(x) = \begin{cases} \inf f(A) & \text{for} \quad x = \inf A, \\ f(x) & \text{for} \quad x \in A, \\ \sup f(A) & \text{for} \quad x = \sup A. \end{cases} \tag{2.17}$$

We shall prove that the function f_1 is continuous. Let $x = \inf A$ and let V be an arbitrary neighbourhood of $f_1(x)$. Since $f_1(x) = \inf f(A)$, there exists a point $y \in f(A) \cap V$. Hence there exists a point $x' \in A$ such that

$$f(x') = y.$$

From (2.14)

$$\inf A < x'.$$

Hence the neighbourhood

$$U = \begin{cases} (-\infty, x') & \text{for} \quad x \neq \inf X, \\ [x, x') & \text{for} \quad x = \inf X \end{cases}$$

of x satisfies the inclusion $f_1(U \cap \bar{A}) \subset V$. This proves that the function f_1 is continuous at the point $x = \inf A$. The proof of the continuity of f_1 at $x = \sup A$ is similar. Now suppose that $x \in A$. Then, according to the continuity of f, for every neighbourhood V of $f(x)$ there exists a neighbourhood U of x such that $f(U \cap A) \subset V$. The set $U_1 \overset{\mathrm{df}}{=} U \cap (\inf A, \sup A)$ is a neighbourhood of x and $f_1(U_1 \cap \bar{A}) = f_1(U \cap A) = f(U \cap A) \subset V$. Hence the function f_1 is continuous on $\bar{A}$.

The inclusion

$$f^n(A) \cap [a, f(a)] \subset f_1^n(\bar{A}) \cap [a, f(a)] \tag{2.18}$$

is evident. Suppose that $n \geqslant 1$ and that there exists an $x \in [f_1^n(\bar{A}) \setminus f^n(A)] \cap \cap [a, f(a)]$. Then there exists an $x' \in \bar{A} \setminus A$ such that $f_1^n(x') = x$. If $x' = \sup A$ then we would have, since $n \geqslant 1$,

$$f_1(\sup A) \leqslant f_1^n(\sup \bar{A}) \leqslant f(a). \tag{2.19}$$

In view of the monotonicity of f_1 and by (2.19) the function f is constant on $A \cap [a, +\infty)$ which is impossible. If $x' = \inf A$, then $x = \inf f^n(A)$ and

$$\inf f^{n+2}(A) = f^2(x) \in [f^2(a), f^3(a)]. \tag{2.20}$$

This implies according to (2.15) and (2.20) that $B_{n+2} = \emptyset$ which contradicts the hypothesis of the lemma. Thus we have shown that the set

$$[f_1^n(\bar{A}) \setminus f^n(A)] \cap [a, f(a)]$$

is empty. Hence the set $\big(f_1^n(\bar{A}) \cap [a, f(a)]\big) \setminus \big(f^n(A) \cap [a, f(a)]\big)$ is empty and in view of (2.18) we have the equality

$$f^n(A) \cap [a, f(a)] = f_1^n(\bar{A}) \cap [a, f(a)] \quad \text{for} \quad n = 1, 2, \ldots \tag{2.21}$$

Suppose that $n = 0$, then

$$\begin{aligned} f^0(A) \cap [a, f(a)] &= A \cap [a, f(a)] = (\bar{A} \setminus \{\inf A, \sup A\}) \cap [a, f(a)] \\ &= \big(\bar{A} \cap [a, f(a)]\big) \setminus \{\inf A, \sup A\}. \end{aligned}$$

But $\inf A, \sup A \notin [a, f(a)]$. Consequently equality (2.21) is satisfied also for $n = 0$.

Since the sequence $\{B_n\}$ of closed and non-empty sets (cf. (2.15), (2.21)) is decreasing, there exists (cf. [5], p. 91) a point x_0 such that

$$x_0 \in B_n \quad \text{for} \quad n = 0, 1, \ldots \tag{2.22}$$

According to (2.22) every set

$$C_n \overset{\mathrm{df}}{=} f^{-n}(\{x_0\}), \quad n = 0, 1, \ldots \tag{2.23}$$

is non-empty. Moreover

(2.24) $$f(C_{n+1}) \subset C_n \quad \text{for} \quad n = 1, 2, \ldots$$

We denote by P the set of sequences $y = \{y_n\}$ such that $y_n \in A$ and $f^n(y_n) = x_0$ for $n = 1, 2, \ldots$ In view of (2.15), (2.22), (2.23) P is not empty. The set P is ordered by the relation

(2.25) $$y \leqslant y' \overset{\text{df}}{=} y_n \leqslant y'_n \quad \text{for every positive integer } n.$$

Let $S \subset P$ and $\bar{y}_n = \sup\limits_{y \in S} y_n$. We shall show that $\bar{y} = \{\bar{y}_n\} \in P$. For every $y \in S$ and for every n the inequality $y_n \leqslant \bar{y}_n$ is fulfilled, whence

$$x_0 = f^n(y_n) = f_1^n(y_n) \leqslant f_1^n(\bar{y}_n).$$

Let n be a fixed positive integer. Suppose that $x_0 < f_1^n(\bar{y}_n)$. Since the function $f_1^n(x)$ is continuous, there exists a neighbourhood U of $\bar{y}_n$ such that $f_1^n(U \cap \bar{A}) \subset (x_0, +\infty)$. On the other hand the equality $\bar{y}_n = \sup\limits_{y \in S} y_n$ implies that there exists a $y \in S$ such that $y_n \in U$ and

$$x_0 = f^n(y_n) \in f_1^n(U \cap \bar{A}) \subset (x_0, +\infty).$$

We obtain a contradiction. Therefore

(2.26) $$x_0 = f_1^n(\bar{y}_n).$$

Suppose that $\bar{y}_n \notin A$, i.e.

(2.27) $$\bar{y}_n = \sup A.$$

From (H_2), (2.14) and (2.17) we have that the function f_1 maps $\bar{A}$ into itself and for every $x \in \bar{A}$

$$x \leqslant f_1(x).$$

Hence by (2.27) we get

$$f_1^n(\bar{y}_n) = \sup A.$$

It is a contradiction with (2.26). Thus we have shown that $\bar{y}_n \in A$. From (2.17) and (2.26) we obtain that $f^n(\bar{y}_n) = x_0$ and

$$\bar{y} \in P.$$

According to the definition of $\bar{y}$ we have $y \leqslant \bar{y}$ for every $y \in S$. Suppose that for every $y \in S$ and for certain $\bar{\bar{y}} \in P$ we have $y \leqslant \bar{\bar{y}}$. It means that for every $y \in S$ and for every n we have $y_n \leqslant \bar{\bar{y}}_n$. Hence $\bar{y}_n = \sup\limits_{y \in S} y_n \leqslant \bar{\bar{y}}_n$ and $\bar{y} \leqslant \bar{\bar{y}}$. This implies $\bar{y} = \sup S$.

Let

(2.28) $$y_n^* \overset{\text{df}}{=} \inf C_n,$$

where C_n is defined by (2.23). Sets C_n, as counter-images of closed sets

by a continuous function, are closed sets. Since the infimum of a set belongs to the closure of this set, we have

$$y_n^* \in C_n. \tag{2.29}$$

From (2.24), (2.28) and (2.29) we obtain

$$y_n^* \leqslant f(y_{n+1}^*). \tag{2.30}$$

We denote by T a mapping of P into P such that for $y = \{y_n\}$, $Ty = \{\tilde{y}_n\}$, where $\tilde{y}_n = f(y_{n+1})$. The mapping T is increasing. According to (2.25) and (2.30) the set

$$\{y \in P\colon\ y \leqslant Ty\}$$

is non-empty. In virtue of Lemma 1.1 there exists a sequence $x = \{x_n\}$ such that $Tx = x$. This sequence satisfies condition (2.16). This completes the proof.

THEOREM 2.3. *Let hypotheses* (H_1)–(H_3) *be fulfilled and assume that the set Q is closed, lower-bounded and Q is a continuous subspace of Y. Let for every $x \in A$ the set Q_x be bounded and let the set A satisfy* (2.14). *If the function $H(x, y)$ is continuous with respect to y and if f is continuous, then equation* (2.1) *has an increasing solution on A.*

Proof. Let $\sigma\colon X \to X^*$ be a similitude and let $A_1 \overset{\mathrm{df}}{=} \sigma(A)$. The function $f_1\colon A_1 \to A_1$ which is defined by the formula

$$f_1(x) = \sigma\big(f[\sigma^{-1}(x)]\big) \tag{2.31}$$

is continuous as a superposition of continuous functions. Moreover, this function has all the properties assured about f by hypothesis (H_2). Similarly, the function

$$H_1(x, y) = H[\sigma^{-1}(x), y] \tag{2.32}$$

has all the properties assured about H by hypotesis (H_3). The set A_1 fulfils condition (2.14).

Let b be an arbitrarily fixed point of A_1. Suppose that each set

$$B_n = f_1^n(A_1) \cap [b, f_1(b)] \tag{2.33}$$

is non-empty. Then in virtue of Lemma 2.1 there exists a sequence $\{x_n\}$ such that $x_n \in A_1$ and

$$f_1(x_n) = x_{n-1}. \tag{2.34}$$

Hence and by (H_2) the sequence $\{x_n\}$ is strictly decreasing.

If there existed $x^* \in A_1$ such that $x^* \leqslant x_n$ for every positive integer n, then by (2.34) we should get the inequality

$$f_1^n(x^*) \leqslant x_0.$$

The sequence $\{f_1^n(x^*)\}$ would have an upper bound in A_1, which is impossible. Thus $\{x_n\}$ has not a lower bound in A_1. Hence

$$A_1 = \bigcup_{n=0}^{\infty} A_1 \cap [x_n, +\infty). \tag{2.35}$$

In every set $A_1 \cap [x_n, +\infty)$ the hypotheses of Theorem 2.1 are fulfilled, so there exists an increasing solution of equation (2.1), where the functions f and H are replaced by f_1 and H_1, respectively. We denote by P_n the family of these solutions. We have

$$\varphi(x) \leqslant \varphi[f_1(x)] = H_1\big(x, \varphi(x)\big),$$

or $\varphi(x) \in Q_x$, since every function $\varphi \in P_n$ is increasing. The family P_n is bounded at every point. Consequently there exists

$$\varphi_n(x) \overset{\mathrm{df}}{=} \sup_{\varphi \in P_n} \varphi(x), \tag{2.36}$$

and φ_n is increasing in $[x_n, +\infty) \cap A_1$. For every $x \in [x_n, +\infty) \cap A_1$ we have $\varphi_n(x) \in Q$, since the set Q is closed. We shall prove that φ_n fulfils equation (2.1) (with f_1 and H_1 instend of f and H) in $[x_n, +\infty) \cap A_1$. Let $x \in A_1 \cap [x_n, \infty)$. Then

$$\varphi_n[f_1(x)] = \sup_{\varphi \in P_n} \varphi[f_1(x)] = \sup_{\varphi \in P_n} H_1\big(x, \varphi_1(x)\big). \tag{2.37}$$

The function $H_1(x, y)$ is increasing with respect to y, therefore for every $\varphi \in P_n$ we have

$$H_1\big(x, \varphi(x)\big) \leqslant H_1\big(x, \varphi_n(x)\big)$$

and

$$\sup_{\varphi \in P_n} H_1\big(x, \varphi(x)\big) \leqslant H_1\big(x, \varphi_n(x)\big). \tag{2.38}$$

On the other hand there exists a sequence of functions $\varphi_k \in P_n$ such that $\varphi_k(x) \to \varphi_n(x)$. Hence

$$H_1\big(x, \varphi_k(x)\big) \leqslant \sup_{\varphi \in P_n} H_1\big(x, \varphi(x)\big)$$

and by the continuity of H_1 with respect to the second variable we obtain

$$H_1\big(x, \varphi_n(x)\big) \leqslant \sup_{\varphi \in P_n} H_1\big(x, \varphi(x)\big). \tag{2.39}$$

By (2.37), (2.38) and (2.39) we get

$$\varphi_n[f_1(x)] = H_1\big(x, \varphi_n(x)\big). \tag{2.40}$$

According to (2.36) and the monotonicity of $\{x_n\}$ we get

$$\varphi_{n+1}(x) \leqslant \varphi_n(x) \quad \text{for} \quad x \in [x_n, +\infty) \cap A_1.$$

The sequence $\{\varphi_n(x)\}$ is decreasing and bounded, thus it is convergent. Let $\psi(x) = \lim_{n\to\infty} \varphi_n(x)$. Since the set Q is closed, $\psi(x) \in Q$ for every $x \in A_1$. The function $\psi(x)$ as the limit of a sequence of increasing functions is increasing. By the continuity of H_1 with respect to the second variable and by (2.40) we have

$$\psi[f_1(x)] = H_1(x, \psi(x)) \quad \text{for} \quad x \in A_1. \tag{2.41}$$

It follows by (2.31), (2.32) and (2.41) that the function

$$\varphi(x) \overset{\text{df}}{=} \psi[\sigma(x)] \tag{2.42}$$

is a solution of equation (2.1).

Now we assume that there exists the least positive integer n_0 such that the set $B_{n_0} = \emptyset$. If there existed a point $x \in f_1^{n_0}(A_1)$ such that $x < b$, then, since the sequence $f_1^k(x)$ is strictly increasing and without an upper bound in A_1, there would exist k_0 such that

$$f_1^{k_0-1}(x) < b \quad \text{and} \quad b \leqslant f_1^{k_0}(x) \leqslant f_1(b).$$

Since

$$f_1^{n_0+k}(A_1) \subset f^{n_0}(A_1),$$

we would have $f_1^k(x) \in B_{n_0}$, a contradiction. Hence in the interval $(-\infty, f_1(b)]$ there is no element of the set $f_1^{n_0}(A_1)$, thus

$$f_1(b) \leqslant \inf f_1^{n_0}(A_1). \tag{2.43}$$

We suppose that

$$f_1(b) < \inf f_1^{n_0}(A_1). \tag{2.44}$$

Since B_{n_0-1} is non-empty, there exists $y \in f_1^{n_0-1}(A_1)$ and $y \leqslant f_1(b)$. Hence and by (2.44) we have

$$\inf f_1^{n_0-1}(A_1) < \inf f_1^{n_0}(A_1). \tag{2.45}$$

If $f_1(b)$ is the least element of $f_1^{n_0}(A_1)$, then in view of the monotonicity of the function f_1, b is the least element of $f_1^{n_0-1}(A_1)$. Also in this case inequality (2.45) is fulfilled.

Let

$$A_2 \overset{\text{df}}{=} A_1 \cup \{\inf A_1\} \cup \{\sup A_1\} \tag{2.46}$$

and

$$f_2(x) = \begin{cases} f_1(x) & \text{for} \quad x \in A_1, \\ \inf f_1(A_1) & \text{for} \quad x = \inf A_1, \\ \sup f_1(A_1) & \text{for} \quad x = \sup A_1. \end{cases} \tag{2.47}$$

The function $f_2\colon A_2 \to A_2$ is increasing in A_2, fulfils the condition $f_2(a_0) \in A_1$. On account of the monotonicity of f_2 we have

$$\inf f_2^{n_0-1}(A_2) = f_2^{n_0-1}(\inf A_2) \tag{2.48}$$

and

$$\inf f_2^{n_0-1}[f_2(A_2)] = f_2^{n_0-1}(\inf f_2(A_2)]. \tag{2.49}$$

By (2.45), (2.48) and (2.49)we get

$$f_2^{n_0-1}(a_0) < f_2^{n_0-1}[f_2(a_0)].$$

Hence

$$a_0 < f_2(a_0).$$

Let

$$H_2(x, y) = \begin{cases} H_1(x, y) & \text{for} \quad (x, y) \in A_1 \times Q, \\ \inf\limits_{x' \in f(A_1)} H_1(x', y) & \text{for} \quad x = a_0,\ y \in Q. \end{cases}$$

Since $H_2\colon A_2 \times Q \to Q$ and $\inf Q \in Q$, we have

$$\inf Q \leqslant H(x, \inf Q) \quad \text{for every } x \in A_2.$$

The functions f_2 and H_2 fulfil in A_2 all the hypotheses Theorem 2.1. Consequently there exists an increasing function ψ such that

$$\psi[f_2(x)] = H_2(x, \psi(x)) \quad \text{for} \quad x \in A_2.$$

Hence ψ fulfils (2.41) in A_1. By (2.31) and (2.32) the function φ defined by (2.42) is an increasing solution of equation (2.1). This finishes the proof.

§ 2. We introduce the following hypothesis

(H_4) $F\colon A \times Y \to Y$ is an increasing function with respect to each variable. Moreover there exists a set $Q \subset Y$ such that for every pair $(x, y) \in A \times Q$ we have

$$F(x, y) \leqslant y \tag{2.50}$$

and there exists a $z \in Q$ such that

$$F(x, z) = y. \tag{2.51}$$

THEOREM 2.4. *If hypotheses* (H_1), (H_2), (H_4) *are fulfilled and there is* $a_0 = \inf A \in A$, *then for every* $y_0 \in Q$ *there exists an increasing solution of equation* (2.2) *such that* $\varphi(a_0) = y_0$.

Proof. We denote by y_1 one of the numbers z fulfilling (2.51) for $y = y_0$, $x = a_0$. It fulfils the inequality

$$y_0 = F(a_0, y_1) \leqslant y_1.$$

Suppose that we know y_n. By y_{n+1} we denote one of the solutions of equation (2.51) for $y = y_n$, $x = a_n = f^n(a_0)$. By (2.50) and (2.51) $y_n = F(a_n, y_{n+1}) \leqslant y_{n+1}$. Thus the sequence $\{y_n\}$ is increasing.

We denote by P the family of all increasing functions in A such that

$$\varphi(a_n) = y_n.$$

This family is ordered by the relation

$$\varphi \leqslant \psi \overset{\mathrm{df}}{\Leftrightarrow} \varphi(x) \leqslant \psi(x) \quad \text{for every } x \in A.$$

The functions

$$\bar{\varphi}(x) = \begin{cases} y_0 & \text{for} \quad x = a_0, \\ y_{n+1} & \text{for} \quad x \in (a_n, a_{n+1}] \cap A, \ n = 0, 1, \ldots, \end{cases}$$

and

$$\underline{\varphi}(x) = y_n \quad \text{for} \quad x \in (a_n, a_{n+1}] \cap A, \ n = 0, 1, \ldots,$$

are the greatest and the least element in the space P. Each subset Z of the family P has the supremum. Namely $\sup Z = \varphi_0$, where $\varphi_0(x) = \sup\limits_{\varphi \in Z} \varphi(x)$. The function φ_0 so defined is an element of the space P. Let $x \in [a_n, a_{n+1})$, $n \geqslant 0$. Then $f(x) \in [a_{n+1}, a_{n+2})$ and

$$F\big(x, \underline{\varphi}[f(x)]\big) \geqslant F\big(x, \underline{\varphi}(a_{n+1})\big) = F(x, y_{n+1}) \geqslant F(a_n, y_{n+1}) = y_n = \underline{\varphi}(x).$$

Hence if

$$(T\varphi)(x) \overset{\mathrm{df}}{=} F\big(x, \varphi[f(x)]\big),$$

then

$$T\underline{\varphi} \geqslant \underline{\varphi}. \tag{2.52}$$

Moreover T is increasing. Since

$$(T\varphi)(a_n) = F\big(a_n, \varphi(a_{n+1})\big) = F(a_n, y_{n+1}) = y_n$$

and $(T\varphi)(x)$ is increasing, T maps P into P. It follows by (2.52) that the set $\{\varphi \in P\colon T\varphi \geqslant \varphi\}$ is non-empty. This set has the supremum in P, so in virtue of Lemma 1.1 there exists a function $\varphi \in P$ such that

$$T\varphi = \varphi.$$

This completes the proof.

THEOREM 2.5. *Suppose that hypotheses* (H_1), (H_2), (H_4) *are fulfilled. If f is continuous and the set A fulfils* (2.14), *then equation* (2.2) *has an increasing solution in A.*

Proof. Similarly as in the proof of Theorem 2.3 the functions f_1 defined by (2.31) and F_1 defined by

$$F_1(x, y) = F\big(\sigma^{-1}(x), y\big) \tag{2.53}$$

have on the set $A_1 = \sigma(A)$ all the properties which are required by hypotheses (H_2) and (H_4) from the functions f and F in A. The function f_1 is continuous, the set A_1 fulfils (2.14).

Let b be an arbitrary point of the set A_1. We suppose that no set B_n (cf. (2.33)) is empty. Then there exists a strictly decreasing sequence $\{x_n\}$, $x_n \in A_1$, which does not possess a lower bound in A_1, and fulfils the condition $x_{n-1} = f_1(x_n)$, $n = 1, 2, \ldots$ Moreover formula (2.35) holds. The hypotheses of Theorem 2.4 are fulfilled in the set $[x_0, +\infty) \cap A_1$, so there is an increasing function $\psi_0(x)$ which fulfils the equation

$$\psi_0(x) = F_1\big(x, \psi_0[f_1(x)]\big)$$

for $x \in [x_0, +\infty) \cap A_1$. We define for $x \in [x_1, x_0]$ the function

$$\varphi_1(x) = F_1\big(x, \psi_0[f_1(x)]\big).$$

The function ψ_1 is increasing as the superposition of increasing functions, moreover

$$\psi_1(x_0) = F_1\big(x_0, \psi_0[f_1(x_0)]\big) = \psi_0(x_0).$$

By induction, for $x \in [x_n, x_{n-1}]$ we define the function

$$\psi_n(x) = F_1\big(x, \psi_{n-1}[f_1(x)]\big). \tag{2.54}$$

The functions $\psi_n(x)$ are increasing and fulfil the conditions

$$\psi_n(x_{n-1}) = \psi_{n-1}(x_{n-1}). \tag{2.55}$$

By (2.54) and (2.55) the function

$$\psi(x) \overset{\mathrm{df}}{=} \begin{cases} \psi_0(x) & \text{for} \quad x \in [x_0, +\infty) \cap A_1, \\ \psi_n(x) & \text{for} \quad x \in [x_n, x_{n-1}) \cap A_1, \quad n = 1, 2, \ldots, \end{cases}$$

is an increasing solution of the equation

$$\psi(x) = F_1\big(x, \psi[f_1(x)]\big) \quad \text{for} \quad x \in A_1. \tag{2.56}$$

Hence and by (2.31) and (2.53) the function

$$\varphi(x) = \psi[\sigma(x)] \tag{2.57}$$

is an increasing solution of equation (2.2) in A.

Now let us suppose that there is the least positive integer n_0 such that $B_{n_0} = \emptyset$.

Let $x \in f_1^{n_0-1}(A_1) \cap [b, f_1(b)]$. There exists a $y \in A_1$ such that $x = f_1^{n_0-1}(y)$. Suppose that there exists a $z \in A_1$, $z < y$ and such that $f_1(z) < y$. Then

$$f_1^{n_0}(z) \leqslant f_1^{n_0-1}(y) = x.$$

Since $x \in [b, f_1(b)]$, there exists a $k \geqslant 0$ such that

$$f_1^{n_0+k}(z) \in [b, f_1(b)]. \tag{2.58}$$

Moreover,

$$f_1^{n_0+k}(z) \in f_1^{n_0+k}(A_1) \subset f_1^{n_0}(A_1). \tag{2.59}$$

It follows by (2.58) and (2.59) that

$$f_1^{n_0+k}(z) \in f_1^{n_0}(A_1) \cap [b, f_1(b)] = B_{n_0},$$

which contradict, the hypothesis that $B_{n_0} = \emptyset$. Thus we have shown that for every $z \in A_1$, $y \leqslant f_1(z)$. In the set $A_1 \cap [y, +\infty)$ all the hypotheses of Theorem 2.4 are fulfilled. Hence there exists an increasing function ψ_0 fulfilling the equation

$$\psi_0(x) = F_1\big(x, \psi_0[f_1(x)]\big) \quad \text{in} \quad A_1 \cap [y, +\infty).$$

The function

$$\psi(x) = \begin{cases} \psi_0(x) & \text{for} \quad x \in A_1 \cap [y, +\infty), \\ F_1\big(x, \psi_0[f_1(x)]\big) & \text{for} \quad x \in A_1 \cap (-\infty, y] \end{cases}$$

is an increasing solution of equation (2.56) in A_1. Hence on account of (2.31), (2.53) the function φ given by (2.57) is an increasing solution of equation (2.2) in A.

CHAPTER III

MONOTONIC SOLUTIONS OF SOME SPECIAL LINEAR FUNCTIONAL EQUATIONS

In this chapter we shall be concerned with the existence and uniqueness of monotonic solutions of the equations

$$\varphi[f(x)] = \varphi(x) + h(x) \tag{3.1}$$

and

$$\varphi[f(x)] = g(x)\varphi(x). \tag{3.2}$$

The functions φ, g, h are real-valued, whereas the variable x lies in a linearly ordered space X.

§ 1. Uniqueness. Let X be a linearly ordered space and let $A \subset X$. Further, suppose that we are given a σ-ideal Ω which does not contain X. In the sequel we shall assume that some of the following hypotheses are fulfilled.

(H_5) $A \subset X$, $A \notin \Omega$;

(H_6) if $S \in \Omega$ and $S \subset f(A)$, then $f^{-1}(S) \in \Omega$;

(H_7) if $S \in \Omega$ and $S \subset A$, then $f(S) \in \Omega$;

(H_8) the function h is defined Ω-a.e. in A;

(H_9) the function g is defined and positive Ω-a.e. in A.

THEOREM 3.1. ([2]) *Suppose that hypotheses* (H_2), (H_5)-(H_8) *are fulfilled, f is strictly monotonic,*

$$f([x_0, f(x_0)] \cap A) = [f(x_0), f^2(x_0)] \cap A \quad \textit{for every} \quad x_0 \in A, \tag{3.3}$$

and

$$\operatorname*{infess}_{x \in A\ (\Omega)} h(x) = 0. \tag{3.4}$$

([2]) The results of the present chapter have mostly been known previously in the particular case, where $X = R$ is the real line, A is an interval and $\Omega = \Omega_0$ is the class of sets of the Lebesgue measure zero or $\Omega = \{\emptyset\}$ consists of the empty set only. Regarding Theorem 3.1 cf. [6] ($f(x) = x+1$, $\Omega = \{\emptyset\}$) and [18] ($\Omega = \{\emptyset\}$).

Then every two monotonic functions, defined Ω-a.e. *in* A *and satisfying equation* (3.1) Ω-a.e. *in* A *differ by a constant* Ω-a.e. *in* A.

Proof. Let φ_1, φ_2 be two monotonic functions satisfying equation (3.1) Ω-a.e. in A. Let $S \in \Omega$, $S \subset A$, be such a set that in $A \setminus S$ the functions φ_1, φ_2, h are defined and φ_1, φ_2 are monotonic and satisfy equation (3.1). Then

$$S_0 \stackrel{\mathrm{df}}{=} \bigcup_{i=0}^{\infty} f^{-i}[S \cap f^i(A)] \cup \bigcup_{i=1}^{\infty} f^i(S) \in \Omega. \tag{3.5}$$

Since the functions φ_1 and φ_2 are monotonic solutions of equation (3.1) in the set $A \setminus S_0$, the function h has a constant sign in this set. By (3.4) and Lemma 1.13 there exists a set $S^* \in \Omega$ such that

$$\inf_{A \setminus S^*} h(x) = 0.$$

Thus we have $0 \leqslant h(x)$ on $A \setminus S^*$, and hence also on $A \setminus S_0 \setminus S^* \subset A \setminus S_0$. Since, by (H_5), $A \setminus S_0 \setminus S^* \neq \emptyset$ and h has a constant sign on $A \setminus S_0$, we have

$$0 \leqslant h(x) \quad \text{for every } x \in A \setminus S_0.$$

In view of (3.4) and of the above inequality we have

$$\inf_{A \setminus S_0} h(x) = 0. \tag{3.6}$$

The set

$$\tilde{A} \stackrel{\mathrm{df}}{=} A \setminus S_0 \tag{3.7}$$

fulfils the condition

$$f(\tilde{A}) \subset \tilde{A}. \tag{3.8}$$

In fact, suppose that $x \in \tilde{A}$ and $f(x) \notin \tilde{A}$. Then

$$f(x) \in \bigcup_{i=0}^{\infty} f^{-i}\big(S \cap f^i(A)\big) \cup \bigcup_{i=1}^{\infty} f^i(S).$$

If $f(x) \in f^{-i}\big(S \cap f^i(A)\big)$ for a certain i, then

$$f^{i+1}(x) \in S \cap f^i(A).$$

Hence, since

$$f^{i+1}(x) \in f^{i+1}(A),$$

we get

$$x \in f^{-1-i}\big(S \cap f^{1+i}(A)\big) \subset S_0,$$

i.e., $x \notin \tilde{A}$. If, on the other hand, $f(x) \in f^i(S)$ for a certain i, then there exists an $x' \in f^{i-1}(S)$ such that $f(x') = f(x)$. Since f is strictly increasing, it follows that $x' = x$. Thus $x \in f^{i-1}(S) \subset S_0$, i.e., $x \notin \tilde{A}$. This completes the proof of (3.8).

Let us fix an arbitrary $x_0 \in \tilde{A}$. By (3.3), (3.7) and (3.8) we obtain

$$f\big([x_0, f(x_0)] \cap \tilde{A}\big) \subset [f(x_0), f^2(x_0)] \cap \tilde{A}. \tag{3.9}$$

In order to prove the converse inclusion let us take a $y \in [f(x_0), f^2(x_0)] \cap \tilde{A}$, whence it follows in particular that $y \in \tilde{A}$. Since $\tilde{A} \subset A$, we have

$$y \in [f(x_0), f^2(x_0)] \cap A = f\big([x_0, f(x_0)] \cap A\big).$$

Consequently there exists an $x \in [x_0, f(x_0)] \cap A$ such that $f(x) = y$. Suppose that $x \notin \tilde{A}$. If there exists a positive integer i such that $x \in f^{-i}\big(f^i(A) \cap S\big)$, then $f^i(x) \in S \cap f^i(A)$ and

$$f^{i-1}(y) = f^i(x) \in f^i(A) \cap S \subset f^{i-1}(A) \cap S.$$

Hence

$$y \in f^{1-i}\big(f^{i-1}(A) \cap S\big) \subset S_0.$$

If, on the other hand, there exists a non-negative integer i such that $x \in f^i(S)$, then

$$y = f(x) \in f^{i+1}(S) \subset S_0.$$

The condition $y \in S_0$ contradicts the fact that $y \in \tilde{A}$. Thus we have shown that

$$\tilde{A} \cap [f(x_0), f^2(x_0)] \subset f\big(\tilde{A} \cap [x_0, f(x_0)]\big). \tag{3.10}$$

By (3.9) and (3.10) we obtain the equality

$$f\big(\tilde{A} \cap [x_0, f(x_0)]\big) = \tilde{A} \cap [f(x_0), f^2(x_0)]. \tag{3.11}$$

The function $\varphi(x) = \varphi_1(x) - \varphi_2(x)$ is a solution of the equation

$$\varphi[f(x)] = \varphi(x). \tag{3.12}$$

By (3.6) the functions φ_1 and φ_2 are increasing and thus they are bounded on every set of the form $[x_0, f(x_0)] \cap \tilde{A}$. Namely, we have $\varphi_1(x_0) \leqslant \varphi_1(x) \leqslant \varphi_1[f(x_0)]$ and $\varphi_2(x_0) \leqslant \varphi_2(x) \leqslant \varphi_2[f(x_0)]$ for $x \in [x_0, f(x_0)] \cap \tilde{A}$. The function φ, as a difference of bounded functions, is itself bounded on $[x_0, f(x_0)] \cap \tilde{A}$. Relation (3.12) implies that φ is bounded on $\tilde{A}$.

Let $K = \sup\limits_{x \in \tilde{A}} \varphi(x)$, $k = \inf\limits_{x \in \tilde{A}} \varphi(x)$ and $K' = \sup\limits_{x \in [x_0, f(x_0)] \cap \tilde{A}} \varphi(x)$, $k' = \inf\limits_{x \in [x_0, f(x_0)] \cap \tilde{A}} \varphi(x)$. Then $K' \leqslant K$. Suppose that $K' < K$. Then there exists an $x^* \in \tilde{A}$ such that $K' < \varphi(x^*) < K$. Further, by (3.11), there exists an integer n such that $f^n(x^*) \in [x_0, f(x_0)] \cap \tilde{A}$ and by (3.12)

$$\varphi[f^n(x^*)] = \varphi(x^*) > K'.$$

The contradiction obtained proves that $K' = K$. Similarly we can prove that $k' = k$.

We shall prove that $\varphi(x) = \text{const}$ in the set $\tilde{A}$, i.e., that $k = K$. Supposing the contrary, let ε be an arbitrary positive number such that

$$0 < \varepsilon < \frac{K-k}{2}. \tag{3.13}$$

Since according to (3.6) and (3.7) $\inf_{\tilde{A}} h(x) = 0$, we may assume that x_0 has been chosen in such a manner that $h(x_0) \leqslant \varepsilon$. The function φ_1 is an increasing solution of equation (3.1), whence

$$\varphi_1(d) - \varphi_1(c) \leqslant \varphi_1[f(x_0)] - \varphi_1(x_0) = h(x_0) \tag{3.14}$$

for all $c, d \in \tilde{A}$ such that $x_0 \leqslant c \leqslant d \leqslant f(x_0)$. Since φ_2 is increasing, we have for $c, d \in \tilde{A}$, $x_0 \leqslant c \leqslant d \leqslant f(x_0)$,

$$\begin{aligned} \varphi(d) - \varphi(c) &\leqslant [\varphi_1(d) - \varphi_2(d)] - [\varphi_1(c) - \varphi_2(c)] + [\varphi_2(d) - \varphi_2(c)] \\ &= \varphi_1(d) - \varphi_1(c). \end{aligned} \tag{3.15}$$

Relations (3.14) and (3.15) imply that

$$\varphi(d) - \varphi(c) \leqslant \varepsilon \quad \text{for} \quad c, d \in \tilde{A},\ x_0 \leqslant c \leqslant d \leqslant f(x_0). \tag{3.16}$$

Setting in (3.16) $d = f(x_0)$ we get by (3.12) the inequality

$$\varphi(x_0) - \varphi(c) \leqslant \varepsilon$$

and

$$\varphi(x_0) - k = \varphi(x_0) - \inf_{[x_0, f(x_0)] \cap \tilde{A}} \varphi(c) = \sup_{[x_0, f(x_0)] \cap \tilde{A}} [\varphi(x_0) - \varphi(c)] \leqslant \varepsilon. \tag{3.17}$$

Similarly, we obtain from (3.16) for $c = x_0$

$$K - \varphi(x_0) = \sup_{[x_0, f(x_0)] \cap \tilde{A}} [\varphi(d) - \varphi(x_0)] \leqslant \varepsilon. \tag{3.18}$$

Relations (3.17) and (3.18) yield

$$\frac{K-k}{2} \leqslant \varepsilon,$$

a contradiction with (3.13). This completes the proof.

In the above theorem we may replace condition (3.4) by $\sup_{x \in A} \operatorname{ess}_{(\Omega)} h(x) = 0$ (it is enough to multiply the equation by -1). It is easily seen that if h is non-negative, then the monotonic solutions of equation (3.1) must be increasing, and if h is non-positive, then the monotonic solutions of (3.1) must be decreasing.

THEOREM 3.2. (3) *Let hypotheses* (H_2), (H_5), (H_6), (H_8) *be fulfilled and*

$$\lim_{n \to \infty} h[f^n(x)] = 0 \tag{3.19}$$

for Ω-almost all $x \in A$. If φ is monotonic Ω-a.e. in A and satisfies equation (3.1) *Ω-a.e. in A, then*

(3) Cf. [12] ($\Omega = \{\emptyset\}$) and [8] ($\Omega = \Omega_0$). [Compare footnote (3).]

$$\varphi(x) = \varphi(x_0) + \sum_{i=0}^{\infty} \{h[f^i(x_0)] - h[f^i(x)]\} \tag{3.20}$$

Ω-a.e. in A, where x_0 is an arbitrarily fixed point of the set $A \setminus S_1$, and S_1 is a certain set from the family Ω.

Proof. Let $S \in \Omega$ be such a set that in $A \setminus S$ the functions φ and h are defined, φ satisfies equation (3.1) and condition (3.19) holds. In virtue of hypothesis (H_6) the set

$$S_1 = \bigcup_{i=0}^{\infty} f^{-i}\big(S \cap f^i(A)\big) \in \Omega. \tag{3.21}$$

One can prove (cf. the proof of Theorem 3.1) that the set

$$A_0 \overset{\mathrm{df}}{=} A \setminus S_1 \tag{3.22}$$

fulfils the condition

$$f(A_0) \subset A_0.$$

Let us fix an $x_0 \in A_0$ and let $x_n \overset{\mathrm{df}}{=} f^n(x_0)$, $n = 0, 1, 2, \ldots$ For every $x \in A_0 \cap [x_0, \infty)$ we denote by $\bar{x} = \bar{x}(x)$ the largest x_n such that $x_n \leqslant x$. The function $\bar{x}(x)$ fulfils the condition $\overline{f(x)} = f(\bar{x})$. Let

$$\psi(x) \overset{\mathrm{df}}{=} \varphi(x) - \varphi(\bar{x}), \qquad h_0(x) \overset{\mathrm{df}}{=} h(x) - h(\bar{x}). \tag{3.23}$$

Then the function ψ satisfies in $A_0 \cap [x_0, \infty)$ the equation

$$\psi[f(x)] = \psi(x) + h_0(x). \tag{3.24}$$

Since φ is a monotonic function, we have

$$|\psi[f^n(x)]| = |\varphi[f^n(x)] - \varphi[f^n(\bar{x})]| \leqslant |\varphi[f^{n+1}(\bar{x})] - \varphi[f^n(\bar{x})]| = |h[f^n(\bar{x})]|.$$

Hence in view of (3.19)

$$\lim_{n\to\infty} \psi[f^n(x)] = 0. \tag{3.25}$$

By (3.24)

$$\psi(x) = \psi[f^n(x)] - \sum_{i=0}^{n-1} h_0[f^i(x)]$$

for $n = 1, 2, \ldots$, and by (3.25)

$$\psi(x) = -\sum_{i=0}^{\infty} h_0[f^i(x)] = \sum_{i=0}^{\infty} \{h[f^i(\bar{x})] - h[f^i(x)]\}.$$

Hence we get according to (3.23)

$$\varphi(x) = \varphi(\bar{x}) + \sum_{i=0}^{\infty} \{h[f^i(\bar{x})] - h[f^i(x)]\} \tag{3.26}$$

for $x \in A_0 \cap [x_0, \infty)$. Let $x \in [x_n, x_{n+1})$; then $\bar{x} = x_n = f^n(x_0)$ and by (3.26) we have

$$\varphi(x) = \varphi[f^n(x_0)] + \sum_{i=0}^{\infty} \{h[f^{i+n}(x_0)] - h[f^i(x)]\}. \tag{3.27}$$

By (3.1)

$$\varphi[f^n(x_0)] = \varphi(x_0) + \sum_{i=0}^{n-1} h[f^i(x_0)]$$

and by (3.27)

$$\begin{aligned} \varphi(x) &= \varphi(x_0) + \sum_{i=0}^{n-1} h[f^i(x_0)] + \lim_{m\to\infty} \sum_{i=0}^{m} \{h[f^{i+n}(x_0)] - h[f^i(x)]\} \\ &= \varphi(x_0) + \lim_{m\to\infty} \sum_{i=0}^{m} \{h[f^i(x_0)] - h[f^i(x)]\} + \lim_{m\to\infty} \sum_{i=m+1}^{m+n} h[f^i(x_0)]. \end{aligned} \tag{3.28}$$

In view of (3.28) and (3.19) we obtain formula (3.20) for $x \in A_0 \cap [x_0, \infty)$. If $x < x_0$, $x \in A_0$, then there exists a positive integer k such that $f^k(x) \in [x_0, x_1) \cap A_0$, whence

$$\varphi[f^k(x)] = \varphi(x_0) + \sum_{i=0}^{\infty} \{h[f^i(x_0)] - h[f^{i+k}(x)]\}. \tag{3.29}$$

We obtain from equation (3.1)

$$\varphi[f^k(x)] = \varphi(x) + \sum_{i=0}^{k-1} h[f^i(x)]. \tag{3.30}$$

By (3.29) and (3.30)

$$\begin{aligned} \varphi(x) &= \varphi(x_0) - \sum_{i=0}^{k-1} h[f^i(x)] + \lim_{m\to\infty} \sum_{i=0}^{m} \{h[f^i(x_0)] - h[f^{i+k}(x)]\} \\ &= \varphi(x_0) + \lim_{m\to\infty} \sum_{i=0}^{m} \{h[f^i(x_0)] - h[f^i(x)]\} - \lim_{m\to\infty} \sum_{i=m+1}^{m+k} h[f^i(x)]. \end{aligned}$$

Hence, according to (3.19), we obtain (3.20), which completes the proof.

THEOREM 3.3. *Suppose that hypotheses* (H_2), (H_5), (H_6), (H_7), (H_9) *and condition* (3.3) *are fufilled, f is strictly increasing and*

$$\inf_{x \in A\ (\Omega)} \operatorname{ess} g(x) = 1. \tag{3.31}$$

Then any two monotonic functions, defined Ω-a.e. in A and satisfying equation (3.2) *Ω-a.e. in A, differ Ω-a.e. in A by a constant factor.*

Proof. Let φ_1 and φ_2 be monotonic solutions of equation (3.2) Ω-a.e. in A and let $\varphi_1 \neq \varphi_2$. Let $S \in \Omega$ be such a set that in $A \setminus S$ the functions φ_1, φ_2, g are defined, g is positive, φ_1, φ_2 are monotonic and satisfy equation (3.2) and let us define the set A_0 by (3.21) and (3.22). If one of the functions

φ_1 and φ_2, say φ_1, vanishes at a point $x_0 \in A_0$, then we have by (3.2) $\varphi_1(x_n) = 0$ (where $x_n = f^n(x_0)$) for $n = 0, 1, 2, \ldots$, and in virtue of the monotonicity of the function φ_1 we get $\varphi_1(x) = 0$ for $x \in [x_0, \infty) \cap A_0$. If $x < x_0$, $x \in A_0$, then $f^n(x) \in [x_0, \infty) \cap A_0$ for a suitable positive integer n. Then it follows by (3.2)

$$\varphi_1[f^n(x)] = \varphi_1(x) g(x) \ldots g[f^{n-1}(x)]$$

and, since $\varphi_1[f^n(x)] = 0$ and $g(x), \ldots, g[f^{n-1}(x)] \neq 0$, we obtain $\varphi_1(x) = 0$. Hence $\varphi_1(x) \equiv 0$ in A_0 and $\varphi_1(x) = 0 \cdot \varphi_2(x)$ in A_0.

Now let us assume that $\varphi_1(x) \neq 0$, $\varphi_2(x) \neq 0$ for $x \in A_0$. These functions cannot change the sign, for if for instance φ_1 had a change of sign, then, according to equation (3.2), φ_1 would change the sign infinitely many times, which contradicts the monotonicity of φ_1. Let

$$\psi_1(x) = \ln|\varphi_1(x)|, \qquad \psi_2(x) = \ln|\varphi_2(x)|.$$

These functions are monotonic solutions of the equation

$$\psi[f(x)] = \psi(x) + \ln g(x)$$

in A_0, i.e. Ω-a.e. in A. Condition (3.31) implies that

$$\inf_{x \in A\ (\Omega)} \operatorname{ess} \ln g(x) = 0.$$

By Theorem 3.1 $\psi_1(x) - \psi_2(x) = \text{const}$ Ω-a.e. in A, whence

$$\left| \frac{\varphi_1(x)}{\varphi_2(x)} \right| = \text{const } \Omega\text{-a.e. in } A.$$

But φ_1 and φ_2 have constant signs and thus also

$$\frac{\varphi_1(x)}{\varphi_2(x)} = \text{const } \Omega\text{-a.e. in } A,$$

which completes the proof.

In the above theorem we may replace condition (3.31) by

$$\sup_{x \in A\ (\Omega)} \operatorname{ess} g(x) = 1.$$

THEOREM 3.4 (4). *Let hypotheses* (H_2), (H_5), (H_6), (H_9) *be fulfilled and*

$$\lim_{n \to \infty} g[f^n(x)] = 1 \tag{3.32}$$

for Ω-almost all $x \in A$. If φ is monotonic Ω-a.e. in A and satisfies equation

(4) Cf. [2] ($\Omega = \{\emptyset\}$). [Compare footnote (2)].

(3.2) Ω-a.e. *in* A, *then*

$$\varphi(x) = \varphi(x_0)\gamma(x),$$

Ω-a.e. *in* A, *where*

$$\gamma(x) = \prod_{i=0}^{\infty} \frac{g[f^i(x_0)]}{g[f^i(x)]}, \tag{3.33}$$

x_0 *is an arbitrarily fixed point of the set* $A \setminus S_1$, *and* S_1 *is a certain set from the family* Ω.

The proof, based on Theorem 3.2, is similar to that of Theorem 3.3.

§ 2. Existence.

THEOREM 3.5.(5) *Let hypotheses* (H_2), (H_5), (H_6), (H_8) *be fulfilled, let* $h \in S_f(A \cap \mathrm{Domain}\, h)$ *be non-negative, and increasing* Ω-a.e. *in* A. *If* $a_0 = \inf A \in A$, *then equation* (3.1) *has an increasing solution* Ω-a.e. *in* A.

Proof. Let $S \in \Omega$ be such a set that in $A \setminus S$ the function h is defined, non-negative and increasing. The set S_1 defined by (3.21) belongs to Ω, and the set A_0 defined by (3.22) fulfils the condition $f(A_0) \subset A_0$. Let $a_n = f^n(a_0)$, $n = 0, 1, 2, \ldots$, and $A_1 \overset{\mathrm{df}}{=} A_0 \cup \{a_n\}$. Then $f(A_1) = f(A_0) \cup \cup f(\{a_n\}) \subset A_0 \cup \{a_n\} = A_1$. The function

$$\bar{h}(x) \overset{\mathrm{df}}{=} \begin{cases} h(x) & \text{for those } x \in A_1 \text{ for which } h(x) \text{ is defined,} \\ \inf\limits_{a_n < x,\ x' \in A_0} h(x') & \text{for } x = a_n \text{ whenever } h(a_n) \text{ is not defined,} \end{cases}$$

is increasing in A_1 and belongs to $S_f(A_1)$. The function $H(x, y) = y + \bar{h}(x)$ is increasing with respect to either variable. Since $h(x) \geqslant 0$, we have $y \leqslant y + \bar{h}(x) = H(x, y)$ for $(x, y) \in A_1 \times R$. Since all the hypotheses of Theorem 2.1 are fulfilled (with A_1 instead of A, and $Q = R$), there exists an increasing solution of the equation

$$\varphi[f(x)] = \varphi(x) + \bar{h}(x)$$

for $x \in A_1$. Consequently φ satisfies equation (3.1) in A_0, i.e., since $A \setminus A_0 \in \Omega$, φ satisfies (3.1) Ω-a.e. in A.

In the sequel the following definition will be useful.

DEFINITION. Let X be a linearly ordered space, $A \subset X$, $A \notin \Omega$, and let Q be a subset of the space of the real numbers. Further let $f\colon A \to A$, $H\colon A \times Q \to Q$ and $x < f(x)$ for $x \in A$. We say that equation (2.1) has Ω-a.e. in A a *solution depending on an arbitrary function* iff there exists a set $\tilde{A} \subset A$ such that $A \setminus \tilde{A} \in \Omega$ and equation (2.1) has in $\tilde{A}$ a solution depending on an arbitrary function.

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(5) Cf. [2] ($\Omega = \{\emptyset\}$). [Compare footnote (2).]

THEOREM 3.6.(6) *Let X be a linearly ordered continuous space, $A = \{x \in X\colon a_0 \leqslant x\}$, let hypotheses* ($H_2$), ($H_5$)–($H_8$) *be fulfilled and let f be continuous. Further we assume that h is non-negative and increasing* Ω-a.e. *in A, $h \in S_f(\mathrm{Domain}\, h)$, and for every positive integer n there exists $h[f^n(a_0)]$.*

If $h(a_0) > 0$, then equation (3.1) *has* Ω-a.e. *in A an increasing solution depending on an abitrary function.*

If $h(a_0) = 0$, then equation (3.1) *has* Ω-a.e. *in A a unique increasing solution up to an additive constant.*

Proof. Since every interval $[a_n, a_{n+1}]$, where $a_n = f^n(a_0)$, $n = 0, 1, 2, \ldots$, is a connected set, and since a continuous function on a connected set has the Darboux property, we have

$$f([a_n, a_{n+1}]) = [a_{n+1}, a_{n+2}].$$

Let $S \in \Omega$, $S \subset A$, be such a set that in $A \setminus S$ the function h is defined, non-negative and increasing. The set S_0 defined by (3.5) belongs to Ω and if we put $\tilde{A} = A \setminus S_0$, then

$$f(\tilde{A}) \subset \tilde{A} \tag{3.34}$$

and

$$f([a_n, a_{n+1}] \cap \tilde{A}) = [a_{n+1}, a_{n+2}] \cap \tilde{A}. \tag{3.35}$$

The proofs of relations (3.34) and (3.35) are analogous to those of relations (3.8) and (3.11) in the proof of Theorem 3.1. It follows from (3.35) that every function φ_0, defined and increasing on the set $[a_0, a_1] \cap \tilde{A}$ and such that

$$\varphi_0(a_1) = \varphi_0(a_0) + h(a_0),$$

can be uniquely extended to an increasing solution φ of equation (3.1) on the set $\tilde{A}$. This extension is furnished by the formula

$$\varphi(x) = \varphi_0[f^{-n}(x)] + \sum_{i=0}^{n-1} h[f^{i-n}(x)] \quad \text{for} \quad x \in [a_n, a_{n+1}] \cap \tilde{A},\ n = 0, 1, 2, \ldots$$

Now, if $h(a_0) > 0$, then the family Φ (cf. p. 19) depends on an infinite number of parameters. If, on the other hand, $h(a_0) = 0$, then the function φ_0 must be constant.

Let us note that in the analogous theorem in [2] the authors have overlooked the fact that in the case $h(a_0) = 0$ equation (3.1) has only a one-parameter family of monotonic solutions.

THEOREM 3.7.(7) *Let hypotheses* (H_2), (H_5), (H_6), (H_8) *be fulfilled, let h be non-negative and decreasing* Ω-a.e. *in A and suppose that there*

(6) Cf. [2] ($\Omega = \{\emptyset\}$). [Compare footnote (2).]
(7) Cf. [2] ($\Omega = \{\emptyset\}$). [Compare footnote (2).]

exists $a_0 = \inf A \in A$. *Then for every real number* y_0 *there exists an increasing solution* φ *of equation* (3.1) Ω-a.e. *in* A *such that* $\varphi(a_0) = y_0$.

Proof. Let $S \in \Omega$ be such a set that in $A \setminus S$ the function h is defined, non-negative and decreasing. We define S_1 and A_0 by (3.21) and (3.22), respectively, and we put

$$A_1 = A_0 \cup \{a_n\}, \qquad a_n = f^n(a_0) \tag{3.36}$$

and

$$\underline{h}(x) = \begin{cases} h(x) & \text{for those } x \in A_1 \text{ for which } h(x) \text{ is defined,} \\ \inf\limits_{x' < a_n,\, x' \in A_0} h(x') & \text{for } x = a_n \text{ whenever } h(a_n) \text{ is not defined.} \end{cases} \tag{3.37}$$

The function $\underline{h}$ is decreasing in A_1. The function

$$F(x, y) = y - \underline{h}(x)$$

is increasing with respect to either variable and

$$F(x, y) \leqslant y$$

for every $(x, y) \in A_1 \times R$, since $\underline{h}(x) \geqslant 0$. If $(x, y) \in A_1 \times R$, then there exists a $z = y + \underline{h}(x)$ fulfilling (2.51). Since $f(A_0) \subset A_0$ and $f(a_n) = a_{n+1}$, we have

$$f(A_1) = f(A_0 \cup \{a_n\}) = f(A_0) \cup f(\{a_n\}) \subset A_0 \cup \{a_n\} = A_1.$$

Consequently in the set A_1 all conditions of Theorem 2.4 are fulfilled. Thus, for every y_0, there exists an increasing function defined and satisfying the equation

$$\varphi[f(x)] = \varphi(x) + \underline{h}(x)$$

in the set A_1 and fulfilling condition $\varphi(a_0) = y_0$. The function φ satisfies equation (3.1) in the set A_0 and hence, since $A \setminus A_0 \in \Omega$, φ is an increasing solution of equation (3.1) Ω-a.e. in A.

THEOREM 3.8.(8) *Let hypotheses* (H_2), (H_5), (H_6), (H_8) *be fulfilled, let* h *be non-negative and decreasing* Ω-a.e. *in* A, *let* f *be a continuous function in* A, *and let* A *fulfil condition* (2.14). *Then there exists a function* φ *which is defined and increasing* Ω-a.e. *in* A *and satisfies equation* (3.1) Ω-a.e. *in* A.

Proof. Let $S \in \Omega$ be such a set that in $A \setminus S$ the function h is defined, non-negative and decreasing. Let S_1 be defined by (3.21) and A_0 by (3.22). We have $f(A_0) \subset A_0$.

If $\inf A_0 \in A_0$, then by Theorem 3.7 there exists an increasing solution of equation (3.1) in A_0, i.e., Ω-a.e. in A.

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(8) Cf. [2] ($\Omega = \{\emptyset\}$). [Compare footnote (8).]

Now let us suppose that

$$\inf A_0 \notin A_0. \tag{3.38}$$

Since, for an arbitrary $a_0 \in A_0$, the sequence $f^n(a_0)$ has no upper bound in A, we have also

$$\sup A_0 \notin A_0. \tag{3.39}$$

Let

$$A_2 \overset{\mathrm{df}}{=} \bar{A}_0 \setminus \{\inf A_0, \sup A_0\}.$$

It follows by (3.38) and (3.39) that

$$\sup A_2 = \sup A_0, \quad \inf A_2 = \inf A_0.$$

Hence

$$\begin{aligned} \bar{A}_2 \setminus \{\inf A_2, \sup A_2\} &= \overline{\bar{A}_0 \setminus \{\inf A_0, \sup A_0\}} \setminus \{\inf A_0, \sup A_0\} \\ &= \bar{A}_0 \setminus \{\inf A_0, \sup A_0\} = A_2. \end{aligned}$$

Consequently the set A_2 fulfils condition (2.14). For $x \in A_2 \subset \bar{A}_0$ we define a function h_2 as follows:

$$h_2(x) = \begin{cases} \inf\limits_{x' \leqslant x, x' \in A_0} h(x') & \text{if } x \text{ is the supremum of a subset of } A_0, \\ \sup\limits_{x' \leqslant x, x' \in A_0} h(x') & \text{if } x \text{ is not the supremum of any subset of } A_0, \\ & \text{but it is the infimum of a subset of } A_0. \end{cases}$$

The function h_2 is non-negative and decreasing, moreover $h_2(x) = h(x)$ for $x \in A_0$. Hence the set A_2, the function f and the function $F(x, y) = y - h_2(x)$ fulfil the hypotheses of Theorem 2.5 and consequently there exists an increasing function φ satisfying the equation

$$\varphi[f(x)] = \varphi(x) + h_2(x)$$

for $x \in A_2$. The function φ is an increasing solution of equation (3.1) in A_0. This completes the proof.

THEOREM 3.9. (9) *Let hypotheses* (H_2), (H_5), (H_6), (H_8) *be fulfilled and let h be decreasing Ω-a.e. in A and fulfil* (3.19) *for Ω-almost all $x \in A$. Then every function*

$$\varphi(x) = \eta_0 + \sum_{n=0}^{\infty} \{h[f^i(x_0)] - h[f^i(x)]\} \tag{3.40}$$

is defined and increasing Ω-a.e. in A and satisfies equation (3.1) *Ω-a.e. in A. Here x_0 is an arbitrarily fixed point of the set $A \setminus S_1$ and S_1 is a certain set from the family Ω.*

Proof. Let $S \in \Omega$ be such a set that in $A \setminus S$ the function h is defined and decreasing and fulfils condition (3.19). We define S_1 by (3.21) and

(9) Cf. [12] ($\Omega = \{\emptyset\}$) and [8] ($\Omega = \Omega_0$). [Compare footnote (2).]

A_0 by (3.22); then $f(A_0) \subset A_0$. Let us fix an arbitrary point $x_0 \in A_0$. We take $x \in A_0$ and write $y = \min(x, x_0)$. Since the sequence $f^n(y)$ has no upper bound in A, there exists a positive integer N such that $x_0, x \in [y, f^N(y)]$. Hence

$$|h[f^i(x)] - h[f^i(x_0)]| \leqslant h[f^i(y)] - h(f^{i+N}(y)]. \tag{3.41}$$

Since

$$\sum_{i=0}^{\infty} \{h[f^i(y)] - h[f^{i+N}(y)]\} = \lim_{m\to\infty} \sum_{i=0}^{m} \{h[f^i(y)] - h[f^{i+N}(y)]\}$$

$$= \lim_{m\to\infty} \Big(\sum_{i=0}^{N-1} h[f^i(y)] - \sum_{i=m+1}^{m+N} h[f^i(y)]\Big),$$

the series

$$\sum_{i=0}^{\infty} \{h[f^i(y)] - h[f^{i+N}(y)]\}$$

converges in view of (3.19). Hence and by (3.41) results the convergence of the series

$$\sum_{i=0}^{\infty} \{h[f^i(x_0)] - h[f^i(x)]\}.$$

Thus the function $\varphi(x)$ is defined by (3.40) in A_0 and by (3.19) satisfies equation (3.1) in A_0. Since the function h is decreasing, the function $h[f^i(x_0)] - h[f^i(x)]$ is increasing for every i, and consequently also the function $\varphi(x)$ in increasing in A_0. This completes the proof.

If in Theorem 3.9 we replace the assumption "h is decreasing" by "h is increasing", then formula (3.40) will represent a decreasing solution of equation (3.1) Ω-a.e. in A.

Similarly one can prove the following

THEOREM 3.9'. *Let hypothesis* (H_5) *be fulfilled, and let us assume that* $f\colon A \to A$ *is increasing, fulfils the inequality* $f(x) < x$ *for every* $x \in A$ *and that the sequence* $\{f^n(x)\}$ *has no lower bound in* A *for any* $x \in A$; *further, assume that* h *is defined, non-positive and decreasing* Ω-a.e. *in* A *and fulfils* (3.19) *for* Ω-*almost all* $x \in A$. *If hypothesis* (H_6) *is fulfilled, then every function* φ *given by* (3.40) *is defined and satisfies equation* (3.1) Ω-a.e. *in* A.

Next we have the following

THEOREM 3.10.(10) *Let hypotheses* (H_2), (H_5), (H_6), (H_9) *be fulfilled and let* g *be increasing* Ω-a.e. *in* A, $g(x) \geqslant 1$ Ω-a.e. *in* A, $g \in S_f$(Domain g).

(10) Cf. [2] ($\Omega = \{\emptyset\}$). [Compare footnote (3).]

If $a_0 \inf A \in A$, *then equation* (3.2) *has an increasing solution* Ω-a.e. *in* A.

Proof. Let

$$h(x) = \ln g(x).$$

The function $h(x)$ fulfils hypothesis (H_8). By Theorem 3.5 there exists an increasing solution ψ of the equation

$$\psi[f(x)] = \psi(x) + h(x)$$

Ω-a.e. in A. The function

$$\varphi(x) = \exp\psi(x)$$

is an increasing solution of equation (3.2) Ω-a.e. in A.

Similarly, making use of Theorems 3.6-3.9, we obtain the following results regarding the existence of increasing solutions of equation (3.2)([11]).

THEOREM 3.11. *Let* X *be a linearly ordered continuous space,* $A = \{x \in X: a_0 \leqslant x\}$, *let hypotheses* (H_2), (H_5), (H_6), (H_7), (H_9) *be fulifilled and let* f *be continuous. Further we assume that* $g(x) \geqslant 1$ Ω-a.e. *in* A, g *is increasing* Ω-a.e. *in* A, $g \in S_f(\mathrm{Domain}\, g)$, *and for every positive integer* n *there exists* $g[f^n(a_0)]$.

If $g(a_0) > 1$, *then equation* (3.2) *has* Ω-a.e. *in* A *an increasing solution depending on an arbitrary function.*

If $g(a_0) = 1$, *then equation* (3.2) *has* Ω-a.e. *in* A *a unique increasing solution up to a multiplicative constant.*

THEOREM 3.12. *Let hypotheses* (H_2), (H_5), (H_6), (H_9) *be fulfilled, let* g *be decreasing* Ω-a.e. *in* A, $g(x) \geqslant 1$ Ω-a.e. *in* A, *and suppose that there exists* $a_0 = \inf A \in A$. *Then for every positive number* y_0 *there exists an increasing solution of equation* (3.2) Ω-a.e. *in* A *such that* $\varphi(a_0) = y_0$.

THEOREM 3.13. *Let hypotheses* (H_2), (H_5), (H_6), (H_9) *be fulfilled, let* g *be decreasing* Ω-a.e. *in* A, $g(x) \geqslant 1$ Ω-a.e. *in* A, *let* f *be a continuous function in* A, *and let* A *fulfil condition* (2.14). *Then there exists a function* φ *which is defined and increasing* Ω-a.e. *in* A *and satisfies equation* (3.2) Ω-a.e. *in* A.

THEOREM 3.14. *Let hypotheses* (H_2), (H_5), (H_6), (H_9) *be fulfilled and let* g *be monotonic* Ω-a.e. *in* A *and fulfil* (3.32) *for* Ω-*almost all* $x \in A$. *Then every function*

$$\varphi(x) = \eta\gamma(x),$$

where $\gamma(x)$ *is given by* (3.33) *and* η *is an arbitrary constant, is defined and monotonic* Ω-a.e. *in* A *and satisfies equation* (3.2) Ω-a.e. *in* A.

In the next theorem we shall have to make the following assumption.

(H_{10}) F is a family of functions from A into A such that for every

([11]) Cf. [2] ($\Omega = \{\emptyset\}$). [Compare footnote ([8]).]

$x, y \in A$, $x < y$, there exists a function $f \in F$ strictly increasing, fulfilling hypothesis (H_2) and the condition

$$f(x) = y.$$

THEOREM 3.15 (Cf. [18]). *Let h be defined in A and fulfil the condition*

$$\inf_{x \in A} h(x) = 0, \tag{3.42}$$

and let F be a family fulfilling hypothesis (H_{10}). *Then equation* (3.1) *has an increasing solution in A for every function $f \in F$ if and only if there exists a section $c = (C, C')$ of the set A such that h is decreasing in C and increasing in C'. One of the sets C, C' may be empty.*

Proof. I. Necessity. Let σ be a similarity of the space X into a subspace of the space X^*, which exists in virtue of Lemma 1.6. The functions of the family

$$F_1 = \{f_1 \colon f_1 = \sigma \circ f \circ \sigma^{-1},\ f \in F\}$$

fulfil hypothesis (H_{10}) in the set $A_1 = \sigma(A)$. If an increasing function φ satisfies equation (3.1) in A, then the function $\psi(x) = \varphi[\sigma^{-1}(x)]$ is an increasing solution of the equation

$$\psi[f_1(x)] = \psi(x) + h_1(x), \tag{3.43}$$

where $h_1(x) = h[\sigma^{-1}(x)]$. Let $x, y, z \in A_1$ and $x < y < z$. Then, according to (H_{10}), there exists a function $f_1 \in F_1$ such that

$$f_1(x) = z. \tag{3.44}$$

Since the function ψ is increasing and satisfies (3.43), we have

$$\psi[f_1(y)] \leqslant \psi[f_1(z)] = \psi(x) + h_1(x) + h_1[f_1(x)]$$

and

$$\psi(x) \leqslant \psi(y).$$

Hence by (3.43) and (3.44)

$$h_1(y) = \psi[f_1(y)] - \psi(y) \leqslant h_1(x) + h_1[f_1(x)] = h_1(x) + h_1(z),$$

i.e.,

$$h_1(y) \leqslant h_1(x) + h_1(z) \tag{3.45}$$

for every $x, y, z \in A_1$ such that $x < y < z$.

According to (3.42) the function $h_1(x)$ fulfils the condition

$$\inf_{x \in A_1} h_1(x) = 0.$$

By Lemma 1.14 there exists a point $x_0 \in \bar{A}_1$ in which the infimum is

attained. Hence by Lemma 1.8 there exists a monotonic sequence $\{x_n\}$, $x_n \in A_1$, $n = 1, 2, \ldots$, such that

$$\lim_{n\to\infty} x_n = x_0 \quad \text{and} \quad \lim_{n\to\infty} h_1(x_n) = 0.$$

Let $x_0 \neq \inf A_1$ and let us take arbitrary $x, y \in A_1 \cap (-\infty, x_0)$. Then there exists a number a such that $x_n \in (y, +\infty)$ for $n > a$. Consequently for $n > a$ the inequality

$$x < y < x_n$$

holds and by (3.45)

$$h_1(y) \leqslant h_1(x) + h_1(x_n).$$

Passing to the limit we get hence

$$h_1(y) \leqslant h_1(x) \quad \text{for} \quad x, y \in A_1,\ x < y < x_0. \tag{3.46}$$

Now let $x_0 \neq \sup A_1$ and let us take arbitrary $x, y \in A_1$, $x_0 < x < y$. Then there exists a number β such that $x_n \in (-\infty, x)$ for $n > \beta$. Consequently for $n > \beta$ we have

$$x_n < x < y$$

and by (3.45)

$$h_1(x) < h_1(y) + h_1(x_n).$$

Passing to the limit we get hence

$$h_1(x) \leqslant h_1(y) \quad \text{for} \quad x, y \in A_1,\ x_0 < x < y. \tag{3.47}$$

It follows from (3.46) and (3.47) that if $x_0 = \inf A_1$, then h_1 increases in A_1, and if $x_0 = \sup A_1$, then h_1 decreases in A_1 ([12]). If $\inf A_1 < x_0 < \sup A_1$, then the function h_1 decreases on the left side of the point x_0 and increases on the right side of the point x_0. In this last case we still have to investigate the behaviour of h_1 at the point x_0.

So let $x_0 \in A_1$, $\inf A_1 < x_0 < \sup A_1$, and let us suppose that

$$\inf_{x<x_0,\, x\in A_1} h_1(x) < h_1(x_0) \quad \text{and} \quad \inf_{x_0<x,\, x\in A_1} h_1(x) < h_1(x_0), \tag{3.48}$$

and that the sequence $\{x_n\}$ is increasing. Given an $\varepsilon > 0$, there exists an n such that

$$h_1(x_n) < \varepsilon. \tag{3.49}$$

By (3.48) there exists an $x' \in A_1$, $x_0 < x'$, such that

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([12]) More exactly, h_1 is monotonic in $A_1 \setminus \{x_0\}$. But, if $x_0 = \sup A_1$, then in view of (H_2) $x_0 \notin A_1$, and if $x_0 = \inf A_1$ and $x_0 \in A_1$, then h_1 may be regarded as decreasing in $\{x_0\}$ and increasing in $A_1 \setminus \{x_0\}$, and the assertion follows.

$$h_1(x') < \inf_{x_0<x,\, x\in A_1} h_1(x)+\varepsilon. \tag{3.50}$$

By (3.45), (3.49) and (3.50) we get

$$h_1(x_0) \leqslant h_1(x_n)+h_1(x') < \inf_{x_0<x,\, x\in A_1} h_1(x)+2\varepsilon,$$

i.e.,

$$h_1(x_0) < \inf_{x_0<x,\, x\in A_1} h_1(x)+2\varepsilon.$$

Letting ε tend to zero we get hence

$$h_1(x_0) \leqslant \inf_{x_0<x,\, x\in A_1} h_1(x).$$

This is a contradiction with (3.48).

In the case of a decreasing sequence $\{x_n\}$ the argument is similar.

Supposition (3.48) has led us to a contradiction, thus we must have

$$h_1(x_0) \leqslant \inf_{x<x_0,\, x\in A_1} h_1(x) \quad \text{or} \quad \inf_{x_0<x,\, x\in A_1} h_1(x) \leqslant h_1(x_0). \tag{3.51}$$

Now, if $\inf A_1 < x_0 < \sup A_1$, $x_0 \notin A_1$, then in the set $C_1 = (-\infty, x_0) \cap A_1$ the function h_1 decreases, and in $C_1' = (x_0, +\infty) \cap A_1$ it increases. The pair (C_1, C_1') is a section of the set A_1. If $\inf A_1 < x_0 < \sup A_1$, $x_0 \in A_1$, and $h_1(x_0) \leqslant \inf_{x<x_0,\, x\in A_1} h_1(x)$ (cf. (3.51)), then the function h_1 is decreasing in $C_1 = (-\infty, x_0)] \cap A_1$ and increasing in $C_1' = (x_0, +\infty) \cap A_1$. Lastly, if $\inf A_1 < x_0 < \sup A_1$, $x_0 \in A_1$, and $\inf_{x_0<x,\, x\in A_1} h_1(x) \leqslant h_1(x_0)$, then the function h_1 is decreasing in $C_1 = (-\infty, x_0) \cap A_1$ and increasing in $C_1' = [x_0, +\infty) \cap A_1$.

As pointed out previously, if $x_0 = \inf A_1$, $x_0 \in A_1$, we assume $C_1 = \{x_0\}$, $C_1' = (x_0, +\infty) \cap A_1$. If $x_0 = \inf A_1$, $x_0 \in A_1$, we assume $C_1 = \emptyset$, $C_1' = A_1$. And if $x_0 = \sup A_1$, then $x_0 \notin A_1$ and we assume $C_1 = A_1$, $C_1' = \emptyset$.

The couple of sets $C = \sigma^{-1}(C_1)$, $C' = \sigma^{-1}(C_1')$ is a section of the set A. In the set C the function $h(x) = h_1[\sigma(x)]$ is decreasing, and in the set C' it is increasing.

II. Sufficiency. Let us suppose that the function h is decreasing. Then by (3.42)

$$\lim_{n\to\infty} h[f^n(x)] = 0$$

for all $x \in A$ and according to Theorem 3.9 there exists an increasing solution of equation (3.1) in A.

Now suppose that the function h is increasing. Then the function $h_1(x) = h[\sigma^{-1}(x)]$ is defined and increasing in $A_1 = \sigma(A)$, moreover $\inf_{x\in A_1} h_1(x) = 0$. The function $f_1(x) \overset{\mathrm{df}}{=} \sigma(f[\sigma^{-1}(x)])$ fulfils hypothesis (H_2) in A_1.

First we assume that $\inf A_1 < \inf f_1(A_1)$. Let $a_n = \inf f_1^n(A_1)$. Since f_1 is strictly increasing, we have $f_1(a_n) = a_{n+1}$ whenever $f_1(a_n)$ exists. We put

$$A_2 = A_1 \cup \{a_n\}$$

and

$$f_2(x) \overset{\mathrm{df}}{=} \begin{cases} f_1(x) & \text{for} \quad x \in A_1, \\ a_{n+1} & \text{for} \quad x = a_n. \end{cases}$$

The set A_2 fulfils the inclusion

$$f_2(A_2) \subset A_2.$$

Let

$$h_2(x) = \inf_{x \leqslant x', x' \in A_1} h_1(x').$$

The assumptions of Theorem 3.5 are fulfilled with the functions f_2 and $h_2(\Omega = \{\emptyset\})$. Thus there exists a function ψ increasing in A_1 and satisfying the equation

$$\psi[f_2(x)] = \psi(x) + h_2(x)$$

for $x \in A_2$. Hence it follows that ψ is an increasing solution of the equation

$$\psi[f_1(x)] = \psi(x) + h_1(x) \tag{3.52}$$

for $x \in A_1$, and the function $\varphi(x) \overset{\mathrm{df}}{=} \psi[\sigma(x)]$ is an increasing solution of equation (3.1) in A.

Now we assume that h is increasing and $\inf A_1 = \inf f_1(A_1)$; then $\inf A_1 \notin A_1$. The functions $f_3(x) = f^{-1}(x)$ and $h_3(x) = -h_1[f_3(x)]$ fulfil the assumptions of Theorem 3.9'$(\Omega = \{\emptyset\})$. Hence it follows that the equation

$$\psi[f_3(x)] = \psi(x) + h_3(x)$$

has an increasing solution in $f_1(A_1)$. Thus the function ψ satisfies equation (3.52) in A_1 and the function $\varphi(x) \overset{\mathrm{df}}{=} \psi[\sigma(x)]$ is an increasing solution of equation (3.1) in A.

Now let us suppose that there exists such a proper section (C, C') of the set A that h is decreasing in C and increasing in C'. We write

$$\bar{h}(x) = \begin{cases} h(x) & \text{for} \quad x \in C, \\ 0 & \text{for} \quad x \in C', \end{cases}$$

$$\underline{h}(x) = \begin{cases} 0 & \text{for} \quad x \in C, \\ h(x) & \text{for} \quad x \in C'. \end{cases} \tag{3.53}$$

The function $\bar{h}(x)$ is decreasing in A, and the function $\underline{h}(x)$ is increasing in A and both fulfil condition (3.42). By what has been shown above,

there exist functions $\bar{\varphi}(x)$ and $\underline{\varphi}(x)$ which are increasing in A and satisfy the equations

$$\bar{\varphi}[f(x)] = \bar{\varphi}(x)+\bar{h}(x), \qquad \underline{\varphi}[f(x)] = \underline{\varphi}(x)+\underline{h}(x),$$

respectively. Hence it follows that the function

$$\varphi(x) \overset{\mathrm{df}}{=} \bar{\varphi}(x)+\underline{\varphi}(x)$$

is an increasing solution of equation (3.1) in A. This completes the proof.

In the proof of the sufficiency it would be enough to assume that h is defined almost everywhere in A.

CHAPTER IV

MONOTONIC SOLUTIONS OF THE LINEAR FUNCTIONAL EQUATION

The present chapter is devoted to the investigation of the existence and uniqueness of monotonic solutions of the equation

$$\varphi[f(x)] = g(x)\varphi(x)+h(x). \tag{4.1}$$

§ 1. Uniqueness.

THEOREM 4.1. *If hypotheses* (H_2), (H_5), (H_6), (H_9) *are fulfilled, h is defined Ω-a.e. in A, g fulfils condition* (3.32), *and equation* (4.1) *has an increasing solution φ_1 and a decreasing solution φ_2 Ω-a.e. in A, then there exists a number η such that*

$$\varphi_1(x)-\varphi_2(x) = \eta\gamma(x) \tag{4.2}$$

Ω-a.e. in A, where $\gamma(x)$ is defined by (3.33).

Proof. The function $\varphi(x) = \varphi_1(x)-\varphi_2(x)$ is an increasing solution of equation (3.2) and according to Theorem 3.4 $\varphi(x) = \eta\gamma(x)$, where $\eta = \varphi(x_0)$.

THEOREM 4.2. *Suppose that hypotheses* (H_2), (H_5)–(H_9) *are fulfilled, f is strictly increasing, g is monotonic and fulfils condition* (3.32), *moreover* (3.3) *holds and*

$$\operatorname*{infess}_{x\in A\ (\Omega)} \frac{h(x)}{\gamma[f(x)]} = 0.$$

Then any two non-negative functions which are equally monotonic [13] *as g and satisfy equation* (4.1) *Ω-a.e. in A, fulfil* (4.2) *Ω-a.e. in A.*

Proof. The function $\gamma(x)$ exists by Theorem 3.14. It is positive and conversely monotonic as g. The functions

$$\psi_1(x) = \frac{\varphi_1(x)}{\gamma(x)}, \qquad \psi_2(x) = \frac{\varphi_2(x)}{\gamma(x)} \tag{4.3}$$

are defined, non-negative, and monotonic Ω-a.e. in A. Moreover, the functions ψ_1 and ψ_2 satisfy the equation

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[13] Two functions are equally monotonic if they are both increasing or both decreasing; and conversely monotonic if one is increasing and the other decreasing.

$$\psi[f(x)] = \psi(x) + \frac{h(x)}{\gamma[f(x)]}$$

Ω-a.e. in A. According to Theorem 3.1 there exists an η such that

$$\psi_1(x) - \psi_2(x) = \eta$$

Ω-a.e. in A. Hence and by (4.3) we have (4.2).

Let

$$(4.4)\quad \begin{cases} G_0(x) = 1,\ G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)] & \text{for} \quad n = 1, 2, \ldots, \\ G(x) = \prod_{i=0}^{\infty} g[f^i(x)] & \text{(provided it exists)}, \\ \gamma_n(x) = \prod_{i=n}^{\infty} \dfrac{g[f^i(x_0)]}{g[f^i(x)]}. \end{cases}$$

THEOREM 4.3. *We suppose that hypotheses* (H_2), (H_5), (H_6), (H_8), (H_9) *and condition* (3.32) *are fulfilled, g is monotonic, and*

$$(4.5)\qquad \lim_{n\to\infty} \frac{h[f^n(x)]}{G_{n+1}(x)} = 0$$

Ω-a.e. *in A. If φ is a solution of equation* (4.1) Ω-a.e. *in A which is non-negative and equally monotonic with g, then*

$$(4.6)\qquad \varphi(x) = \varphi(x_0)\gamma(x) + \sum_{i=0}^{\infty} \frac{h[f^i(x_0)]\gamma_{i+1}(x) - h[f^i(x)]}{G_{i+1}(x)},$$

Ω-a.e. *in A, where x_0 is an arbitrarily fixed point of the set $A \setminus S_1$ and S_1 is a certain set from the family Ω.*

Proof. The function γ exists by Theorem 3.14. The function

$$(4.7)\qquad \psi(x) = \frac{\varphi(x)}{\gamma(x)}$$

is monotonic and non-negative Ω-a.e. in A and fulfils the equation

$$\psi[f(x)] = \psi(x) + \frac{h(x)}{\gamma[f(x)]}$$

Ω-a.e. in A. By (3.33) and (4.5)

$$\lim_{n\to\infty} \frac{h[f^n(x)]}{\gamma[f^n(x)]} = \lim_{n\to\infty} \frac{h[f^n(x)]}{G_n(x)\gamma(x)} = \frac{1}{\gamma(x)} \lim_{n\to\infty} \frac{h[f^n(x)]}{G_n(x)} = 0$$

and thus, according to Theorem 3.2, the function ψ is expressed Ω-a.e. in A by the formula

$$\psi(x) = \psi(x_0) + \sum_{i=0}^{\infty} \left\{ \frac{h[f^i(x_0)]}{\gamma[f^{i+1}(x_0)} - \frac{h[f^i(x)]}{\gamma[f^{i+1}(x)]} \right\}.$$

Hence and by (4.7) and (4.4)

$$\varphi(x) = \varphi(x_0)\gamma(x) + \gamma(x) \sum_{i=0}^{\infty} \left\{ \frac{h[f^i(x_0)]}{G_{i+1}(x_0)} - \frac{h[f^i(x)]}{G_{i+1}(x)} \frac{1}{\gamma(x)} \right\}.$$

By (3.33) and (4.4) we get (4.6).

THEOREM 4.4. *We suppose that hypotheses* (H_2), (H_5), (H_6), (H_8), (H_9) *are fulfilled, g is monotonic and fulfils condition* (3.32), *moreover the series*

$$\sum_{i=0}^{\infty} \frac{h[f^i(x)]}{G_{i+1}(x)} \tag{4.8}$$

is convergent Ω-a.e. in A. If φ is a solution of equation (4.1) *which is non-negative and conversely monotonic as g, then*

$$\varphi(x) = \eta\gamma(x) - \sum_{i=0}^{\infty} \frac{h[f^i(x)]}{G_{i+1}(x)} \tag{4.9}$$

Ω-a.e. *in A.*

Proof. Let $S \in \Omega$ be a set such that in $A \setminus S$ the functions g, h and φ are defined, series (4.8) converges, and the function φ fulfils (4.1). The set S_1 defined by (3.21) belongs to Ω and the set A_0 defined by (3.22) fulfils the inclusion $f(A_0) \subset A_0$. By (4.1) we get

$$\frac{\varphi[f^n(x)]}{G_n(x)} = \varphi(x) + \sum_{i=0}^{n-1} \frac{h[f^i(x)]}{G_{i+1}(x)} \tag{4.10}$$

for $x \in A_0$. The functions

$$\varphi_n(x) = \frac{\varphi[f^n(x)]}{G_n(x)} \tag{4.11}$$

are conversely monotonic as g and non-negative in A_0. Since the right-hand side of (4.10) has a finite limit for $n \to \infty$, the sequence φ_n converges to a finite limit in A_0. The limit $\bar{\varphi}$ of this sequence is monotonic and non-negative. The functions φ_n fulfil the condition

$$\varphi_n[f(x)] = g(x)\varphi_{n+1}(x).$$

Hence the function $\bar{\varphi}$ satisfies the functional equation (3.2). It follows by Theorem 3.4 that there exits a number η such that $\bar{\varphi}(x) = \eta\gamma(x)$. Hence and by (4.10), (4.11) we get the assertion of this theorem.

Suppose that hypothesis (H_9) is fulfilled. Let S be the set of all those points from A for which $g(x)$ does not exist. Let S_1 be defined by (3.21),

and A_0 by (3.22). All the functions $G_n(x)$ exist in A_0 and form a monotonic sequence, which therefore converges (to a finite limit or not). Thus $G(x)$ exists Ω-a.e. in A (finite or not).

LEMMA 4.1. *We suppose that hypotheses* (H_2), (H_5), (H_7), (H_9) *are fulfilled. If g is incresing and fulfils* (3.32) Ω-a.e. *in A, then either* $G(x) \equiv 0$ Ω-a.e. *in A or* $\{x\colon G(x) = 0\} \in \Omega$.

Proof. Let the set $S \in \Omega$ be such that $g(x)$ is defined, increasing and fulfils (3.32) in $A \setminus S$. Let S_1 be defined by (3.21) and A_0 by (3.22). For the set A_0 we have $f(A_0) \subset A_0$. If $G(x) \neq 0$ for every $x \in A_0$, then the set $\{x \in A\colon G(x) = 0\}$ belongs to Ω as a subset of the set $S_1 \in \Omega$.

We suppose that there is a point $x_1 \in A$ such that $G(x_1) = 0$. The function $G(x)$ is increasing and non-negative. Thus $G(x) = 0$ for $x \in A_0 \cap \cap (-\infty, x_1]$. Let $x \in A_0 \cap (x_1, +\infty)$. Then, since the sequence $f^n(x_1)$ has no upper bound in A, there is a positive number k such that $x \leqslant f^k(x_1)$. We have on account of the monotonicity of G

$$0 \leqslant G(x) \leqslant G[f^k(x_1)] = \lim_{n\to\infty} G_n[f^k(x_1)] = \lim_{n\to\infty} \frac{G_{n+k}(x_1)}{G_k(x_1)} = \frac{G(x_1)}{G_k(x_1)} = 0.$$

This completes the proof.

THEOREM 4.5. *Let hypotheses* (H_2), (H_5), (H_6), (H_8), (H_9) *be fulfilled, let g be a monotonic function and* $0 < G(x) \leqslant +\infty$ Ω-a.e. *in A. If φ is a non-negative solution of equation* (4.1) *conversely monotonic as g and bounded, then φ is given by formula* (4.9) Ω-a.e. *in A.*

Proof. The sequence φ_n given by (4.11) converges being a quotient of two sequences the first of which has a limit, and the other has a positive limit. Hence by (4.10) series (4.8) converges and the assertion results by Theorem 4.4.

The last theorem is closely related to paper [3].

LEMMA 4.2. *Let hypotheses* (H_2), (H_5), (H_6), (H_8), (H_9) *be fulfilled and let g be an increasing function and* $G(x) = 0$ Ω-a.e. *in A. If equation* (4.1) *has a decreasing and non-negative solution φ* Ω-a.e. *in A, then there is a number l such that*

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} h[f^i(x)] \frac{G_n(x)}{G_{i+1}(x)} = l \tag{4.12}$$

Ω-a.e. *in A.*

Proof. Let S be a set such that the functions g, h, φ are defined in $A \setminus S$, $G(x) = 0$ for $x \in A \setminus S$, and φ fulfils (4.1) in $A \setminus S$. Let S_1 and A_0 be defined by (3.21) and (3.22). Since φ is non-negative, there exists

$$l = \inf_{A_0} \varphi(x).$$

It follows by hypothesis (H_2) and by the monotonicity of φ that

$$\lim_{n\to\infty} \varphi[f^n(x)] = l \tag{4.13}$$

for $x \in A_0$. We get by (4.1)

$$\varphi[f^n(x)] = \varphi(x)G_n(x) + \sum_{i=0}^{n-1} h[f^i(x)] \frac{G_n(x)}{G_{i+1}(x)}. \tag{4.14}$$

Hence relation (4.12) results for $x \in A_0$ in virtue of (4.13) and (4.14).

THEOREM 4.6. *Let hypotheses* (H_2), (H_5), (H_6), (H_8), (H_9) *be fulfilled, let g be increasing and fulfil condition* (3.32). *Moreover we suppose that* $G(x) = 0$ Ω-a.e. *in A and*

$$\frac{h(x)}{1-g(x)} \leqslant l, \tag{4.15}$$

where l is given by formula (4.12). *If a function φ is a non-negative decreasing solution of equation* (4.1), *then*

$$\varphi(x) = l + \eta\gamma(x) - \sum_{i=0}^{\infty} \frac{l\big(g[f^i(x)]-1\big) + h[f^i(x)]}{G_{i+1}(x)}$$

Ω-a.e. *in A.*

Proof. Let S be the set of those points belonging to A at which the functions g, h, φ are not defined. We define A_0 by (3.22). The function $\psi(x) = \varphi(x) - l$ is a non-negative decreasing solution of the equation

$$\psi[f(x)] = g(x)\psi(x) + H(x), \tag{4.16}$$

where $H(x) \overset{\mathrm{df}}{=} l\big(g(x)-1\big) + h(x)$. We have by (4.15)

$$H(x) \leqslant 0 \tag{4.17}$$

for $x \in A_0$. By (4.16)

$$\frac{\psi[f^n(x)]}{G_n(x)} = \psi(x) + \sum_{i=0}^{n-1} \frac{H[f^i(x)]}{G_{i+1}(x)}, \tag{4.18}$$

for $x \in A_0$. The left-hand side of (4.18) is non-negative thus

$$-\psi(x) \leqslant \sum_{i=0}^{n-1} \frac{H[f^i(x)]}{G_{i+1}(x)}.$$

The components of the right-hand side of the last inequality are non-positive by (4.17). Therefore the series

$$\sum_{i=0}^{\infty} \frac{H[f^i(x)]}{G_{i+1}(x)}$$

is convergent. We get the assertion by Theorem 4.4.

§ 2. Existence.

THEOREM 4.7. *Suppose that hypotheses* (H_2), (H_5), (H_6), (H_8), (H_9) *are fulfilled, g and h are increasing Ω-a.e. in A and $g \in S_f(\mathrm{Domain}\, g)$, $h \in S_f(\mathrm{Domain}\, h)$. Moreover, suppose that $g(x) \geqslant 1$, $h(x) \geqslant 0$ Ω-a.e. in A and there exists $a_0 = \inf A \in A$. Then equation* (4.1) *has an increasing solution Ω-a.e. in A.*

Proof. Let S be such a set that g and h are defined and $g(x) \geqslant 1$, $h(x) \geqslant 0$ in $A \setminus S$. Let S_1 be defined by (3.21), A_0 by (3.22) and $A_1 = A_0 \cup \cup \{a_n\}$. We define functions $\bar{g}$ and $\bar{h}$ in A_1 by the formulas

$$\bar{g}(x) = \sup_{x' \leqslant x,\, x' \in A} g(x'), \qquad \bar{h}(x) = \sup_{x' \leqslant x,\, x' \in A} h(x')$$

for $x \in A_1$. These functions are increasing in A_1 and constant in those subsets of the set A_1 in which f is constant, $\bar{g}(x) \geqslant 1$, $\bar{h}(x) \geqslant 0$ for $x \in A_1$. Let

$$H(x, y) = \bar{g}(x) y + \bar{h}(x);$$

then

$$H\colon\ A_1 \times [0, +\infty) \to [0, +\infty)$$

is an increasing function with respect to each variable. This function is constant for every fixed $y \in [0, +\infty)$ in those subsets of the set A_1 in which f is constant. We have $H(x, y) \geqslant y$ for every $x \in A_1$ and for every $y \in [0, +\infty)$. In virtue of Theorem 2.1 (with $Q = [0, +\infty)$) there exists an increasing function φ which fulfils the equation

$$\varphi[f(x)] = \bar{g}(x)\varphi(x) + \bar{h}(x)$$

for $x \in A_1$. Hence φ is an increasing solution of equation (4.1) in A_0, thus Ω-a.e. in A.

Similarly, making use of Theorem 2.1 and assuming $Q = [0, +\infty)$, $Q_x = \left[0, \dfrac{h(x)}{1-g(x)}\right]$, we can prove the following

THEOREM 4.8. *Suppose that hypotheses* (H_2), (H_5), (H_6), (H_8), (H_9) *are fulfilled, g and h are increasing Ω-a.e. in A and $g \in S_f(\mathrm{Domain}\, g)$, $h \in S_f(\mathrm{Domain}\, h)$. Moreover, suppose that $g(x) < 1$, $h(x) \geqslant 0$ Ω-a.e. in A and there exists $a_0 = \inf A \in A$. Then equation* (4.1) *has an increasing solution Ω-a.e. in A.*

THEOREM 4.9. *Let hypotheses* (H_2), (H_5), (H_6), (H_8), (H_9) *be fulfilled and let the set* A *fulfil the condition*

$$\inf_{x\in A\ (\Omega)} \operatorname{ess} x = \inf A \notin A. \tag{4.19}$$

Moreover we suppose that f *is continuous,* $g(x) < 1$, $h(x) \geqslant 0$, g *and* h *are increasing, and* $g \in S_f(\mathrm{Domain}\, g)$, $h \in S_f(\mathrm{Domain}\, h)$. *Then equation* (4.1) *has an increasing solution* Ω-a.e. *in* A.

Proof. Let S be a set such that the functions g and h are defined in $A \setminus S$. Let S_1 be defined by formula (3.21) and A_0 by (3.22). The set A_0 fulfils the inclusion $f(A_0) \subset A_0$. It follows by (4.19) that $\inf A_0 = \inf A$. Hence, we have

$$\inf A_1 = \inf A_0 = \inf A \tag{4.20}$$

for

$$A_1 \overset{\mathrm{df}}{=} \bar{A}_0 \setminus \{\inf A_0, \sup A_0\}.$$

Since the sequence $\{f^n(x)\}$ has no upper bound in A, we have

$$\sup A_1 = \sup A_0 = \sup A. \tag{4.21}$$

We get by (4.20) and (4.21)

$$\begin{aligned} \bar{A}_1 \setminus \{\inf A_1, \sup A_1\} &= \overline{\bar{A}_1 \setminus \{\inf A_1, \sup A_1\}} \setminus \{\inf A_0, \sup A_0\} \\ &= \bar{A}_0 \setminus \{\inf A_0, \sup A_0\} = A_1. \end{aligned}$$

Thus the set A_1 fulfils condition (2.14).

Let

$$h_1(x) = \begin{cases} \sup\limits_{x' \leqslant x} h(x') & \text{if } x \in A_1 \text{ is an upper bound of a subset of } A_0, \\ \inf\limits_{x \leqslant x'} h(x') & \text{if } x \in A_1 \text{ is not an upper bound of any subset of } A_0, \end{cases}$$

$$g_1(x) = \begin{cases} \sup\limits_{x' \leqslant x} g(x') & \text{if } x \in A_1 \text{ is an upper bound of a subset of } A_0, \\ \inf\limits_{x \leqslant x'} g(x') & \text{if } x \in A_1 \text{ is not an upper bound of any subset of } A_0. \end{cases}$$

The functions g_1 and h_1 are increasing in A_1, $g_1(x) < 1$ and $h_1(x) \geqslant 0$. Furthermore $h_1(x) = h(x)$ and $g_1(x) = g(x)$ for $x \in A_0$. The function

$$H(x, y) = g_1(x)y + h_1(x)$$

is increasing with respect to each variable in $A_1 \times [0, +\infty)$ and is constant in those subsets of A_1 in which f is constant. Moreover

$$Q_x = \{y \in [0, +\infty) \colon H(x, y) \geqslant y\} = \left[0, \frac{h_1(x)}{1 - g_1(x)}\right].$$

In virtue of Theorem 2.3 there exists an increasing function φ fulfilling the equation

$$\varphi[f(x)] = g_1(x)\varphi(x) + h_1(x)$$

for $x \in A_2$. The function φ fulfils equation (4.1) in A_0.

THEOREM 4.10. *Let hypotheses* (H_2), (H_5), (H_6), (H_8), (H_9) *be fulfilled and let* g, h *be decreasing,* $h(x) \geqslant 0$, $g(x) \geqslant 1$. *If there exists* $\inf A = a_0 \in A$, *then for every positive number* y_0 *there is an increasing solution* φ *of equation* (4.1) *defined* Ω-a.e. *in* A *such that* $\varphi(a_0) = y_0$.

Proof. It follows from Theorem 3.12 that there exists a function ω, defined, increasing, fulfilling equation (3.2) Ω-a.e. in A and such that $\omega(a_0) = 1$. Consequently the function

$$h_1(x) = \frac{h(x)}{\omega[f(x)]}$$

is defined and decreasing Ω-a.e. in A. In virtue of Theorem 3.7 the equation

$$\psi[f(x)] = \psi(x) + h_1(x) \tag{4.22}$$

has an increasing solution defined Ω-a.e. in A and such that $\psi(a_0) = y_0$. A product of positive and increasing functions is positive and increasing, thus by (4.22) the function

$$\varphi(x) = \omega(x)\psi(x)$$

is an increasing and positive solution of equation (4.1) defined Ω-a.e. in A. This completes the proof.

THEOREM 4.11. *Let the hypotheses of Theorem* 4.9 *and conditions* (3.32) *and* (4.5) *be fulfilled. If there exist* $\inf A = a_0 \in A$ *and* $h(a_n)$, $g(a_n)$, *where* $a_n = f^n(a_0)$, $n = 0, 1, \ldots$, *then equation* (4.1) *has an increasing positive solution defined* Ω-a.e. *in* A *by formula* (4.6).

Proof. It follows by Theorem 3.14 that there exists an increasing function γ, defined by (3.33) Ω-a.e. in A, satisfying equation (3.2) Ω-a.e. in A and such that $\gamma(a_0) = 1$ $(x_0 = a_0)$. The function

$$h_1(x) = \frac{h(x)}{\gamma[f(x)]} \tag{4.23}$$

is decreasing, non-negative and defined for Ω-almost all $x \in A$. It follows by Theorem 3.9 that there exists an increasing function ψ, given by the formula

$$\psi(x) = y_0 + \sum_{i=0}^{\infty} \{h_1[f^i(x_0)] - h_1[f^i(x)]\}, \tag{4.24}$$

satisfying the equation

$$\psi[f(x)] = \psi(x) + h(x) \tag{4.25}$$

for Ω-almost all $x \in A$ and such that $\psi(a_0) = y_0$. We get by (4.23) and (4.25) that the function $\varphi(x) = \gamma(x)\psi(x)$ is an increasing solution of equation (4.1) such that $\varphi(a_0) = y_0$. It follows by (4.23) and (4.24) that

$$\psi(x) = y_0 + \sum_{i=0}^{\infty} \left\{ \frac{h[f^i(a_0)]}{\gamma[f^{i+1}(a_0)]} - \frac{h[f^i(x)]}{\gamma[f^{i+1}(x)]} \right\},$$

whence

$$\varphi(x) = \gamma(x)\psi(x) = y_0\gamma(x) + \gamma(x) \sum_{i=0}^{\infty} \left\{ \frac{h[f^i(a_0)]}{\gamma[f^{i+1}(a_0)]} - \frac{h[f^i(x)]}{\gamma[f^{i+1}(x)]} \right\}.$$

Hence and by (4.4) we obtain the assertion.

THEOREM 4.12. *Let X be a linearly ordered continuous space, $A = \{x \in X: a_0 \leqslant x\}$, let hypotheses* (H_2), (H_5)–(H_9) *be fulfilled and let f be continuous. We assume that g and h are increasing and $g \in S_f(\mathrm{Domain}\, g)$, $h \in S_f(\mathrm{Domain}\, h)$. Further we assume that $g(x) \geqslant 1$, $h(x) \geqslant 0$ and for every integer $n \geqslant 0$ there exist $g[f^n(a_0)]$ and $h[f^n(a_0)]$.*

If $g(a_0) > 1$ or $h(a_0) > 0$, then equation (4.1) *has Ω-*a.e. *in A an increasing solution depending on an arbitrary function.*

If $g(a_0) = 1$ and $h(a_0) = 0$, then equation (4.1) *has Ω-*a.e. *in A a unique one-parameter family of, increasing solutions.*

Proof. Since every interval $[a_n, a_{n+1}]$, $a_n = f^n(a_0)$, $n = 0, 1, \ldots$, is a connected set, and since a continuous function on a connected set has the Darboux property, we have

$$f([a_n, a_{n+1}]) = [a_{n+1}, a_{n+2}].$$

Let $S \in \Omega$, $S \subset A$, be such a set that in $A \setminus S$ the functions g and h are defined and $a_0 \notin S$. We define the set S_0 by (3.5). The set $\tilde{A} = A \setminus S_0$ fulfils (3.34) and (3.35). The set $\tilde{A}$ contains its infimum a_0. The function $H(x, y) \overset{\mathrm{df}}{=} g(x)y + h(x)$ is increasing in $\tilde{A} \times [0, +\infty)$ with respect to either variable. For every fixed y the function $H(x, y)$ belongs to $S_f(\tilde{A})$. Moreover, $H(A \times [0, +\infty)) \subset [0, +\infty)$ and for every $x \in \tilde{A}$ and for every y

$$y \leqslant H(x, y) = g(x)y + h(x).$$

It follows from Theorem 2.1 that for every function φ_0 defined, increasing, non-negative on the set $[a_0, a_1] \cap \tilde{A}$, and constant in those subsets of $[a_0, a_1] \cap \tilde{A}$ in which f is constant, and such that

$$\varphi_0(a_1) = g(a_0)\varphi_0(a_0) + h(a_0), \tag{4.26}$$

there exists a function φ defined and increasing in A, satisfying (4.1) in $\tilde{A}$ and such that $\varphi(x) = \varphi_0(x)$ for $x \in [a_0, a_1] \cap \tilde{A}$. It follows by (3.35)

that for a given function φ_0 fulfilling the above mentioned conditions the function φ is unique.

If $g(a_0) > 1$ or $h(a_0) > 0$, then the family Φ (cf. Chapter II) of functions φ_0 is not a finite parameter family.

If, on the other hand, $g(a_0) = 1$ and $h(a_0) = 0$, then by (4.26) the function φ_0 must be constant.

REFERENCES

[1] D. Brydak, J. Kordylewski, *On monotonic sequences defined by a recurrence relation*, Publ. Math. Debrecen 9 (1962), pp. 189–197.

[2] J. Burek, M. Kuczma, *Einige Bemerkungen über monotone und konvexe Lösungen gewisser Funktionalgleichungen*, Math. Nachr. 36 (1968), pp. 121–134.

[3] B. Choczewski, M. Kuczma, *On the "indeterminate case" in the theory of theory of a linear functional equation*, Fund. Math. 58 (1966), pp. 163–175.

[4] Á. Császár, *Sur les fonctions localement monotones au sens généralisé*, Acta Math. Acad. Sci. Hung. 6 (1955), pp. 452–461.

[5] R. Engelking, *Zarys topologii ogólnej*, Warszawa 1965.

[6] F. John, *Special solutions of certain difference equations*, Acta Math. 71 (1939), pp. 175–189.

[7] M. Kuczma, *Functional equations is a single variable*, Monografie Matematyczne 46, Warszawa 1968.

[8] — *On the Schröder equation*, Rozprawy Mat. 34 (1963).

[9] — *Remark on a difference equation*, Roczniki P.T.M., Seria I., Prace Matematyczne 9 (1965), pp. 2–8.

[10] — *Remarks on some functional equations*, Ann. Polon. Math. 8 (1960), pp. 277–284.

[11] — *Remarques sur quelques théorèmes de J. Anastassiadis*, Bull. Sci. Math. 84 (1960), pp. 98–102.

[12] — *Sur une équation fonctionelle*, Mathematica, Cluj, 3 (26) (1961), pp. 79–87.

[13] K. Kuratowski, M. Mostowski, *Teoria mnogości*, Warszawa–Wrocław 1952.

[14] A. Pelczar, *On invariant points of monotone transformations in partially ordered space*, Ann. Polon. Math. 17 (1965), pp. 50–53.

[15] — *On the invariant points of a transformation*, Ann. Polon. Math. 11 (1961), pp. 199–202.

[16] R. Sikorski, *Funkcje rzeczywiste*, Tom II, Monografie Matematyczne 37, Warszawa 1959.

[17] A. Smajdor, *On monotonic solutions of a recurrence relation*, Ann. Polon. Math. 19 (1967), pp. 169–176.

[18] — *Remarks on monotonic solutions of a functional equation*, Aequationes Math. 1 (1968), pp. 87–93.