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ON A PAPER BY KAREL PRIKRY CONCERNING ULAM'S PROBLEM ON FAMILIES OF MEASURES

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Marczewski in [8] introduced the notion of an $\mathscr{A}$-hull of a set $X \subset S$, where $\mathscr{A}$ is an arbitrary field of subsets of S. By modification of Prikry's proofs and definitions in [3] we obtain*, with the help of an $\mathscr{A}$-hull, a strengthening and a common generalization of all main theorems of papers [2], [3], [5], and [6]. We deal with theorems on the existence of a partition of a set X into sets as "large" as X, e.g. in a sense of measure or category. For understanding the proofs of Theorems 2 and 3 the reader is not expected to be familiar with any of the papers quoted in the References. For understanding the proof of Theorem 1 the knowledge of [3] is sufficient.

0. Terminology and notation. Let $|X|$ denote the cardinality of a set X. In the sequel, lower case Greek letters denote ordinals with $\varkappa$ and λ always standing for infinite cardinals, and δ for any (finite or infinite) cardinal. By $\mathrm{cf}(\varkappa)$ we denote the cofinality of $\varkappa$, and by $\varkappa^+$ the cardinal successor of $\varkappa$. An ordinal α is considered to be the set of all ordinals smaller than α. The set of natural numbers is denoted by ω, and $\omega_1 = \omega^+$, $\omega_2 = \omega_1^+$. A cardinal $\varkappa$ is called *regular* if $\mathrm{cf}(\varkappa) = \varkappa$, otherwise it is called *singular*. A cardinal $\varkappa > \omega$ is called *weakly inaccessible* if it is both regular and limit. $\mathscr{P}(X)$ denotes the power set of X, and

$$[X]^{<\delta} = \{Y \subset X : |Y| < \delta\}.$$

A field of subsets of a set S will be called, shortly, a *field* on S. Similarly, if $\mathscr{I}$ is an ideal in the field $\mathscr{P}(S)$, then $\mathscr{I}$ will be called an *ideal* on S. For an ideal $\mathscr{I}$ we always assume that $\emptyset \in \mathscr{I}$. We say that an ideal $\mathscr{I}$ on S is *non-trivial* if $S \notin \mathscr{I}$ and $[S]^{<\omega} \subset \mathscr{I}$. A family $\mathscr{R} \subset \mathscr{P}(S)$ is *$\varkappa$-complete* if for every family $\mathscr{Q} \subset \mathscr{R}$ such that $|\mathscr{Q}| < \varkappa$ we have $\bigcup \mathscr{Q} \in \mathscr{R}$.

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Let $\mathcal{I}$ be an ideal on S. We say that subsets X, Y of S are *$\mathcal{I}$-almost disjoint* if $X \cap Y \in \mathcal{I}$. A family $\mathcal{R} \subset \mathcal{P}(S)$ is called *$\mathcal{I}$-almost disjoint* if any two different sets in $\mathcal{R}$ are $\mathcal{I}$-almost disjoint. An indexed family $\{X_\xi : \xi \in \delta\}$ is called *$\mathcal{I}$-almost disjoint* if $X_\xi \cap X_\eta \in \mathcal{I}$ for every $\xi, \eta \in \delta$ such that $\xi \neq \eta$. It should be noted that we do not require that $X_\xi \neq X_\eta$ for $\xi \neq \eta$. If $\mathcal{I} = \{\emptyset\}$, then instead of "$\mathcal{I}$-almost disjoint" we say, shortly, *disjoint*. If $\mathcal{R}$ is any subfamily of $\mathcal{P}(S)$, then by $\mathrm{Sat}[\mathcal{R}|\mathcal{I}]$ we denote the smallest cardinal δ such that every $\mathcal{I}$-almost disjoint family contained in $\mathcal{R} - \mathcal{I}$ is of cardinality less than δ. If $\mathcal{I} = \{\emptyset\}$, then instead of $\mathrm{Sat}[\mathcal{R}|\mathcal{I}]$ we write $\mathrm{Sat}[\mathcal{R}]$. If $\mathcal{A}$ is a field on S, then by $I(\mathcal{A})$ we denote the family of all $A \in \mathcal{A}$ such that $X \in \mathcal{A}$ for every $X \subset A$. The almost disjoint transversal hypothesis for $\varkappa^+$ is denoted by $\mathrm{TH}(\varkappa^+)$. Let us recall that $\mathrm{TH}(\varkappa^+)$ follows from Gödel's axiom of constructibility (for more information see [3]). CH denotes the continuum hypothesis, and GCH — the generalized continuum hypothesis.

1. Theorems and corollaries. In this note we prove three theorems. The proof of Theorem 1 will not be presented in details, since it is very long and needs only a minor modification of Prikry's proof of Theorem 1 in [3]. The proof can be completed with the help of our generalization of an eventual hull (see Section 2). It will be clear from the proof of our Theorem 2 how to modify Prikry's proof of his Theorem 1 in [3] in order to obtain our Theorem 1 (see also Added in proof). The proof of Theorems 2 and 3 is a modification of Prikry's proof of his Theorem 2 in [3]. The main result of our note is Theorem 2.

THEOREM 1. *Assume* $\mathrm{TH}(\omega_1)$. *Let* $\mathfrak{M}$ *be a family of* ω_1*-complete fields on* ω_1 *satisfying* $|\mathfrak{M}| \leqslant \omega_1$ *and such that* $[\omega_1]^{<\omega} \subset \mathcal{A}$ *and* $\mathrm{Sat}[\mathcal{A} - I(\mathcal{A})] \leqslant \omega_1$ *for every* $\mathcal{A} \in \mathfrak{M}$. *Then for every* $X \subset \omega_1$ *with* $X \notin \bigcup \{I(\mathcal{A}) : \mathcal{A} \in \mathfrak{M}\}$ *there exists a disjoint family* $\mathcal{Q} \subset \mathcal{P}(X) - \bigcup M$ *such that* $|\mathcal{Q}| = \omega_1$.

THEOREM 2. *Let* $\varkappa$ *be a cardinal,* λ *a regular cardinal with* $\varkappa \leqslant \lambda \geqslant \omega_1$, *and let* $\mathfrak{M}$ *be a family of* $\varkappa$*-complete fields on* S *satisfying* $|\mathfrak{M}| \leqslant \omega$ *and such that* $\mathrm{Sat}[\mathcal{A} - I(\mathcal{A})] \leqslant \varkappa$ *for every* $\mathcal{A} \in \mathfrak{M}$. *Suppose that* $\mathcal{I} \subset \bigcap \{I(\mathcal{A}) : \mathcal{A} \in \mathfrak{M}\}$ *is a* $(\max(\varkappa, \omega_1))$*-complete ideal on* S *such that* $\mathrm{Sat}[\mathcal{P}(X) - I(\mathcal{A})|\mathcal{I}] > \lambda$ *for every* $\mathcal{A} \in \mathfrak{M}$ *and every* $X \in \mathcal{P}(S) - I(\mathcal{A})$. *Then for every* $X \subset S$ *there exists an* $\mathcal{I}$*-almost disjoint family* $\{X_\xi : \xi \in \lambda\} \subset \mathcal{P}(X)$ *satisfying the following condition*:

$$(*) \quad \forall(\xi \in \lambda)\, \forall(\mathcal{A} \in \mathfrak{M})\, \forall(A \in \mathcal{A})\, \big[(X_\xi - A \in I(\mathcal{A})) \Rightarrow (X - A \in I(\mathcal{A}))\big].$$

THEOREM 3. *Let* λ *be a regular cardinal and let* $\mathfrak{M}$ *be a family of fields on* S *satisfying* $|\mathfrak{M}| < \omega$ *and such that* $\mathrm{Sat}[\mathcal{A} - I(\mathcal{A})] \leqslant \omega$ *for every* $\mathcal{A} \in \mathfrak{M}$. *Suppose that* $\mathcal{I} \subset \bigcap \{I(\mathcal{A}) : \mathcal{A} \in \mathfrak{M}\}$ *is an ideal on* S *such that* $\mathrm{Sat}[\mathcal{P}(X) - I(\mathcal{A})|\mathcal{I}] > \lambda$ *for every* $\mathcal{A} \in \mathfrak{M}$ *and every* $X \in \mathcal{P}(S) - I(\mathcal{A})$. *Then for every* $X \subset S$ *there exists an* $\mathcal{I}$*-almost disjoint family* $\{X_\xi : \xi \in \lambda\} \subset \mathcal{P}(X)$ *satisfying condition* (*) *of Theorem* 2.

In [3] Prikry has proved Theorems 1 and 2 only in case $X = S$, $\varkappa = \omega_1$ and with $\mathfrak{M}$ consisting of ω_1-complete fields allowing complete probabilities which vanish on all finite sets. Theorems 1 and 2 of our note generalize Theorems 1 and 2 of [3]. To see this, use fact (a) from this section and the Sublemma. Our generalization of Theorem 2 of [3] may be motivated by the fact that it gives a strengthening and a common generalization of known theorems for measures ([6] and [3]), for outer measure [2] and for Baire category [5]. To see that the above-mentioned classical results are a very weak consequence of our Theorem 2 we recall the following facts:

(a) For every $\varkappa^+$-complete non-trivial $\mathcal{I}$ on $\varkappa^+$

$$\mathrm{Sat}[\mathcal{P}(X)-\mathcal{I}] > \varkappa^+ \quad \text{for every } X \in \mathcal{P}(\varkappa^+)-\mathcal{I}$$

(see [9]).

(b) If $\varkappa$ is less than the first weakly inaccessible cardinal and $\omega_1 \leqslant \lambda \leqslant \varkappa$, then for every λ-complete non-trivial ideal $\mathcal{I}$ on $\varkappa$

$$\mathrm{Sat}[\mathcal{P}(X)-\mathcal{I}] > \lambda \quad \text{for every } X \in \mathcal{P}(\varkappa)-\mathcal{I}$$

(this is an easy consequence of (a)). From the result of Solovay in [7] this holds if $\varkappa$ is even larger.

(c) Assume $\mathrm{TH}(\varkappa^+)$. Then for every $\varkappa^+$-complete non-trivial ideal $\mathcal{I}$ on $\varkappa^+$

$$\mathrm{Sat}[\mathcal{P}(X)-\mathcal{I} \mid [\varkappa^+]^{<\varkappa^+}] > \varkappa^{++} \quad \text{for every } X \in \mathcal{P}(\varkappa^+)-\mathcal{I}$$

(since $\mathrm{TH}(\varkappa^+)$ is equivalent to the existence of a Kurepa matrix for $\varkappa^+$; see [3]).

(d) Assume $\mathrm{TH}(\omega_1)$, GCH, and suppose that there is no two-valued measurable cardinal (e.g., the assumption is satisfied if we assume Gödel's axiom of constructibility). Then for every ω_1-complete non-trivial ideal $\mathcal{I}$ on $\varkappa \geqslant \omega_1$

$$\mathrm{Sat}[\mathcal{P}(X)-\mathcal{I} \mid \mathcal{I}] > \omega_2 \quad \text{for every } X \in \mathcal{P}(\varkappa)-\mathcal{I}$$

(see [3a]).

(e) Assume CH and suppose that there is no two-valued measurable cardinal. Then for every ω_1-complete non-trivial ideal $\mathcal{I}$ on $\varkappa \geqslant \omega_1$

$$\mathrm{Sat}[\mathcal{P}(X)-\mathcal{I}] > \omega_1 \quad \text{for every } X \in \mathcal{P}(\varkappa)-\mathcal{I}$$

(see, e.g., [1]).

A very weak consequence of Theorem 2, the Sublemma, and fact (b) is the following

COROLLARY 1. *Assume that 2^ω is less than the first weakly inaccessible cardinal. Then for every subset X of the real line there exists a disjoint family*

$\{X_\xi\colon \xi \in \omega_1\} \subset \mathscr{P}(X)$ *such that, for every* $\xi \in \omega_1$ *and every Borel set* $B \supset X_\xi$, $X - B$ *is of first category and of Lebesgue measure* 0.

Corollary 1 is a strengthening and a common generalization of theorems of papers [6] and [5]. A more interesting application of Theorem 2 and fact (b) is that for a countable family consisting of arbitrary finite complete measures vanishing on all finite sets and of topologies satisfying the second axiom of countability and without isolated points. (The case of measures which are not complete can easily be reduced to the case of complete measures; see Corollary 3 in Section 4.)

Another weak consequence of Theorem 2, the Sublemma, and fact (b) is the result of Poprużenko in [2]. This result can also be obtained more directly from Section 2 of [3].

It should be noted that if we assume that each set of cardinality less than 2^ω is of Lebesgue measure 0 and of first category, then even stronger result than Corollary 1 can easily be proved directly in Zermelo-Fraenkel set theory with the axiom of choice.

2. An eventual hull of a family of sets with respect to a field of sets. The main tool in Prikry's proofs in [3] is the notion of an eventual hull of a family of sets with respect to a measure, introduced on p. 43 in [3]. We slightly generalize this notion by considering an eventual hull with respect to a $\varkappa$-complete field $\mathscr{A}$ of sets such that $\operatorname{Sat}[\mathscr{A} - I(\mathscr{A})] \leqslant \varkappa$. It is also possible to generalize this notion to an eventual hull with respect to a $\varkappa$-complete ideal $\mathscr{I}$ in $\mathscr{A}$ such that $\operatorname{Sat}[\mathscr{A} - \mathscr{I}] \leqslant \varkappa$.

Let $\mathscr{A}$ be a field on S. If $X, Y \subset S$, then we write $X \Subset Y$ if $X - Y \in I(\mathscr{A})$, and we write $X \equiv Y$ if $X \Subset Y$ and $Y \Subset X$. It is clear that the relations $\Subset$ and $\equiv$ depend on $\mathscr{A}$. If $X, A \subset S$, then we write $A \in O[\mathscr{A}](X)$ whenever $X \Subset A$, $A \in \mathscr{A}$ and, for every $B \in \mathscr{A}$ such that $X \Subset B$, we have $A \Subset B$. Any $A \in O[\mathscr{A}](X)$ will be called an *$\mathscr{A}$-hull* of X. If $\mathscr{A}$ is fixed, then instead of $O[\mathscr{A}](X)$ we write $O(X)$. Observe that for every $A, B \in O(X)$ and every $C \subset S$ such that $C \equiv A$ we have $A \equiv B$ and $C \in O(X)$.

We omit an easy proof of the following well-known lemma (see, e.g., [8]):

LEMMA 0. *Let* $\mathscr{A}$ *be a* $\varkappa$*-complete field on* S *such that*

$$\operatorname{Sat}[\mathscr{A} - I(\mathscr{A})] \leqslant \varkappa.$$

Then for every $X \subset S$ *there exists an* $\mathscr{A}$*-hull of* X.

Let $\mathscr{A}$ be as in the assumption of Lemma 0. Let $\mathscr{R} \subset \mathscr{P}(S)$, $|\mathscr{R}| = \lambda$, and let $\mathscr{R} = \{\mathscr{R}_\xi\colon \xi \in \lambda\}$ be a one-to-one enumeration of $\mathscr{R}$. For every $\varrho \in \lambda$ put

$$\mathscr{A}_\varrho = \{A \in \mathscr{A}\colon \exists(\mathscr{S})\forall(\mathscr{T})[\mathscr{S} \subset \mathscr{R} - \{\mathscr{R}_\xi\colon \xi \in \varrho\} \text{ and } |\mathscr{S}| < \varkappa \text{ and } ((\mathscr{T} \subset \mathscr{R} - \{\mathscr{R}_\xi\colon \xi \in \varrho\} \text{ and } |\mathscr{T}| < \varkappa) \Rightarrow A \in O[\bigcup \mathscr{S} \cup \mathscr{T}])]\}.$$

LEMMA 1. *Let $\mathscr{A}$ and $\mathscr{R}$ be as above. Then*

(i) $A \equiv B$ *for every* $\varrho \in \lambda$ *and all* $A, B \in \mathscr{A}_\varrho$.

(ii) $\mathscr{A}_\varrho \neq \emptyset$ *for every* $\varrho \in \lambda$.

(iii) *If* $\varrho \in \varphi \in \lambda$, *then* $B \Subset A$ *for every* $A \in \mathscr{A}_\varrho$ *and every* $B \in \mathscr{A}_\varphi$.

(iv) *If* $\mathrm{cf}(\lambda) \geqslant \varkappa$, *then there exists a* $\varrho \in \lambda$ *such that* $\mathscr{A}_\varrho = \mathscr{A}_\varphi$ *for all* $\varphi \geqslant \varrho$.

Proof. The proof of (ii) is based on the following observation. Let $\mathscr{A}$ be a $\varkappa$-complete field on S, let $\mathscr{F}$ be a subfamily of $\mathscr{P}(S)$ such that $|\mathscr{F}| < \varkappa$, and let $A_F \in O[\mathscr{A}](F)$ for every $F \in \mathscr{F}$. Then

$$\bigcup \{A_F\colon F \in \mathscr{F}\} \in O[\mathscr{A}]\big(\bigcup \mathscr{F}\big).$$

We omit easy proofs of (i), (iii) and (iv).

If $\mathscr{A}$ and $\mathscr{R}$ are as in Lemma 1 and if $A \in \mathscr{A}_\varrho$, where ϱ is as in Lemma 1 (iv), then A is called an *eventual $\mathscr{A}$-hull* of the family $\mathscr{R}$, and we put

$$O[\mathscr{A}](\mathscr{R}) = \mathscr{A}_\varrho.$$

If $X \subset S$ and $\mathscr{A}, \mathscr{R}$ are as above, then we put

$$O[\mathscr{A}](\mathscr{R}|X) = O[\mathscr{A}](\mathscr{R} \cap X).$$

It is easy to see that for regular λ the collection of eventual $\mathscr{A}$-hulls of $\mathscr{R}$ does not depend on the choice of the enumeration of $\mathscr{R}$ in the procedure above. (The last assertion is not essential for our purposes; cf. [3], p. 44.)

The following two observations are useful in reducing the proofs of Theorems 2 and 3 (and a few lemmas which Prikry uses in his proof of Theorem 2 of [3]) to the case $X = S$.

LEMMA 2. *Let $\mathscr{A}$ be a $\varkappa$-complete field on S such that*

$$\mathrm{Sat}[\mathscr{A} - I(\mathscr{A})] \leqslant \varkappa \quad \text{and} \quad \mathrm{Sat}[\mathscr{P}(X) - I(\mathscr{A})] > \varkappa$$

for every $X \subset S$ with $X \notin I(\mathscr{A})$. Then

$$I(\mathscr{A} \cap X) = I(\mathscr{A}) \cap X \quad \text{for every } X \subset S.$$

First we prove the following

SUBLEMMA. *Let $\mathscr{B}$ be a field on X. Let $\mathscr{I}$ be an ideal on X such that $\mathscr{I} \subset \mathscr{B}$, $\mathrm{Sat}[\mathscr{B} - \mathscr{I}] \leqslant \varkappa$, and $\mathrm{Sat}[\mathscr{P}(Y) - \mathscr{I}] > \varkappa$ for every $Y \in \mathscr{B} - \mathscr{I}$. Then $\mathscr{I} = I(\mathscr{B})$.*

Proof. It is evident that $\mathscr{I} \subset I(\mathscr{B})$. To prove the inclusion $I(\mathscr{B}) \subset \mathscr{I}$ suppose, on the contrary, that there exists a $Y_0 \in I(\mathscr{B}) - \mathscr{I}$. Since $Y_0 \in \mathscr{B} - \mathscr{I}$, we have $\mathrm{Sat}[\mathscr{P}(Y_0) - \mathscr{I}] > \varkappa$.

Since $Y_0 \in I(\mathscr{B})$, we have $\mathscr{P}(Y_0) - \mathscr{I} \subset \mathscr{B} - \mathscr{I}$, and hence

$$\mathrm{Sat}[\mathscr{P}(Y_0) - \mathscr{I}] \leqslant \varkappa,$$

which is a contradiction.

Proof of Lemma 2. Fix any $X \subset S$. We must prove that $I(\mathscr{A}\cap X) = I(\mathscr{A})\cap X$. In the Sublemma put $\mathscr{B} = \mathscr{A}\cap X$ and $\mathscr{I} = I(\mathscr{A})\cap X$. It is easy to check that the assumption of the Sublemma is satisfied for our $\mathscr{B}$ and $\mathscr{I}$, and so the conclusion holds.

LEMMA 3. *Let $\mathscr{A}$ be a $\varkappa$-complete field on S such that*

$$\mathrm{Sat}[\mathscr{A}-I(\mathscr{A})] \leqslant \varkappa \quad \text{and} \quad \mathrm{Sat}[\mathscr{P}(X)-I(\mathscr{A})] > \varkappa$$

for every $X \notin I(\mathscr{A})$. Let $Y \subset X \subset S$ be such that $X \in O[\mathscr{A}\cap X](Y)$. Then

$$O[\mathscr{A}](X) = O[\mathscr{A}](Y).$$

Proof. By Lemma 0 there exists an $A \in O[\mathscr{A}](Y)$. To prove that $A \in O[\mathscr{A}](X)$, it is enough to show that $X-A \in I(\mathscr{A})$ because $Y \subset X$. Since $Y-A \in I(\mathscr{A})$, we have

$$Y-A\cap X \in I(\mathscr{A}\cap X).$$

Hence

$$X-A\cap X \in I(\mathscr{A}\cap X)$$

because $X \in O[\mathscr{A}\cap X](Y)$. Consequently, by Lemma 2,

$$X-A\cap X \in I(\mathscr{A})\cap X,$$

and so $X-A \in I(\mathscr{A})$, which completes the proof of the lemma.

3. Proof of Theorems 2 and 3. First we prove some lemmas.

LEMMA 4. *Let $\varkappa$ and λ be cardinals such that $\lambda \geqslant \varkappa$ and λ is regular. Let $\mathscr{B}$ be a $\varkappa$-complete field on X and $\mathscr{J}$ a $\varkappa$-complete ideal on X such that*

$$\mathscr{J} \subset I(\mathscr{B}), \quad \mathrm{Sat}_{\mathscr{J}}[\mathscr{B}-I(\mathscr{B})] \leqslant \varkappa, \quad \text{and} \quad \mathrm{Sat}_{\mathscr{J}}[\mathscr{P}(B)-I(\mathscr{B})\,|\,\mathscr{J}] > \lambda$$

for every $B \in \mathscr{B}-I(\mathscr{B})$. Then for every $B \in \mathscr{B}-I(\mathscr{B})$ there exist $B_0 \subset B$ with $B_0 \in \mathscr{B}-I(\mathscr{B})$ and a $\mathscr{J}$-almost disjoint family $\{X_\xi\colon \xi \in \lambda\} \subset \mathscr{P}(B_0)$ such that $B_0 \in O[\mathscr{B}](X_\xi)$ for every $\xi \in \lambda$.

Proof. Fix $B \in \mathscr{B}-I(\mathscr{B})$. Let $\mathscr{R} \subset \mathscr{P}(B)-\mathscr{J}$ be a $\mathscr{J}$-almost disjoint family such that $|\mathscr{R}| = \lambda$. Put $B_0 \in O[\mathscr{B}](\mathscr{R})$. Since $\mathscr{R} \subset \mathscr{P}(B)$ and $B \in \mathscr{B}$, we can find such a B_0 which satisfies additionally the inclusion $B_0 \subset B$. By the definition of an eventual $\mathscr{B}$-hull and the fact that λ is regular it is easy to find a disjoint family $\{\mathscr{S}_\xi\colon \xi \in \lambda\} \subset \mathscr{P}(\mathscr{R})$ such that $|\mathscr{S}_\xi| < \varkappa$ and $B_0 \in O[\mathscr{B}]\big(\bigcup \mathscr{S}_\xi\big)$ for every $\xi \in \lambda$. Put $X_\xi = \bigcup \mathscr{S}_\xi \cap B_0$ for every $\xi \in \lambda$.

LEMMA 5. *Under the assumptions as in Lemma 4 there exists a disjoint family $\mathscr{R} \subset \mathscr{B}$ such that $|\mathscr{R}| < \varkappa$, $\bigcup \mathscr{R} = X$ and for every $B \in \mathscr{R}$*

(**) *there exists a $\mathscr{J}$-almost disjoint family $\{X_\xi^B\colon \xi \in \lambda\} \subset \mathscr{P}(B)$ such that $B \in O[\mathscr{B}](X_\xi^B)$ for every $\xi \in \lambda$.*

Proof. Consider the set

$\{\mathscr{Q}\colon \mathscr{Q} \subset \mathscr{B}-I(\mathscr{B})$, $\mathscr{Q}$ is a disjoint family, every $B \in \mathscr{Q}$ has property (**)$\}$

partially ordered by inclusion. Observe that this set is non-empty, since it contains the empty family. By the Kuratowski-Zorn lemma, in that set there exists a maximal element $\mathcal{Q}_0$. Since $\mathrm{Sat}[\mathcal{B}-I(\mathcal{B})] \leqslant \varkappa$, $|\mathcal{Q}_0| < \varkappa$. Hence $X - \bigcup \mathcal{Q}_0 \in \mathcal{B}$. Since $\mathcal{Q}_0$ is maximal, we have $X - \bigcup \mathcal{Q}_0 \in I(\mathcal{B})$ by Lemma 4. Adding the set $\{X - \bigcup \mathcal{Q}_0\}$ to the family $\mathcal{Q}_0$ we obtain a new family, say $\mathcal{R}$. It is clear that such a family $\mathcal{R}$ is what we need.

LEMMA 6. *Under the assumptions as in Lemma 4 there exists a $\mathcal{J}$-almost disjoint family $\{X_\xi\colon \xi \in \lambda\} \subset \mathcal{P}(X)$ such that $X \in O[\mathcal{B}](X_\xi)$ for every $\xi \in \lambda$.*

Proof. Let $\mathcal{R}$ and $\{X_\xi^B\colon \xi \in \lambda\}$ for every $B \in \mathcal{R}$ be as in Lemma 5. For every $\xi \in \lambda$ put

$$X_\xi = \bigcup \{X_\xi^B\colon B \in \mathcal{R}\}.$$

For any $\varkappa$-complete field $\mathcal{B}$ the union of $\mathcal{B}$-hulls of all sets from a family of cardinality less than $\varkappa$ is a $\mathcal{B}$-hull of the union of all sets from this family. Therefore, $B \in O[\mathcal{B}](X_\xi^B)$ for every $B \in \mathcal{R}$ implies

$$\bigcup \{B\colon B \in \mathcal{R}\} \in O[\mathcal{B}]\big(\bigcup \{X_\xi^B\colon B \in \mathcal{R}\}\big)$$

and hence $X \in O[\mathcal{B}](X_\xi)$ because $\bigcup \{B\colon B \in \mathcal{R}\} = X$.

LEMMA 6*. *Let $\varkappa$ and λ be cardinals such that $\lambda \geqslant \varkappa$ and λ is regular. Let $\mathcal{A}$ be a $\varkappa$-complete field on S and $\mathcal{J}$ a $\varkappa$-complete ideal on S with the properties: $\mathcal{J} \subset I(\mathcal{A})$, $\mathrm{Sat}[\mathcal{A}-I(\mathcal{A})] \leqslant \varkappa$ and $\mathrm{Sat}[\mathcal{P}(X)-I(\mathcal{A})|\mathcal{J}] > \lambda$ for every $X \in \mathcal{P}(S)-I(\mathcal{A})$. Let $X \in \mathcal{P}(S)$. Then there exists an $\mathcal{J}$-almost disjoint family $\{X_\xi\colon \xi \in \lambda\} \subset \mathcal{P}(X)$ such that $O[\mathcal{A}](X) = O[\mathcal{A}](X_\xi)$ for every $\xi \in \lambda$.*

Proof. Let $X \in \mathcal{P}(S)$. In Lemma 6 put $\mathcal{B} = \mathcal{A} \cap X$ and $\mathcal{J} = \mathcal{J} \cap X$. It is easy to check that in this case the assumptions of Lemma 6 are satisfied (to do this use Lemma 2), and hence also the conclusion holds. Hence there exists a $\mathcal{J}$-almost disjoint family $\{X_\xi\colon \xi \in \lambda\} \subset \mathcal{P}(X)$ such that $X \in O[\mathcal{A} \cap X](X_\xi)$ for every $\xi \in \lambda$. Thus, by Lemma 3, the sets X_ξ, $\xi \in \lambda$, have the required properties.

The next Lemmas 7 and 8 as well as the end of the proof of Theorems 2 and 3 are almost rewritten proofs of Lemmas 2 and 3 on p. 55 in Prikry's paper [3]. For reader's convenience we give complete proofs of Lemmas 7 and 8.

LEMMA 7. *Let the assumptions of Theorem 2 be satisfied. Let $\mathcal{A}_0 \in \mathfrak{M}$ and let $X \subset S$ be such that $S \in O[\mathcal{A}](X)$ for every $\mathcal{A} \in \mathfrak{M}$. Then there exists a $Y \subset X$ such that $S \in O[\mathcal{A}_0](Y)$ and $S \in O[\mathcal{A}](X-Y)$ for every $\mathcal{A} \in \mathfrak{M}$.*

Proof. By Lemma 6* there exists an $\mathcal{J}$-almost disjoint family $\{X_\xi\colon \xi \in \lambda\} \subset \mathcal{P}(X)$ such that $O[\mathcal{A}_0](X) = O[\mathcal{A}_0](X_\xi)$ for every $\xi \in \lambda$. We claim that there is a $\xi \in \lambda$ such that $S \in O[\mathcal{A}](X - X_\xi)$ for every $\mathcal{A} \in \mathfrak{M}$. Suppose

that this is not so. Then we can find $Z \subset \lambda$ with $|Z| = \lambda$ and $\mathscr{A}_1 \in \mathfrak{M}$ such that $S \notin O[\mathscr{A}_1](X - X_\xi)$ for every $\xi \in Z$. Hence for all $\xi \in Z$ we can choose $A_\xi \subset X_\xi \cup (S - X)$ such that $A_\xi \in \mathscr{A}_1 - I(\mathscr{A}_1)$. Since $\operatorname{Sat}[\mathscr{A}_1 - I(\mathscr{A}_1)] \leqslant \varkappa$, $\lambda \geqslant \varkappa$, there are $\xi \neq \eta$ such that $A_\xi \cap A_\eta \in \mathscr{A}_1 - I(\mathscr{A}_1)$. Since

$$A_\xi \cap A_\eta \subset (S - X) \cup (X_\xi \cap X_\eta), \quad X_\xi \cap X_\eta \in \mathscr{I} \subset I(\mathscr{A}_1),$$

and

$$A_\xi \cap A_\eta \in \mathscr{A}_1 - I(\mathscr{A}_1),$$

we have $S \notin O[\mathscr{A}_1](X)$. So we have a contradiction. Hence we can set $Y = X_\xi$ for some ξ.

LEMMA 7'. *Under the assumptions as in Theorem* 3 *the conclusion of Lemma* 7 *holds.*

The proof is similar to that of Lemma 7 and we omit it.

LEMMA 8. *Under the assumptions as in Theorem* 2 *or in Theorem* 3 *there exist pairwise disjoint sets* $X(\mathscr{A})$, $\mathscr{A} \in \mathfrak{M}$, *such that* $S \in O[\mathscr{A}](X(\mathscr{A}))$ *for every* $\mathscr{A} \in \mathfrak{M}$.

Proof. Let $\mathfrak{M} = \{A_n : n \in \delta\}$, $\delta \leqslant \omega$, be an enumeration of $\mathfrak{M}$. Choose sets $X_n \subset S$ $(n \in \delta)$ by induction as follows. Let

$$X_n \subset S - \bigcup \{X_m : m < n\}, \quad S \in O[\mathscr{A}_n](X_n),$$

and

$$S \in O[\mathscr{A}](S - \bigcup \{X_m : m < n\}) \quad \text{for all } \mathscr{A} \in \mathfrak{M}.$$

This can be done by Lemmas 7 and 7'.

Proof of Theorems 2 and 3. It can easily be seen that the conclusion of the theorems is equivalent to the following:

For every $X \subset S$ *there exists an* $\mathscr{I}$*-almost disjoint family* $\{X_\xi : \xi \in \lambda\} \subset \mathscr{P}(X)$ *such that*

(***) *for every* $\xi \in \lambda$ *and every* $\mathscr{A} \in \mathfrak{M}$ *we have*

$$O[\mathscr{A}](X) = O[\mathscr{A}](X_\xi^-).$$

Using Lemmas 2 and 3 we can reduce the proof of the theorems to the case $X = S$. Let then $X = S$. We have to show that there is an $\mathscr{I}$-almost disjoint family $\{X_\xi : \xi \in \lambda\} \subset \mathscr{P}(S)$ such that for every $\xi \in \lambda$ and every $\mathscr{A} \in \mathfrak{M}$ we have $S \in O[\mathscr{A}](X_\xi)$. Let $X(\mathscr{A})$, for $\mathscr{A} \in \mathfrak{M}$, be such that $S \in O[\mathscr{A}](X(\mathscr{A}))$ for every $\mathscr{A} \in \mathfrak{M}$. By Lemma 8 such a family exists. Let, for every $\mathscr{A} \in \mathfrak{M}$, $\{X_\xi(\mathscr{A}) : \xi \in \lambda\}$ be an $\mathscr{I}$-almost disjoint family of subsets of $X(\mathscr{A})$ such that

$$O[\mathscr{A}](X(\mathscr{A})) = O[\mathscr{A}](X_\xi(\mathscr{A})) \quad \text{for every } \xi \in \lambda.$$

Such a family exists by Lemma 6*. We now set

$$X_\xi = \bigcup \{X_\xi(\mathscr{A}) : \mathscr{A} \in \mathfrak{M}\} \quad (\xi \in \lambda).$$

Then X_ξ ($\xi \in \lambda$) are as desired. This completes the proof of Theorems 2 and 3.

4. Remarks. We begin with the terminology as in [3]. We claim that Corollary 2 in [3], p. 56, is not true even if we consider only $M = \{\mu\}$, where μ is a two-valued measure. More precisely, we can show that the last sentence in Corollary 2 in [3] is not true. To see this let $\varkappa = X_0 \cup X_1$, $X_0 \cap X_1 = \emptyset$, $|X_0| \geqslant \omega_2$, and $|X_1| = \omega_1$. Put

$$\mathscr{I} = \{A \subset \varkappa\colon |A \cap X_1| \leqslant \omega\}.$$

Let μ be the measure on the σ-field

$$S(\mu) = \{Y \subset \varkappa\colon Y \in \mathscr{I} \text{ or } \varkappa - Y \in \mathscr{I}\}$$

such that $\mu(Y) = 0$ and $\mu(\varkappa - Y) = 1$ for every $Y \in \mathscr{I}$. We have $\varkappa > \omega_1$, but, evidently, there is no disjoint family $\{T_\xi\colon \xi \in \omega_2\}$ such that $\mu^*(T_\xi) = 1$ for every $\xi \in \omega_2$.

It should be noted that Corollary 2 in [3] is true and correctly proved if we assume additionally that all measures in M are ω_2-additive.

In connection with Corollary 2 let us formulate the following

PROBLEM. Let $\mathscr{I}$ be an ω_1-complete ideal on ω_2 such that $[\omega_2]^{<\omega_2} \subset \mathscr{I}$ and $\omega_2 \notin \mathscr{I}$. Is it true that there exists a pairwise disjoint family $\mathscr{R} \subset \mathscr{P}(\omega_2) - \mathscr{I}$ such that $|\mathscr{R}| = \omega_2$? **(P 1133)**

Now we use our terminology.

There are many open problems in [3] for measures. It is clear how to reformulate them for $\varkappa$-complete fields $\mathscr{A}$ such that $\mathrm{Sat}[\mathscr{A} - I(\mathscr{A})] \leqslant \varkappa$.

We do not know an asnwer to the following

PROBLEM. Can one replace ω_1 by $\varkappa^+$ in Theorem 1? **(P 1134)**

Our last remarks are connected with those of Rao in [4]. In [4] Rao wrote that he did not know of probability spaces with some property. The following corollary shows that it is possible to prove in ZFC + CH that there are no such spaces.

COROLLARY 2. *Assume that for every non-trivial ω_1-complete ideal $\mathscr{I}$ on 2^ω we have* $\mathrm{Sat}[\mathscr{P}(2^\omega) - \mathscr{I}] > \omega_1$ *(e.g., by fact* (b) *of Section* 1 *the assumption is satisfied if we assume* CH). *Let $\langle S, \mathscr{A}_i, \mu_i\rangle$ ($i \in \omega$) be a family of non-atomic probability spaces. Let, for every $i \in \omega$, μ_i^* be the outer measure induced by μ_i. Then for every $X \subset S$ there exists a disjoint family $\{X_\xi\colon \xi \in \omega_1\} \subset \mathscr{P}(X)$ such that for each $i \in \omega$ and each $\xi \in \omega_1$ we have*

$$\mu_i^*(X_\xi) = \mu_i^*(X).$$

For the proof we need the following lemma (cf. [9]):

LEMMA 9. *Let the assumption about 2^ω as in Corollary 2 be satisfied.*

Let $\langle S, \mathscr{A}, \mu\rangle$ *be a non-atomic complete probability space and let*

$$I(\mu) = \{X \subset S\colon \mu(X) = 0\}.$$

Then $\operatorname{Sat}[\mathscr{P}(X) - I(\mu)] > \omega_1$ *for every* $X \in \mathscr{P}(S) - I(\mu)$.

Proof. Let $X \in \mathscr{P}(S) - I(\mu)$ and let $\nu(B) = \mu^*(B)/\mu^*(X)$ for every $B \in \mathscr{A} \cap X$. It is easy to see that the probability space $\langle X, \mathscr{A} \cap X, \nu\rangle$ is also non-atomic. So we can assume, without loss of generality, that $X = S$. Since μ is non-atomic, there exists a disjoint family

$$\{Y_\xi\colon \xi \in 2^\omega\} \subset I(\mu)$$

such that $\bigcup\{Y_\xi\colon \xi \in 2^\omega\} = S$. Put

$$\mathscr{I} = \{Z \subset 2^\omega\colon \bigcup\{Y_\xi\colon \xi \in Z\} \in I(\mu)\}.$$

Clearly, $\mathscr{I}$ is a non-trivial ω_1-complete ideal on 2^ω. Hence there exists a disjoint family

$$\{Z_\xi\colon \xi \in \omega_1\} \subset \mathscr{P}(2^\omega) - \mathscr{I}.$$

Consequently, $\bigcup\{Y_\xi\colon \xi \in Z_\xi\} \notin I(\mu)$ for every $\xi \in \omega_1$ which implies

$$\operatorname{Sat}[\mathscr{P}(S) - I(\mu)] > \omega_1.$$

Proof of Corollary 2. We can assume, without loss of generality, that all measures μ_i $(i \in \omega)$ are complete and non-atomic. By the Sublemma and by Lemma 9 we infer that, for every $i \in \omega$, $I(\mathscr{A}_i) = \{Y \subset S\colon \mu_i(Y) = 0\}$ and $\operatorname{Sat}[\mathscr{P}(X) - I(\mathscr{A}_i)] > \omega_1$ for every $X \subset S$ with $X \notin I(\mathscr{A}_i)$. So for $\mathfrak{M} = \{\mathscr{A}_i\colon i \in \omega\}$ the assumption of Theorem 2 is satisfied. Let $\{X_\xi\colon \xi \in \omega_1\}$ be as in Theorem 2. It is clear that the sets X_ξ, $\xi \in \omega_1$, have the required properties.

It should be noted that Corollary 2 can also be obtained from the proof of Prikry's Theorem 2 in [3], p. 54.

Observe that if there exists a real-valued measurable cardinal $\varkappa \leqslant 2^\omega$, then, evidently, the conclusion of Corollary 2 does not hold.

By Theorem 2, fact (e) from Section 1 and by the Sublemma we have the following

COROLLARY 3. *Suppose that* CH *holds and that there is no two-valued measurable cardinal. Let* $\langle S, \mathscr{A}_i, \mu_i\rangle$ $(i \in \omega)$ *be a family of probability spaces such that, for every* $i \in \omega$, $[S]^{<\omega} \subset \mathscr{A}_i$ *and* $\mu_i(X) = 0$ *for all* $X \in [S]^{<\omega}$. *Let* $\langle S, \mathscr{T}_i\rangle$ $(i \in \omega)$ *be a family of* T_1*-topologies on* S *satisfying the second axiom of countability and without isolated points. Then for every* $X \subset S$ *there is a disjoint family* $\{X_\xi\colon \xi \in \omega_1\} \subset \mathscr{P}(X)$ *such that for each* $i \in \omega$ *and each* $\xi \in \omega_1$ *we have*

$$\mu_i^*(X_\xi) = \mu_i^*(X)$$

and, for every Borel set B, if $X_\xi - B$ is of first category, then so is $X - B$ (we understand Borel and category with respect to the topology $\mathcal{T}_t$).

Note that Corollary 3 generalizes Corollary 1 of [3].

Replace the first sentence in Corollary 3 by "Suppose that $|S|$ is less than the first weakly inaccessible cardinal". Then so changed Corollary 3 is still true (use fact (b) from Section 1).

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Added in proof (December 5, 1977). Recently, in a preprint *On saturated sets of ideals and Ulam's problem* Alan D. Taylor has obtained among others the following strengthening of Theorem 1:

THEOREM (Taylor). *Assume that for every non-trivial ω_1-complete ideal $\mathcal{I}$ on ω_1 we have*

$$\operatorname{Sat}[\mathcal{P}(\omega_1) - \mathcal{I} | [\omega_1]^{<\omega_1}] > \omega_2$$

(this holds if we assume, e.g., TH_{ω_1}). *Let $\mathfrak{M}$ be a family of ω_1-complete fields on ω_1 satisfying $|\mathfrak{M}| \leqslant \omega_1$ and such that*

$$[\omega_1]^{<\omega_1} \subset \mathcal{A}, \quad \mathcal{A} \neq \mathcal{P}(\omega_1) \quad \text{and} \quad \operatorname{Sat}[\mathcal{A} - I(\mathcal{A}) | [\omega_1]^{<\omega_1}] \leqslant \omega_2$$

for every $\mathcal{A} \in \mathfrak{M}$. Then

$$\operatorname{Sat}[P(\omega_1) - \bigcup \mathfrak{M} | [\omega_1]^{<\omega_1}] > \omega_2.$$

Also recently we have proved that in Taylor's theorem one can remove the assumption "$\operatorname{Sat}[\mathcal{A} - I(\mathcal{A}) | [\omega_1]^{<\omega_1}] \leqslant \omega_2$".

To prove this result we use a general lemma which claims that, in many cases, theorems for families of ideals, related to Ulam's problem on families of measures, are also true for families of arbitrary non-trivial fields of sets (the paper in preparation to Fundamenta Mathematicae). Then we use a particular case of Taylor's theorem.

REFERENCES

[1] H. J. Keisler and A. Tarski, *From accessible to inaccessible cardinals*, Fundamenta Mathematicae 53 (1964), p. 255-308.

[2] J. Popruženko, *Sur une propriété d'une classe de mesures abstraites*, Colloquium Mathematicum 4 (1957), p. 189-194.

[3] K. Prikry, *Kurepa's hypothesis and a problem of Ulam on families of measures*, Monatshefte für Mathematik 81 (1976), p. 41-57.

[3a] — *Kurepa's hypothesis and σ-complete ideals*, Proceedings of the American Mathematical Society 38 (1973), p. 617-620.

[4] B. V. Rao, Review # 5820, Mathematical Reviews 53 (1977), p. 823.

[5] W. Sierpiński, *Sur les ensembles partout de deuxième catégorie*, Fundamenta Mathematicae 22 (1934), p. 1-3.

[6] — *Sur une propriété des ensembles linéaires quelconques*, ibidem 23 (1934), p. 125-134.

[7] R. M. Solovay, *Real valued measurable cardinals*, p. 397-428 in *Axiomatic set theory*, Providence, R. I., 1971.

[8] E. Szpilrajn-Marczewski, *Sur certains invariants de l'opération* (A), Fundamenta Mathematicae 21 (1933), p. 229-235.

[9] S. Ulam, *Zur Masstheorie in der allgemeinen Mengenlehre*, ibidem 16 (1930), p. 140-150.

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