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# BOCHNER FLAT KÄHLERIAN MANIFOLDS

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## § 1. Introduction

Let $M$ be a Kählerian manifold with Kählerian metric $g$ and almost complex structure $J$, $\dim M = n$ $(= 2m)$. Denote by $\mathfrak{X}(M)$ the Lie algebra of vector fields on $M$ and by $\nabla$ the Riemannian connection of $M$. If $W, X \in \mathfrak{X}(M)$, then $R_{WX}$ stands for the curvature operator $[\nabla_W, \nabla_X] - \nabla_{[W,X]}$. Let $R_{WXYZ}$ indicate the value of the curvature tensor of $M$ on vector fields $W, X, Y, Z \in \mathfrak{X}(M)$. We choose the sign convention so that $R_{WXYZ} = g(R_{WX}Y, Z)$. Denote by $Q$, $\varrho$ and $\tau$ the Ricci operator, the Ricci tensor and the scalar curvature of $M$. So we have $g(QX, Y) = \varrho(X, Y) = \sum_i R_{E_iXYE_i}$ and $\tau = \sum_i \varrho(E_i, E_i)$, if $\{E_1, \ldots, E_n\}$ is an orthonormal frame of $M$.

The Bochner curvature tensor of $M$ is defined by (see [2], [9] and [6])

$$
\begin{aligned}
(1)\qquad B_{WXYZ} = R_{WXYZ} - \frac{1}{n+4}\{&g(X, Y)\varrho(W, Z) - g(W, Y)\varrho(X, Z) + \\
&+ g(W, Z)\varrho(X, Y) - g(X, Z)\varrho(W, Y) + g(JX, Y)\varrho(JW, Z) - \\
&- g(JW, Y)\varrho(JX, Z) + g(JW, Z)\varrho(JX, Y) - g(JX, Z)\varrho(JW, Y) - \\
&- 2g(JW, X)\varrho(JY, Z) - 2g(JY, Z)\varrho(JW, X)\} + \\
+ \frac{\tau}{(n+2)(n+4)}\{&g(X, Y)g(W, Z) - g(W, Y)g(X, Z) + \\
&+ g(JX, Y)g(JW, Z) - g(JW, Y)g(JX, Z) - 2g(JW, X)g(JY, Z)\}
\end{aligned}
$$

for any $W, X, Y, Z \in \mathfrak{X}(M)$. Certain aspects of the geometric meaning of this tensor are discussed by Blair [1] and Yano [8].

The Kählerian manifold $M$ is said to be *Bochner flat* if $B$ vanishes identically. This is always so for $n = 2$.

Any Kählerian manifold which is either of constant holomorphic sectional curvature or, locally, a product of two Kählerian manifolds of constant holomorphic sectional curvature $H > 0$ and $-H$, has vanishing Bochner curvature tensor and constant scalar curvature. And conversely (see Matsumoto and Tanno [4], Theorem 3), any Bochner flat Kählerian manifold with $\tau$ = constant is one of the above type.

In [7] Tachibana and Liu gave examples of non-compact Bochner flat Kählerian manifolds with non-constant scalar curvature. It seems to be that examples of Bochner flat Kählerian manifolds being compact and having $\tau \neq$ constant are not known. Concerning the existence of such manifolds, we shall prove the following two theorems, which is the aim of the presented paper.

THEOREM 1. *Let $M$ be a 4-dimensional compact Bochner flat Kählerian manifold. If the scalar curvature $\tau$ of $M$ is non-positive, then $\tau$ = constant and consequently either*

(a) *$M$ is of constant non-positive holomorphic sectional curvature, or*

(b) *$M$ is locally a product of a 2-sphere of constant curvature $K$ and a hyperbolic 2-space of constant curvature $-K$, with the natural Kählerian structures.*

THEOREM 2. *Let $M$ be a compact Bochner flat Kählerian manifold of dimension $n \geqslant 6$. If the scalar curvature $\tau$ of $M$ is non-positive and the square of the length of the Ricci tensor satisfies the inequality*

$$|\varrho|^2 \leqslant \frac{1}{n}\left\{1+\frac{2(n-2)(n+4)^2}{(n+2)^2(n-4)^2}\right\}\tau^2,$$

*then $\tau$ = constant and consequently either*

(a) *$M$ is of constant non-positive holomorphic sectional curvature, or*

(b) *$M$ is locally a product of two Kählerian manifolds of constant holomorphic sectional curvature $H > 0$ and $-H$.*

## § 2. Preliminary result

As it is well known, the Ricci tensor and the Ricci operator of a Kählerian manifold have the properties

$$\varrho(JX, JY) = \varrho(X, Y), \qquad QJ = JQ, \tag{2}$$

for any $Y, X \in \mathfrak{X}(M)$.

In [3] it is shown that the covariant derivative of the Ricci tensor of a Bochner flat Kählerian manifold is given by the formula

$$\begin{aligned}(2n+4)(\nabla_X\varrho)(Y, Z) = g(X, Y)Z\tau+g(X, Z)Y\tau+2g(Y, Z)X\tau-\\ -g(JX, Y)(JZ)\tau-g(JX, Z)(JY)\tau.\end{aligned} \tag{3}$$

In the sequel $\{E_1, \ldots, E_n\}$ will always represent a local orthonormal frame of $M$. Moreover, for a tensor field $T$ on $M$, $\nabla^2 T$ is the tensor field on $M$ defined by $\nabla^2_{XY} T = \nabla_X\nabla_Y T-\nabla_{\nabla_X Y} T$ for $X, Y \in \mathfrak{X}(M)$.

PROPOSITION. *For a Bochner flat Kählerian manifold of dimension $n \geqslant 4$, we have*

$$(4) \qquad 2n(n+2)g(Q^2X, Y)-2n\tau g(QX, Y)-2\{(n+2)\mathrm{tr}Q^2-\tau^2\}g(X, Y) = (n+4)\{-n\nabla^2_{XY}\tau+(\Delta\tau)g(X, Y)\},$$

*for any $X, Y \in \mathfrak{X}(M)$, where* tr *means the trace of the operator and $\Delta$ is the Laplace operator $\sum_i \nabla^2_{E_iE_i}$.*

*Proof.* Firstly from (3) we derive

$$(2n+4)(\nabla^2_{WX}\varrho)(Y, Z) = g(X, Y)\nabla^2_{WZ}\tau+g(X, Z)\nabla^2_{WY}\tau+2g(Y, Z)\nabla^2_{WX}\tau- -g(JX, Y)\nabla^2_{WJZ}\tau-g(JX, Z)\nabla^2_{WJY}\tau.$$

Therefore

$$(5) \qquad (2n+4)\sum_i(R_{E_iX}\varrho)(Y, E_i) = (2n+4)\sum_i\{(\nabla^2_{E_iX}\varrho)(Y, E_i)-(\nabla^2_{XE_i}\varrho)(Y, E_i)\} = -(n-1)\nabla^2_{XY}\tau-\nabla^2_{JXJY}\tau+(\Delta\tau)g(X, Y),$$

where we used the symmetry of $\nabla^2_{XY}\tau$.

On the other hand, we have in general

$$(6) \qquad \sum_i(R_{E_iX}\varrho)(Y, E_i) = -\sum_i\{\varrho(R_{E_iX}Y, E_i)+\varrho(Y, R_{E_iX}E_i)\} = -\sum_i R_{E_iXYQE_i}+g(Q^2X, Y).$$

But by $B=0$, (1) and (2) one gets

$$(n+4)\sum_i R_{E_iXYQE_i} = 4g(Q^2X, Y)+\frac{n}{n+2}\tau g(QX, Y)+ +(\mathrm{tr}Q^2-\tau^2/(n+2))g(X, Y),$$

which used in (6) gives

$$(n+4)\sum_i(R_{E_iX}\varrho)(Y, E_i) = ng(Q^2X, Y)-\frac{n}{n+2}\tau g(QX, Y)- -(\mathrm{tr}Q^2-\tau^2/(n+2))g(X, Y).$$

Comparing the last relation with (5), we obtain

$$2n(n+2)g(Q^2X, Y)-2n\tau g(QX, Y)-2\{(n+2)\mathrm{tr}Q^2-\tau^2\}g(X, Y) = (n+4)\{-(n-1)\nabla^2_{XY}\tau-\nabla^2_{JXJY}\tau+(\Delta\tau)g(X, Y)\},$$

which with the help of (2) leads to (4), completing the proof.

## § 3. Proof of Theorems 1 and 2

At first we prove that for a compact Bochner flat Kählerian manifold of dimension $n \geqslant 4$ it holds good

$$(7)\qquad \int \{n(n+2)\operatorname{tr}Q^3-2(n+1)\tau\operatorname{tr}Q^2+\tau^3\} = \frac{1}{4}(n+4)(n-2)\int|\operatorname{grad}\tau|^2 \geqslant 0,$$

where grad denotes the gradient and $\int$ the integral with respect to natural volume element on $M$.

Indeed, taking $X = E_i$, $Y = QE_i$ into (4) and summing over $i = 1, \dots, n$, we obtain

$$(8)\qquad 2n(n+2)\operatorname{tr}Q^3-4(n+1)\tau\operatorname{tr}Q^2+2\tau^3 = (n+4)\{-n\sum_i \nabla^2_{E_iQE_i}\tau+\tau(\Delta\tau)\}.$$

On the other hand, knowing the following facts $\operatorname{div}X = \sum_i g(\nabla_{E_i}X, E_i)$, where div means the divergence, $X\tau = g(X, \operatorname{grad}\tau)$, for any $X\in\mathfrak{X}(M)$, and $\sum_i(\nabla_{E_i}Q)E_i = \frac{1}{2}\operatorname{grad}\tau$, one can find

$$(9)\qquad \sum_i \nabla^2_{E_iQE_i}\tau = \operatorname{div}(Q\operatorname{grad}\tau)-\tfrac{1}{2}|\operatorname{grad}\tau|^2.$$

Moreover,

$$(10)\qquad \tau(\Delta\tau) = \tfrac{1}{2}\Delta(\tau^2)-|\operatorname{grad}\tau|^2.$$

Now integrating equality (8) on $M$ and using (9), (10) and Green's theorem, one gets (7).

In the sequel we shall consider the operator $S = Q-\tau I/n$ instead of the Ricci operator $Q$. By (2) it is obvious that $S$ commutes with $J$. So $S$ has $m$ $(= n/2)$ eigenvalues, say $a_1, \dots, a_m$, each of multiplicity two.

In [5] Okumura proved the following lemma.

LEMMA. *Let $a_i$ $(i = 1, \dots, m)$ be $m$ real numbers satisfying*

$$\sum_i a_i = 0 \quad \text{and} \quad \sum_i a_i^2 = k^2,$$

*for certain non-negative $k$. Then we have*

$$\Big|\sum_i a_i^3\Big| \leqslant \frac{m-2}{\sqrt{m(m-1)}}\,k^3.$$

As $\operatorname{tr}S = 0$, taking the eigenvalues of the operator $S$ for $a_i$ we find by the above lemma

$$(11)\qquad |\operatorname{tr}S^3| \leqslant \frac{n-4}{\sqrt{2n(n-2)}}(\operatorname{tr}S^2)^{3/2}.$$

Because of (11) and

$$\operatorname{tr}Q^2 = \operatorname{tr}S^2+\tau^2/n,$$

$$\operatorname{tr}Q^3 = \operatorname{tr}S^3+(3\tau/n)\operatorname{tr}S^2+\tau^3/n^2,$$

one can derive

$$n(n+2)\operatorname{tr}Q^3-2(n+1)\tau\operatorname{tr}Q^2+\tau^3 = n(n+2)\operatorname{tr}S^3+(n+4)\tau\operatorname{tr}S^2 \leqslant \frac{n(n+2)(n-4)}{\sqrt{2n(n-2)}}(\operatorname{tr}S^2)^{3/2}+(n+4)\tau\operatorname{tr}S^2. \tag{12}$$

To receive the assertion of Theorem 1 let us assume that $n = 4$ and $\tau \leqslant 0$. Then by (12) we see that the left-hand side of (7) must be non-positive. Therefore, (7) gives $\tau$ = constant. The remaining part of our assertion is obvious in virtue of the theorem of Matsumoto and Tanno quoted in the Introduction.

In a similar manner, using (12) and (7), one obtains the assertion of Theorem 2.

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