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ON ALGEBRAIC OPERATIONS IN BINARY ALGEBRAS

BY

S. FAJTLOWICZ (WROCŁAW)

We adopt the terminology of [2] and [3]. In particular, if $\mathfrak{A} = (A; \boldsymbol{F})$ is an abstract algebra, then by $S(\mathfrak{A})$ we denote the set of all integers n for which there exists an n-ary non-trivial algebraic operation in $\mathfrak{A}$ depending on every variable.

The notion of the set $S(\mathfrak{A})$ has been introduced by E. Marczewski. In [5] and [6] K. Urbanik solved Marczewski's problems concerning description of all possible sets $S(\mathfrak{A})$ in the class of all abstract algebras and also in the class of all symmetrical algebras.

The aim of this note is to solve analogous Marczewski's problem concerning binary algebras ([3], P 528).

Moreover, we give a description of the sets $S(\mathfrak{A})$ for semigroups and algebras with one binary fundamental operation.

Theorem 1 gives some necessary conditions for the sets $S(\mathfrak{A})$ in n-ary algebras but the full description of such sets is so far unknown (**P 677**).

0. Let $\boldsymbol{A}(\mathfrak{A})$ denote the set of all algebraic operations of an algebra $\mathfrak{A}$. Then it is easy to prove

THEOREM 0. *The family $\boldsymbol{A}(\mathfrak{A})$ is the smallest set B such that if an operation $g(x_1, \ldots, x_n)$ belongs to B and if f is a trivial or fundamental operation of the algebra $\mathfrak{A}$, then the operation $g\big(x_1, \ldots, x_{i-1}, f(y_1, \ldots, y_k), x_{i+1}, \ldots, x_n\big)$ also belongs to B.*

1. Let $\mathscr{P}$ be an additive family of non-empty subsets of a set X such that $X \in \mathscr{P}$. Let us denote by $s_n(x_1, \ldots, x_n)$ $(n = 1, 2, \ldots)$ the operation on the set $\mathscr{P}$ defined as follows: $s_n(x_1, \ldots, x_n) = x_1 \cup \ldots \cup x_n$ if $x_1, \ldots, x_n$ are pairwise disjoint and $s_n(x_1, \ldots, x_n) = X$ in the other case.

Moreover, let s_0 be the constant operation equal to X.

(i) *The operations s satisfy equations*

$$s_k\big(s_{n_1}(x_{11}, \ldots, x_{nn_1}), s_{n_2}(x_{21}, \ldots, x_{2n_i}), \ldots, s_{n_k}(x_{k1}, \ldots, x_{kn_k})\big)$$
$$= s_{n_1+n_2+\ldots+n_k}(x_{11}, \ldots, x_{1n_1}, x_{21}, \ldots, x_{2n_2}, \ldots, x_{k1}, \ldots, x_{kn_k}).$$

For any set $K \subseteq \omega$ let us denote by $\mathscr{P}_K$ an algebra $\big(\mathscr{P}; (s_n)_{n\in K}\big)$; if $K = \{n\}$, we write $\mathscr{P}_K = \mathscr{P}_n$.

From proposition (i) it follows easily that

(ii) *every algebraic operation in the algebra $\mathscr{P}_K$ depending on every variable is of the form $s_k(x_1, \ldots, x_k)$, $k = 0, 1, \ldots$*

Denoting by $\iota(\mathscr{P})$ the least upper bound of the numbers of disjoint sets belonging to $\mathscr{P}$, we have the proposition

(iii) *if* $\iota(\mathscr{P}_K) = n+1$, *then* $S(\mathscr{P}_K) \subseteq \{0, 2, 3, \ldots, n\}$ *for every* K.

In the case of $K = \{n\}$ we have

(iv) $S(\mathscr{P}_n) = \{m < \iota\colon m = n+k(n-1)\}$.

Let us denote by $\mathfrak{S}_n$ an algebra $(\{0,1\}^{n+1}-\{\emptyset\}; s_2)$. Then $\mathfrak{S}_n$ is a semigroup (such semigroups were used in [1]). In view of (iii) we have

(v) *for every* $n \geqslant 2$ *there exists a semigroup* $\mathfrak{S}$ *such that* $S(\mathfrak{S}) = \{0, 2, \ldots, n\}$.

Let $s(\mathfrak{A}) = \sup\{n\colon n \in S(\mathfrak{A})\}$. Then for the Cartesian product of algebras of the same type we have

(vi) $s(\mathfrak{A}\times\mathfrak{B}) \leqslant s(\mathfrak{A})+s(\mathfrak{B})$.

In [4] J. Płonka proved the inclusion $S(\mathfrak{D}_n) \subseteq \{2, 3, \ldots, n\}$ for *n-dimensional diagonal algebras* $\mathfrak{D}_n$, i.e. for algebras with one n-ary fundamental operation f satisfying equations $f(x, x, \ldots, x) = x$ and $f\big(f(x_{11}, \ldots, x_{nn}), \ldots, f(x_{n1}, \ldots, x_{nn})\big) = f(x_{11}, \ldots, x_{nn})$.

Two-dimensional diagonal algebra is a semigroup.

It is easy to show by means of (vi) that

(vii) *if* $2 \in S(\mathfrak{D}_2)$, *then* $S(\mathfrak{D}_2\times\mathfrak{S}_n) = \{1, 2, \ldots, n+1\}$.

Of course, there exists a semigroup $\mathfrak{P}$ for which $S(\mathfrak{P}) = \{1, 2\}$, e.g. the free semigroup in the class of semigroups satisfying equations $(xy)z = x(yz) = xz$. Hence from (vii) it follows that

(viii) *For every* $n \geqslant 2$ *there exists a semigroup* $\mathfrak{S}(n)$ *such that* $S\big(\mathfrak{S}(n)\big) = \{1, 2, \ldots, n\}$.

The fact that

(ix) *For every natural number* $n \geqslant 2$ *there exists a semigroup* $\mathfrak{C}_n$ *such that* $S(\mathfrak{C}_n) = \{0, 1, \ldots, n\}$

follows from (v) and from the following proposition:

(x) *Let* $\mathfrak{P}_n = (P; \cdot)$ *be a semigroup satisfying equations* $x^2 = x_1x_2\ldots x_n = 0$. *Then the algebra* $\mathfrak{C}_n = (P \cup \{a, b\}; \circ)$, *where*

$$x \circ y = \begin{cases} xy & \text{if } \{x, y\} \subseteq P, \\ b & \text{if } x = y = a, \\ c & \text{otherwise,} \end{cases}$$

is a semigroup satisfying equations $x^3 = x_1x_2\ldots x_n = 0$ *and* $1 \in S(\mathfrak{C}_n)$.

2. THEOREM 1. *Let $\mathfrak{A} = (A; F)$ be an n-ary algebra, $n \geqslant 2$. If for a natural number $p > 0$ there is $\{p+1, \ldots, p+n-1\} \cap S(\mathfrak{A}) = \emptyset$, then $S(\mathfrak{A}) \subseteq \{0, 1, \ldots, p\}$.*

Proof. In view of Theorem 0 it suffices to show that for every algebraic operation $g(x_1, \ldots, x_s)$ depending on at most p variables and every fundamental or trivial operation f the operation

$$h(x_1, \ldots, x_{i-1}, y_1, \ldots, y_k, x_{i+1}, \ldots, x_p) = g\big(x_1, \ldots, x_{i-1}, f(y_1, \ldots, y_k), x_{i+1}, \ldots, x_p\big)$$

depends on at most p variables (then our theorem follows by induction).

If g does not depend on the variable x_i, this is obvious. If g does, then h depends on at most $p+n-1$ variables (since f depends on at most n variables as being a fundamental or trivial operation).

In view of the assumptions of the theorem, we have $p+k \notin S(\mathfrak{A})$, for $1 \leqslant k < n$ and thus h depends on at most p variables, q.e.d.

3. THEOREM 2. *Let $2 \in S \leqslant \omega$. Then a binary algebra $\mathfrak{A}$ with the property $S(\mathfrak{A}) = S$ exists if and only if the following condition holds*

(b) *if $2 < n \notin S$, then $n+1 \notin S$.*

Proof. The necessity follows immediately from Theorem 1.

As to sufficiency Urbanik has shown that diagonal algebras can be treated as binary algebras ([5], p. 150). Hence, if S is of the form $\{2, 3, \ldots \ldots, n\}$, there exists a binary algebra $\mathfrak{A}$ such that $S(\mathfrak{A}) = S$. And if S is not of that form but still does satisfy (b), then the required algebra is constructed in section 1 (see propositions (iv), (v), (viii) and (iv)).

Thus Theorem 2 is proved.

THEOREM 3. *Let $2 \in S \subseteq \omega$. Then an algebra $\mathfrak{A}$ with the property $S(\mathfrak{A}) = S$ and possesing one fundamental binary operation only does exist if and only if S satisfies* (b) *and $S \neq \{2, 3, \ldots, n\}$, $n = 3, 4, \ldots$*

Proof. By virtue of section 1 and Theorem 2 all we have to show is that there exists no binary algebra $\mathfrak{A}$ with one fundamental operation and such that $S(\mathfrak{A}) = \{2, 3, \ldots, n\}$, $n = 3, 4, \ldots$

Urbanik has shown that if $S(\mathfrak{A}) = \{2, 3, \ldots, n\}$, then $\mathfrak{A}$ is a diagonal algebra ([5], p. 133, Theorem 3). J. Płonka proved that if g is an algebraic operation in the diagonal algebra $(D; f)$, then $(D; g)$ is also a diagonal algebra ([4], p. 309).

Thus, in the class of all diagonal algebras only two-dimensional ones can be treated as algebras with the only binary operation.

The description of the sets $S(\mathfrak{A})$ for semigroups follows from examples of section 1 and from Theorem 3. Namely

THEOREM 4. *If $\mathfrak{A}$ is an algebra with one binary fundamental operation only then there exists a semigroup S such that $S(\mathfrak{S}) = S(\mathfrak{A})$.*

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INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

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