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Scalar and density concomitants of tensor with the valence (1, 2) in a 2-dimensional space

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1. Introduction. Suppose we are given an abstract special purely differential geometric object [9] with a fibre $\mathfrak{M}$ and with the transformation formula:

$$\omega' = F(\omega, L); \qquad \omega \in \mathfrak{M},\ L \in \mathscr{L}^n_s. \tag{1.1}$$

The subset $\overline{\mathfrak{M}}$ of the fibre $\mathfrak{M}$ is called an *allowable set* [8] if the implication

$$\bigwedge_{\omega_1 \in \overline{\mathfrak{M}}} \bigwedge_{L \in \mathscr{L}^n_s} \big(\omega_1 \in \overline{\mathfrak{M}} \Rightarrow \omega_2 = F(\omega_1, L) \in \overline{\mathfrak{M}}\big) \tag{1.2}$$

holds.

An allowable set is the union sum of transitive fibres.

Suppose we are given the geometric object (1.1). Let us take the subobject $\overline{\omega}$ of the object ω with respect to the allowable set $\overline{\mathfrak{M}} \subset \mathfrak{M}$ [10]. Every concomitant of the subobject $\overline{\omega}$ is called the *relative concomitant* of the object ω with respect to the allowable set $\overline{\mathfrak{M}}$ [8].

The purpose of this paper is to determine all scalar concomitants of a tensor $t^{\nu}_{\lambda\mu}$ in the 2-dimensional space. The fibre of the tensor $t^{\nu}_{\lambda\mu}$ has been divided into allowable sets in which the relative scalar concomitants have been determined. In Section 7 there are given some remarks concerning the density concomitants of the tensor $t^{\nu}_{\lambda\mu}$.

2. The allowable sets of a tensor $t^{\nu}_{\lambda\mu}$ in a two-dimensional space. Suppose we are given the tensor $t^{\nu}_{\lambda\mu}$ at some point x_0 of the two-dimensional manifold of class C^1 with the transformation formula:

$$t^{\nu'}_{\lambda'\mu'} = A^{\lambda}_{\lambda'} A^{\mu}_{\mu'} A^{\nu'}_{\nu} t^{\nu}_{\lambda\mu} \qquad (\lambda, \mu, \nu = 1, 2;\ \lambda', \mu', \nu' = 1', 2'). \tag{2.1}$$

The whole space R^8 is the fibre $\mathfrak{M}$ of $t^{\nu}_{\lambda\mu}$. The group $\mathscr{L}^2_1$ acting in our problem is a group of non-singular square matrices of the second order.

In the sequel we introduce some division of the fibre $\mathfrak{M}$ of $t^{\nu}_{\lambda\mu}$ into allowable sets. In these sets we shall determine the relative scalar concomitants of $t^{\nu}_{\lambda\mu}$.

Let us consider two covariant vectors formed by the contraction of the tensor $t^{\nu}_{\lambda\mu}$:

$$\overset{1}{v}_{\lambda} \overset{\mathrm{df}}{=} t^{\mu}_{\lambda\mu}, \qquad \overset{2}{v}_{\lambda} \overset{\mathrm{df}}{=} t^{\mu}_{\mu\lambda}. \tag{2.2}$$

Now let us introduce the following notation:

$$\mathfrak{M}_2 = \{t^{\nu}_{\lambda\mu}\colon\ t^{\nu}_{\lambda\mu} \in \mathfrak{M},\ \det\|\overset{a}{v}_{\lambda}\| \neq 0\}, \tag{2.3}$$

$$\mathfrak{M}_{\varkappa 1} = \{t^{\nu}_{\lambda\mu}\colon\ t^{\nu}_{\lambda\mu} \in \mathfrak{M},\ \overset{1}{v} \neq 0,\ \overset{2}{v} = \varkappa \overset{1}{v}\}, \tag{2.4}$$

$$\mathfrak{M}_{10} = \{t^{\nu}_{\lambda\mu}\colon\ t^{\nu}_{\lambda\mu} \in \mathfrak{M},\ \overset{1}{v} = 0,\ \overset{2}{v} \neq 0\}, \tag{2.5}$$

$$\mathfrak{M}_{00} = \{t^{\nu}_{\lambda\mu}\colon\ t^{\nu}_{\lambda\mu} \in \mathfrak{M},\ \overset{1}{v} = \overset{2}{v} = 0\}. \tag{2.6}$$

The subsets $\mathfrak{M}_2$, $\mathfrak{M}_{\varkappa 1}$, $\mathfrak{M}_{10}$ and $\mathfrak{M}_{00}$ are the allowable sets of the fibre $\mathfrak{M}$ of $t^{\nu}_{\lambda\mu}$. On the basis of [4] and [6] it is known that all transitive fibres of the pair of vectors $\overset{1}{v}_{\lambda}$ and $\overset{2}{v}_{\lambda}$ are $\mathfrak{M}_2$, $\mathfrak{M}_{\varkappa 1}$, $\mathfrak{M}_{10}$ and $\mathfrak{M}_{00}$. Hence it follows that the fibre $\mathfrak{M}$ is the sum of the allowable sets:

$$\mathfrak{M} = \mathfrak{M}_2 \cup \mathfrak{M}_{\varkappa 1} \cup \mathfrak{M}_{10} \cup \mathfrak{M}_{00}. \tag{2.7}$$

3. The scalar concomitants of $t^{\nu}_{\lambda\mu}$ in $\mathfrak{M}_2$. In the case of the scalar concomitants of the tensor $t^{\nu}_{\lambda\mu}$ the functional equation for the required concomitant takes the form:

$$f(t^{\nu}_{\lambda\mu}) = f(t^{\nu}_{\lambda\mu} A^{\lambda}_{\lambda'} A^{\mu}_{\mu'} A^{\nu'}_{\nu}), \qquad t^{\nu}_{\lambda\mu} \in \mathfrak{M},\ A^{\lambda'}_{\lambda} \in \mathscr{L}^2_1. \tag{3.1}$$

Solving equation (3.1) when $t^{\nu}_{\lambda\mu} \in \mathfrak{M}_2$, we determine all relative scalar concomitants of $t^{\nu}_{\lambda\mu}$ in $\mathfrak{M}_2$. As A in equation (3.1) we take the matrix:

$$A \overset{\mathrm{df}}{=} \|A^{\varrho'}_{\lambda}\| = \|\overset{\varrho}{v}_{\lambda}\| \qquad (\lambda, \varrho = 1, 2);\ \varrho = \varrho'. \tag{3.2}$$

Then the elements of the inverse matrix $\overset{-1}{\underset{\sigma}{v}}{}^{a}$ are determined by the relations

$$\overset{\varrho}{v}_{a} \overset{-1}{\underset{\sigma}{v}}{}^{a} = \delta^{\varrho}_{\sigma}, \tag{3.3}$$

where $\delta^{\varrho}_{\sigma}$ are Kronecker symbols. After the substitution (3.2) equation (3.1) takes the form

$$f(t^{\nu}_{\lambda\mu}) = f(\overset{a}{\underset{\varrho\sigma}{\omega}}), \tag{3.4}$$

where $\overset{a}{\underset{\varrho\sigma}{\omega}}$ are scalars

$$\overset{a}{\underset{\varrho\sigma}{\omega}} = t^{\nu}_{\lambda\mu} \overset{-1}{\underset{\varrho}{v}}{}^{\lambda} \overset{-1}{\underset{\sigma}{v}}{}^{\mu} \overset{a}{v}_{\nu} \qquad (\varrho, \sigma, a = 1, 2). \tag{3.5}$$

After the transformation of the coordinate system determined by (3.2) the components of the vectors $\overset{1}{v}_{\lambda'}$ and $\overset{2}{v}_{\lambda'}$ become

$$\overset{1}{v}_{1'} = 1, \quad \overset{1}{v}_{2'} = 0; \quad \overset{2}{v}_{1'} = 0, \quad \overset{2}{v}_{2'} = 1. \tag{3.6}$$

On account of (3.6) among eight scalars of the form (3.5) only four $\underset{11}{\overset{1}{\omega}}$, $\underset{11}{\overset{2}{\omega}}$, $\underset{22}{\overset{1}{\omega}}$ and $\underset{22}{\overset{2}{\omega}}$ are essentially different. Thus we have proved the following

THEOREM 1. *The most general scalar concomitant of $t^{\nu}_{\lambda\mu}$ in $\mathfrak{M}_2$ is a function of the form*

$$f(t^{\nu}_{\lambda\mu}) = \varphi(\underset{11}{\overset{1}{\omega}}, \underset{11}{\overset{2}{\omega}}, \underset{22}{\overset{1}{\omega}}, \underset{22}{\overset{2}{\omega}}),$$

where $\underset{\varrho\sigma}{\overset{a}{\omega}}$ are defined in (3.5) *and φ is arbitrary.*

4. The scalar concomitants of $t^{\nu}_{\lambda\mu}$ in $\mathfrak{M}_{\varkappa 1}$. In $\mathfrak{M}_{\varkappa 1}$ we first of all determine some density $\mathfrak{g}$ formed from the tensor $t^{\nu}_{\lambda\mu}$ and later we divide $\mathfrak{M}_{\varkappa 1}$ with the aid of $\mathfrak{g}$ into the sets $\mathfrak{M}^1_{\varkappa 1}$ and $\mathfrak{M}^2_{\varkappa 1}$. We determine the most general form of the scalar concomitant in $\mathfrak{M}^1_{\varkappa 1}$. Next we divide the set $\mathfrak{M}^2_{\varkappa 1}$ into two allowable subsets in which we determine the relative scalar concomitants of $t^{\nu}_{\lambda\mu}$.

With this aim we define the following objects:

$$\mathfrak{a}^{\lambda} \overset{\mathrm{df}}{=} -\varepsilon^{\lambda\varrho}\overset{1}{v}_{\varrho}, \tag{4.1}$$

where $\varepsilon^{\lambda\varrho}$ are Ricci symbols.

The sign "$-$" in definition (4.1) is used for the simplification of the further calculations.

The object $\mathfrak{a}^{\lambda}$ is a non-vanishing contravariant vector density

$$\mathfrak{a}^{\lambda'} = J^{-1}A^{\lambda'}_{\lambda}\mathfrak{a}^{\lambda}, \tag{4.2}$$

where

$$J = \det[A^{\lambda'}_{\lambda}] \neq 0.$$

We now define the object $\mathfrak{h}^{\nu}$:

$$\mathfrak{h}^{\nu} \overset{\mathrm{df}}{=} t^{\nu}_{\lambda\mu}\mathfrak{a}^{\lambda}\mathfrak{a}^{\mu}, \tag{4.3}$$

which, as can easily be verified, is a contravariant vector density of weight 2:

$$\mathfrak{h}^{\nu'} = J^{-2}A^{\nu'}_{\nu}\mathfrak{h}^{\nu}. \tag{4.4}$$

Now, analogically to (4.1), we define the object $\mathfrak{h}_{\lambda}$:

$$\mathfrak{h}_{\lambda} \overset{\mathrm{df}}{=} -\varepsilon_{\lambda\varrho}\mathfrak{h}^{\varrho}. \tag{4.5}$$

The object $\mathfrak{h}_\lambda$ is a covariant vector density of weight 1:

$$\mathfrak{h}_{\lambda'} = J^{-1} A^{\lambda}_{\lambda'} \mathfrak{h}_\lambda. \tag{4.6}$$

Finally we define three objects, $\mathfrak{g}$, $\mathfrak{f}$ and $\mathfrak{w}$:

$$\mathfrak{g} \overset{\mathrm{df}}{=} \mathfrak{h}^\alpha \overset{1}{v}_\alpha, \tag{4.7}$$

$$\mathfrak{f} \overset{\mathrm{df}}{=} t^{\nu}_{\lambda\mu} \mathfrak{h}^\lambda \mathfrak{h}^\mu \overset{1}{v}_\nu, \tag{4.8}$$

$$\mathfrak{w} \overset{\mathrm{df}}{=} t^{\nu}_{\lambda\mu} \mathfrak{h}^\lambda \mathfrak{h}^\mu \mathfrak{h}_\nu. \tag{4.9}$$

It is easy to see that the objects $\mathfrak{g}$ and $\mathfrak{f}$ are W-densities and $\mathfrak{w}$ is a G-density, their weights being two, four and five, respectively:

$$\mathfrak{g}' = J^{-2}\mathfrak{g}, \quad \mathfrak{f}' = J^{-4}\mathfrak{f}, \quad \mathfrak{w}' = J^{-5}\mathfrak{w}. \tag{4.10}$$

Now we divide the allowable set $\mathfrak{M}_{\varkappa 1}$ into two allowable subsets $\mathfrak{M}^1_{\varkappa 1}$ and $\mathfrak{M}^2_{\varkappa 1}$:

$$\mathfrak{M}^1_{\varkappa 1} = \{t^{\nu}_{\lambda\mu}:\ t^{\nu}_{\lambda\mu} \in \mathfrak{M}_{\varkappa 1},\ \mathfrak{g} \neq 0\}, \tag{4.11}$$

$$\mathfrak{M}^2_{\varkappa 1} = \{t^{\nu}_{\lambda\mu}:\ t^{\nu}_{\lambda\mu} \in \mathfrak{M}_{\varkappa 1},\ \mathfrak{g} = 0\}. \tag{4.12}$$

We shall determine the relative scalar concomitants of the tensor $t^{\nu}_{\lambda\mu}$ in these sets.

We consider the case $t^{\nu}_{\lambda\mu} \in \mathfrak{M}^1_{\varkappa 1}$. Let us take the transformation of the coordinate system, the parameters of which are determined as follows:

$$A = \|A^{\lambda'}_{\lambda}\| \overset{\mathrm{df}}{=} \frac{1}{\mathfrak{g}} \begin{Vmatrix} \overset{1}{v}_1 & \overset{1}{v}_2 \\ \mathfrak{h}_1 & \mathfrak{h}_2 \end{Vmatrix}. \tag{4.13}$$

The inverse matrix is of the form

$$A^{-1} = \|A^{\lambda}_{\lambda'}\| = \begin{Vmatrix} \mathfrak{h}^1 & \mathfrak{a}^1 \\ \mathfrak{h}^2 & \mathfrak{a}^2 \end{Vmatrix}. \tag{4.14}$$

After such a change of the coordinate system the components of vectors $\overset{1}{v}_{\lambda'}$ and $\overset{2}{v}_{\lambda'}$ take the form

$$\overset{1}{v}_{1'} = \mathfrak{g}, \quad \overset{1}{v}_{2'} = 0; \quad \overset{2}{v}_{1'} = \varkappa\mathfrak{g}, \quad \overset{2}{v}_{2'} = 0. \tag{4.15}$$

On the basis of (4.15) we infer as a corollary that after the transformation of the coordinate system determined by (4.13) four components of $t^{\nu}_{\lambda\mu}$, namely $t^{1'}_{1'2'}, t^{2'}_{1'2'}, t^{1'}_{2'1'}$ and $t^{2'}_{2'1'}$, are expressed by the components $t^{1'}_{1'1'}$ and $t^{2'}_{2'2'}$.

So the functional equation of the required scalar concomitants in $\mathfrak{M}^1_{\varkappa 1}$ can be written in the form

$$f(t^{\nu}_{\lambda\mu}) = \varphi\left(\varkappa, \mathfrak{g}, \frac{\mathfrak{f}}{\mathfrak{g}}, \frac{\mathfrak{w}}{\mathfrak{g}}\right), \tag{4.16}$$

where

$$\begin{aligned}
t^{1'}_{1'1'} &= \frac{1}{\mathfrak{g}} t^{\nu}_{\lambda\mu}\mathfrak{h}^{\lambda}\mathfrak{h}^{\mu}\overset{1}{v}_{\nu} = \frac{\mathfrak{f}}{\mathfrak{g}},\\
t^{2'}_{1'1'} &= \frac{1}{\mathfrak{g}} t^{\nu}_{\lambda\mu}\mathfrak{h}^{\lambda}\mathfrak{h}^{\mu}\mathfrak{h}_{\nu} = \frac{\mathfrak{w}}{\mathfrak{g}},\\
t^{1'}_{2'2'} &= \frac{1}{\mathfrak{g}} t^{\nu}_{\lambda\mu}\mathfrak{a}^{\lambda}\mathfrak{a}^{\mu}\overset{1}{v}_{\nu} = 1,\\
t^{2'}_{2'2'} &= \frac{1}{\mathfrak{g}} t^{\nu}_{\lambda\mu}\mathfrak{a}^{\lambda}\mathfrak{a}^{\mu}\mathfrak{h}_{\nu} = 0.
\end{aligned} \tag{4.17}$$

Let us now examine two cases: $\mathfrak{w} \neq 0$ and $\mathfrak{w} = 0$. When $\mathfrak{w} \neq 0$, let us take the transformation of the coordinate system of the form $\xi^{\lambda''} = \xi^{\lambda''}(\xi^{\lambda'})$ for which

$$\det\|A^{\lambda''}_{\lambda'}\| = \operatorname{sgn}\mathfrak{w}\,|\mathfrak{g}|^{1/2}. \tag{4.18}$$

holds.

After such a change of the coordinate system the right-hand side of (4.16) is of the form

$$\varphi\left(\varkappa, \operatorname{sgn}\mathfrak{g}, \frac{\mathfrak{f}\operatorname{sgn}\mathfrak{g}}{\mathfrak{g}^2}, \frac{|\mathfrak{w}|\operatorname{sgn}\mathfrak{g}}{|\mathfrak{g}|^{5/2}}\right), \tag{4.19}$$

and it represents the general scalar concomitants.

In the case $\mathfrak{w} = 0$ it is enough to take the transformation of the coordinate system for which

$$\det\|A^{\lambda''}_{\lambda'}\| = |\mathfrak{g}|^{1/2} \tag{4.20}$$

holds.

After such a change of the coordinate system we have

$$\varphi\left(\varkappa, \mathfrak{g}, \frac{\mathfrak{f}}{\mathfrak{g}}\right) = \varphi\left(\varkappa, \operatorname{sgn}\mathfrak{g}, \frac{\mathfrak{f}\operatorname{sgn}\mathfrak{g}}{\mathfrak{g}^2}\right). \tag{4.21}$$

We state a posteriori that (4.21) is a particular case of formula (4.13). Thus we have proved the following

THEOREM 2. *The most general scalar concomitant of $t^{\nu}_{\lambda\mu}$ in $\mathfrak{M}^1_{\varkappa 1}$ is an arbitrary function of the form*

$$f(t^{\nu}_{\lambda\mu}) = \varphi\left(\varkappa, \operatorname{sgn}\mathfrak{g}, \frac{\mathfrak{f}\operatorname{sgn}\mathfrak{g}}{\mathfrak{g}^2}, \frac{|\mathfrak{w}|\operatorname{sgn}\mathfrak{g}}{|\mathfrak{g}|^{5/2}}\right),$$

where $\varkappa$, $\mathfrak{g}$, $\mathfrak{f}$ *and* $\mathfrak{w}$ *are defined in* (2.4), (4.7), (4.8) *and* (4.9).

In the case $t^{\nu}_{\lambda\mu} \in \mathfrak{M}^2_{\varkappa 1}$ we have the following relation:

$$\mathfrak{g} = t^{\nu}_{\lambda\mu}\mathfrak{a}^{\lambda}\mathfrak{a}^{\mu}\overset{1}{v}_{\nu} = 0. \tag{4.22}$$

If we write

$$a_{\lambda\mu} \overset{\mathrm{df}}{=} t^{\nu}_{\lambda\mu}\overset{1}{v}_{\nu}, \tag{4.23}$$

then relation (4.22) becomes:

$$a_{\lambda\mu}\mathfrak{a}^{\lambda}\mathfrak{a}^{\mu} = 0. \tag{4.24}$$

After a set of elementary but long calculations [12] it can be shown that if $\mathfrak{a}^{\lambda}$ (see (4.1)) fulfils (4.24), then the tensor $a_{\lambda\mu}$ is a symmetric one:

$$a_{\mu\lambda} = a_{\lambda\mu}. \tag{4.25}$$

Now we divide the set $\mathfrak{M}^2_{\varkappa 1}$ into two allowable subsets, $\mathfrak{M}^{21}_{\varkappa 1}$ and $\mathfrak{M}^{22}_{\varkappa 1}$:

$$\mathfrak{M}^{21}_{\varkappa 1} = \{t^{\nu}_{\lambda\mu}\colon\ t^{\nu}_{\lambda\mu} \in \mathfrak{M}^2_{\varkappa 1},\ \det[a_{\lambda\mu}] \neq 0\}, \tag{4.26}$$

$$\mathfrak{M}^{22}_{\varkappa 1} = \{t^{\nu}_{\lambda\mu}\colon\ t^{\nu}_{\lambda\mu} \in \mathfrak{M}^2_{\varkappa 1},\ \det[a_{\lambda\mu}] = 0\}. \tag{4.27}$$

In $\mathfrak{M}^{21}_{\varkappa 1}$ we can construct the tensor $a^{\lambda\mu}$ inverse to the tensor $a_{\lambda\mu}$, i.e. satisfying the relations ([2], p. 107):

$$a^{\alpha\varrho}a_{\varrho\beta} = \delta^{\alpha}_{\beta}. \tag{4.28}$$

Let us consider the following tensor in $\mathfrak{M}^{21}_{\varkappa 1}$:

$$b_{\lambda\mu} \overset{\mathrm{df}}{=} 2t^{[1}_{\lambda[1}t^{2]}_{2]\mu}. \tag{4.29}$$

The tensor $b_{\lambda\mu}$ is also a symmetric one. Let us form the transvection of $b_{\lambda\mu}$ and $a^{\lambda\mu}$:

$$C^{\mu}_{\lambda} \overset{\mathrm{df}}{=} b_{\lambda\varrho}a^{\varrho\mu}. \tag{4.30}$$

As can easily be verified, the trace of tensor c^{μ}_{λ} vanishes on the whole $\mathfrak{M}^{21}_{\varkappa 1}$. Let us write

$$\tau \overset{\mathrm{df}}{=} \det[c^{\mu}_{\lambda}]. \tag{4.31}$$

THEOREM 3. *The most general scalar concomitant of* $t^{\nu}_{\lambda\mu}$ *in* $\mathfrak{M}^{21}_{\varkappa 1}$ *is an arbitrary function of the form*

$$f(t^{\nu}_{\lambda\mu}) = \varphi(\varkappa, \tau),$$

where τ *is defined by* (4.13) *and* $\varkappa$ *by* (2.4).

Proof. Since $\mathfrak{M}^{21}_{\varkappa 1} \subset \mathfrak{M}^2_{\varkappa 1} \subset \mathfrak{M}_{\varkappa 1}$, the relations

$$\begin{gathered} \overset{2}{v}_{\lambda} = \varkappa \overset{1}{v}_{\lambda}, \qquad a_{\lambda\mu}\mathfrak{a}^{\lambda}\mathfrak{a}^{\mu} = 0, \\ \det[a_{\lambda\mu}] \neq 0, \qquad a_{\mu\lambda} = a_{\lambda\mu}. \end{gathered} \tag{4.32}$$

hold.

Let us consider a coordinate system in which the components of the non-vanishing covariant vector $\overset{1}{v}_\lambda$ have the following values:

$$\overset{1}{v}_1 \overset{*}{=} 1, \quad \overset{1}{v}_2 \overset{*}{=} 0 \tag{4.33}$$

(the sign $\overset{*}{=}$ means that the equality does not hold in each allowable coordinate system). Expressing the components of vectors $\overset{1}{v}_\lambda$ and $\overset{2}{v}_\lambda$, the vector density $\mathfrak{a}^\lambda$ and the components of tensor $a_{\lambda\mu}$ occurring in (4.32) by the components of tensor $t^\nu_{\lambda\mu}$, we can write relations (4.32) using the coordinate system determined by (4.33) in the form

$$\begin{aligned} t^2_{12} &\overset{*}{=} 1 - t^1_{11}, \quad & t^1_{21} &\overset{*}{=} t^1_{12}, \quad & t^1_{12} &\overset{*}{\neq} 0, \\ t^2_{21} &\overset{*}{=} \varkappa - t^1_{11}, \quad & t^1_{22} &\overset{*}{=} 0, \quad & t^2_{22} &\overset{*}{=} -t^1_{12}. \end{aligned} \tag{4.34}$$

Let us now take into consideration a transformation of the coordinate system which preserves relations (4.33) and (4.37) [13]. Let us denote the parameters of such a transformation by

$$\|A^{\lambda'}_\lambda\| = \begin{Vmatrix} 1 & 0 \\ \gamma & \delta \end{Vmatrix}, \quad \delta \neq 0. \tag{4.35}$$

Substituting in (4.35) the values of γ and δ,

$$\gamma = \tfrac{1}{2} t^1_{11}, \quad \delta = t^1_{12}, \tag{4.36}$$

we get

$$\begin{aligned} t^{1'}_{1'1'} &= 0, \quad & t^{2'}_{1'1'} &= -\tau, \quad & t^{1'}_{1'2'} &= 1, \quad & t^{2'}_{1'2'} &= 1, \\ t^{1'}_{2'1'} &= 1, \quad & t^{2'}_{2'1'} &= \varkappa, \quad & t^{1'}_{2'2'} &= 0, \quad & t^{2'}_{2'2'} &= -1. \end{aligned} \tag{4.37}$$

Thus our theorem has been proved.

In the set $\mathfrak{M}^{22}_{\varkappa 1}$ (i.e. in the case where $\det[a_{\lambda\mu}] = 0$) we can show [12] that the tensor $a_{\lambda\mu}$ is proportional to the product $\overset{1}{v}_\lambda \overset{1}{v}_\mu$, where $\overset{1}{v}_\lambda$ is the non-vanishing vector determined by (2.2),

$$a_{\lambda\mu} = \eta \overset{1}{v}_\lambda \cdot \overset{1}{v}_\mu, \tag{4.38}$$

where η is the scalar.

Analogically to the proof of Theorem 3 we can prove [12] the following

THEOREM 4. *The most general scalar concomitant of* $t^\nu_{\lambda\mu}$ *in* $\mathfrak{M}^{22}_{\varkappa 1}$ *is an arbitrary function of the form*

$$f(t^\nu_{\lambda\mu}) = \varphi(\varkappa, \eta),$$

where $\varkappa$ *is defined by* (2.4) *and* η *is determined by* (4.38).

5. The scalar concomitants of $t^{\nu}_{\lambda\mu}$ in $\mathfrak{M}_{10}$. Besides the vectors $\overset{1}{v}_{\lambda} = 0$ and $\overset{2}{v}_{\lambda} \neq 0$ determined in (2.2), we form from the tensor $t^{\nu}_{\lambda\mu}$ the following auxiliary concomitants:

$$\overset{2}{\mathfrak{a}}{}^{\lambda} \overset{\mathrm{df}}{=} -\varepsilon^{\lambda\varrho}\overset{2}{v}_{\varrho}, \tag{5.1}$$

$$\overset{2}{\mathfrak{h}}{}^{\nu} \overset{\mathrm{df}}{=} t^{\nu}_{\lambda\mu}\overset{2}{\mathfrak{a}}{}^{\lambda}\overset{2}{\mathfrak{a}}{}^{\mu}, \tag{5.2}$$

$$\overset{2}{\mathfrak{h}}_{\lambda} \overset{\mathrm{df}}{=} -\varepsilon_{\lambda\varrho}\overset{2}{\mathfrak{h}}{}^{\varrho}, \tag{5.3}$$

$$\overset{2}{\mathfrak{g}} \overset{\mathrm{df}}{=} \overset{2}{\mathfrak{h}}{}^{\alpha}\overset{2}{v}_{\alpha}, \tag{5.4}$$

$$\overset{2}{\mathfrak{f}} \overset{\mathrm{df}}{=} t^{\nu}_{\lambda\mu}\overset{2}{\mathfrak{h}}{}^{\lambda}\overset{2}{\mathfrak{h}}{}^{\mu}\overset{2}{v}_{\nu}, \tag{5.5}$$

$$\overset{2}{\mathfrak{w}} \overset{\mathrm{df}}{=} t^{\nu}_{\lambda\mu}\overset{2}{\mathfrak{h}}{}^{\lambda}\overset{2}{\mathfrak{h}}{}^{\mu}\overset{2}{\mathfrak{h}}_{\nu}. \tag{5.6}$$

The transformation formulas of objects (5.1)-(5.6) are the same as the transformation formulas of the suitable objects determined in Section 4 in (4.1), (4.3), (4.5), (4.7), (4.8) and (4.9) and denoted there by the same letters without the index 2 above.

The index 2 above denotes that instead of the vector $\overset{1}{v}_{\lambda}$ we take the vector $\overset{2}{v}_{\lambda}$.

Now we divide the allowable set $\mathfrak{M}_{10}$ into two allowable subsets $\mathfrak{M}^{1}_{10}$ and $\mathfrak{M}^{2}_{10}$,

$$\mathfrak{M}^{1}_{10} = \{t^{\nu}_{\lambda\mu}\colon\ t^{\nu}_{\lambda\mu} \in \mathfrak{M}_{10},\ \overset{2}{\mathfrak{g}} \neq 0\}, \tag{5.7}$$

$$\mathfrak{M}^{2}_{10} = \{t^{\nu}_{\lambda\mu}\colon\ t^{\nu}_{\lambda\mu} \in \mathfrak{M}_{10},\ \overset{2}{\mathfrak{g}} = 0\}. \tag{5.8}$$

Then we can prove the following

THEOREM 5. *The most general scalar concomitant of $t^{\nu}_{\lambda\mu}$ in $\mathfrak{M}^{1}_{10}$ is an arbitrary function of the form*

$$f(t^{\nu}_{\lambda\mu}) = \varphi\left(\operatorname{sgn}\overset{2}{\mathfrak{g}},\ \frac{\overset{2}{\mathfrak{f}}\operatorname{sgn}\overset{2}{\mathfrak{g}}}{\overset{2}{\mathfrak{g}}{}^{2}},\ \frac{|\overset{2}{\mathfrak{w}}|\operatorname{sgn}\overset{2}{\mathfrak{g}}}{|\overset{2}{\mathfrak{g}}|^{5/2}}\right),$$

where $\overset{2}{\mathfrak{g}}$, $\overset{2}{\mathfrak{f}}$ and $\overset{2}{\mathfrak{w}}$ are defined by (5.4), (5.5) *and* (5.6).

The proof of this theorem is analogous to that of Theorem 2.

In the allowable set $\mathfrak{M}^{2}_{10}$ the following relation holds:

$$\overset{2}{\mathfrak{g}} = t^{\nu}_{\lambda\mu}\overset{2}{\mathfrak{a}}{}^{\lambda}\overset{2}{\mathfrak{a}}{}^{\mu}\overset{2}{v}_{\nu} = 0. \tag{5.9}$$

Let us write

$$a^{2}_{\lambda\mu} \overset{\mathrm{df}}{=} t^{\nu}_{\lambda\mu} \overset{2}{v}_{\nu}. \tag{5.10}$$

It is easy to show that from condition (5.9) follows the symmetric

$$\overset{2}{a}_{\mu\lambda} = \overset{2}{a}_{\lambda\mu}. \tag{5.11}$$

Now we divide the allowable set $\mathfrak{M}^{2}_{10}$ into two further allowable subsets $\mathfrak{M}^{21}_{10}$ and $\mathfrak{M}^{22}_{10}$,

$$\mathfrak{M}^{21}_{10} = \{t^{\nu}_{\lambda\mu}\colon\ t^{\nu}_{\lambda\mu} \in \mathfrak{M}^{2}_{10},\ \det[\overset{2}{a}_{\lambda\mu}] \neq 0\}, \tag{5.12}$$

$$\mathfrak{M}^{22}_{10} = \{t^{\nu}_{\lambda\mu}\colon\ t^{\nu}_{\lambda\mu} \in \mathfrak{M}^{2}_{10},\ \det[\overset{2}{a}_{\lambda\mu}] = 0\}. \tag{5.13}$$

In the set $\mathfrak{M}^{21}_{10}$ we can form the tensor $\overset{2}{a}{}^{\lambda\mu}$ inverse to the tensor $\overset{2}{a}_{\lambda\mu}$. Still, let us consider in this set the tensor $b_{\lambda\mu}$ defined by (4.29). Let us form the transvection of $b_{\lambda\mu}$ and $\overset{2}{a}{}^{\lambda\mu}$.

$$\overset{2}{c}{}^{\mu}_{\lambda} \overset{\mathrm{df}}{=} b_{\lambda\varrho} \overset{2}{a}{}^{\varrho\mu}. \tag{5.15}$$

Let us put, further,

$$\overset{2}{\tau} \overset{\mathrm{df}}{=} \det[\overset{2}{c}{}^{\mu}_{\lambda}]. \tag{5.16}$$

THEOREM 6. *The most general scalar concomitant of $t^{\nu}_{\lambda\mu}$ in $\mathfrak{M}^{21}_{10}$ is an arbitrary function of the form*

$$f(t^{\nu}_{\lambda\mu}) = \varphi(\overset{2}{\tau}),$$

where $\overset{2}{\tau}$ is defined by (5.15).

We omit the proof, which is analogous to that of Theorem 3.

In the allowable set $\mathfrak{M}^{22}_{10}$ the relation

$$\overset{2}{a}_{\lambda\mu} = \overset{2}{\eta} \overset{2}{v}_{\lambda} \overset{2}{v}_{\mu} \tag{5.16}$$

holds, where $\overset{2}{v}_{\lambda}$ is the vector defined in (2.2), $\overset{2}{a}_{\lambda\mu}$ the tensor defined in (5.11) and $\overset{2}{\eta}$ some scalar.

THEOREM 7. *The most general scalar concomitant of $t^{\nu}_{\lambda\mu}$ in $\mathfrak{M}^{22}_{10}$ is an arbitrary function of the form*

$$f(t^{\nu}_{\lambda\mu}) = \varphi(\overset{2}{\eta}),$$

where $\overset{2}{\eta}$ is determined by (5.16).

6. The scalar concomitants of $t^{\nu}_{\lambda\mu}$ in $\mathfrak{M}_{00}$. We shall reduce the determination of the scalar concomitants of $t^{\nu}_{\lambda\mu}$ in the set $\mathfrak{M}_{00}$ to the determination of the so-called transitive domains [3]. It will be shown that the allowable set $\mathfrak{M}_{00}$ falls into five transitive domains. On the basis of (2.2) and (2.6) the tensor $t^{\nu}_{\lambda\mu}$ belonging to the allowable set $\mathfrak{M}_{00}$ has only four essential components, $t^1_{11}, t^2_{11}, t^1_{22}$ and t^2_{22}; the remaining components can be expressed by t_{11} and t^2_{22}. We have namely:

$$t^1_{12} = -t^2_{22}, \quad t^2_{12} = -t^1_{11}, \quad t^1_{21} = -t^2_{22}, \quad t^2_{21} = -t^1_{11}. \tag{6.1}$$

For brevity we adopt the following notation:

$$z_1 = t^1_{11}, \quad z_2 = t^2_{11}, \quad z_3 = t^1_{22}, \quad z_4 = t^2_{22}. \tag{6.2}$$

Let us consider the tensor $b_{\lambda\mu}$ defined by (4.29):

$$b_{\lambda\mu} = 2t^{[1}_{\lambda[1}t^{2]}_{2]\mu}. \tag{6.3}$$

Now we will express the components of $b_{\lambda\mu}$ by the essential components of tensor $t^{\nu}_{\lambda\mu}$ taken from the allowable set $\mathfrak{M}_{00}$,

$$\begin{aligned} b_{11} &= z_2z_4 - z_1^2, \\ b_{12} &= b_{21} = \tfrac{1}{2}(z_1z_4 - z_2z_3), \\ b_{22} &= z_1z_3 - z_4^2. \end{aligned} \tag{6.4}$$

The transitive domains of the symmetric tensor $b_{\lambda\mu}$ determine the allowable subsets of $\mathfrak{M}_{00}$.

The transivity domains for an arbitrary twice covariant tensor in the two-dimensional space have been determined by E. Siwek in the paper [11]. Denoting the coordinates of the symmetric twice covariant tensor by x_{ik} $(i, k = 1, 2)$, we have the following transitive domains for it:

$$\begin{aligned} &\mathfrak{N}_1\colon\ x_{11} = x_{12} = x_{22} = 0, \\ &\mathfrak{N}_2\colon\ x_{11} \geqslant 0, \quad x_{22} \geqslant 0, \quad x_{11}^2 + x_{22}^2 > 0, \quad x_{11}x_{22} - x_{12}^2 = 0, \\ &\mathfrak{N}_3\colon\ x_{11} \leqslant 0, \quad x_{22} \leqslant 0, \quad x_{11}^2 + x_{22}^2 > 0, \quad x_{11}x_{22} - x_{12}^2 = 0, \\ &\mathfrak{N}_4\colon\ x_{11} > 0, \quad x_{22} > 0, \quad x_{11}x_{22} - x_{12}^2 > 0, \\ &\mathfrak{N}_5\colon\ x_{11} < 0, \quad x_{22} < 0, \quad x_{11}x_{22} - x_{12}^2 > 0, \\ &\mathfrak{N}_6\colon\ x_{11}x_{22} - x_{12}^2 < 0. \end{aligned} \tag{6.5}$$

If can be proved that the symmetric tensor $b_{\lambda\mu}$ defined by (6.3) for the tensor $t^{\nu}_{\lambda\mu}$ taken from the allowable set $\mathfrak{M}_{00}$ cannot belong to the transitive domains $\mathfrak{N}_2$ and $\mathfrak{N}_4$.

If only $b_{\lambda\mu} \in \mathfrak{N}_2$, then, according to (6.5), the components of the tensor $b_{\lambda\mu}$ ought to satisfy the following relations:

$$b_{11} \geqslant 0, \quad b_{22} \geqslant 0, \quad b_{11}^2 + b_{22}^2 > 0, \quad b_{11}b_{22} - b_{12}^2 = 0. \tag{6.6}$$

It follows from (6.6) that the tensor $b_{\lambda\mu}$ can be put in the canonical form. Let

$$b_{11} \stackrel{*}{=} 1, \quad b_{12} \stackrel{*}{=} 0, \quad b_{21} \stackrel{*}{=} 0, \quad b_{22} \stackrel{*}{=} 0. \tag{6.7}$$

Then on the basis of (6.4) we get the following system of equations:

$$z_2 z_4 - z_1^2 \stackrel{*}{=} 1, \quad z_1 z_4 - z_2 z_3 \stackrel{*}{=} 0, \quad z_1 z_3 - z_4^2 \stackrel{*}{=} 0. \tag{6.8}$$

It can easily be shown that this system is contradictory. Analogically we can show that the tensor $b_{\lambda\mu}$ cannot fulfil the conditions which determine the transitive domains $\mathfrak{N}_4$.

In the case where $b_{\lambda\mu} \in \mathfrak{N}_1$ we get, according to (6.3) and (6.4), the following system of equations:

$$z_2 z_4 - z_1^2 = 0, \quad z_1 z_4 - z_2 z_3 = 0, \quad z_1 z_3 - z_4^2 = 0. \tag{6.9}$$

If $z_2^2 + z_3^2 = 0$, then the only solution of system (6.9) is the vanishing one:

$$z_1 = z_2 = z_3 = z_4 = 0. \tag{6.10}$$

Hence it follows that all components of the tensor $t^{\nu}_{\lambda\mu}$ vanish. Of course it represents a transitive domain for the tensor $t^{\nu}_{\lambda\mu}$ taken from the allowable set $\mathfrak{M}_{00}$.

In the case where $z_2^2 + z_3^2 > 0$ we get another solution of system (6.9). If

$$z_1 \cdot z_2 \neq 0, \tag{6.11}$$

then on the basis of the second equation of (6.9) we have

$$z_2 = k z_1, \quad z_4 = k z_3, \tag{6.12}$$

where k is an arbitrary real number. On the basis of (6.12) and of the third equation of (6.9) we get

$$z_1 = k^2 z_3, \quad z_2 = k^3 z_3, \quad z_4 = k z_3. \tag{6.13}$$

Let us take into consideration the transformation of the coordinate system determined by the matrix

$$\|A_{\lambda}^{\lambda'}\| \stackrel{\mathrm{df}}{=} \begin{Vmatrix} \dfrac{1}{z_3} & 0 \\ -k & 1 \end{Vmatrix}. \tag{6.14}$$

Then on the basis of (2.1) and of (6.1) we get

$$z_1' = 0, \quad z_2' = 0, \quad z_3' = 1, \quad z_4' = 0. \tag{6.15}$$

The cases $z_1 = 0$ or $z_3 = 0$ do not give anything new. In such cases it is easy to show the effective transformation of the coordinate system, after which the essential components of $t^{\nu}_{\lambda\mu}$ taken from the allowable set $\mathfrak{M}_{00}$ have the values (6.15).

Thus we have proved

THEOREM 8. *For the tensor $t^{\nu}_{\lambda\mu}$ from $\mathfrak{M}_{00}$ for which the tensor $b_{\lambda\mu}$ determined by* (6.3) *belongs to the transitive domain $\mathfrak{N}_1$ we have two transitive domains determined by the conditions*

$$\mathfrak{M}^1_{00}\colon\ z_1 = z_2 = z_3 = z_4 = 0,$$

$$\mathfrak{M}^2_{00}\colon\quad \begin{cases} z_2 z_4 - z_1^2 = 0, & z_1 z_4 - z_2 z_3 = 0, \\ z_1 z_3 - z_4^2 = 0, & z_2^2 + z_3^2 = 0, \end{cases}$$

where z_i $(i = 1, 2, 3, 4)$ are the essential components of the tensor $t^{\nu}_{\lambda\mu}$ in $\mathfrak{M}_{00}$ defined by (6.2). *An arbitrary constant function on the domains $\mathfrak{M}^1_{00}$ and $\mathfrak{M}^2_{00}$ is a sclar concomitant.*

In the case where $b_{\lambda\mu} \in \mathfrak{N}_3$ the components of tensor $b_{\lambda\mu}$ must satisfy in consequence of (6.5) the following relations:

$$b_{11} \leqslant 0, \quad b_{22} \leqslant 0, \quad b_{11}^2 + b_{22}^2 > 0, \quad b_{11} b_{22} - b_{12}^2 = 0. \tag{6.16}$$

In this case we can put the tensor $b_{\lambda\mu}$ in the canonical form [11]. Let

$$b_{11} \overset{*}{=} -1, \quad b_{12} = b_{21} \overset{*}{=} 0, \quad b_{22} \overset{*}{=} 0. \tag{6.17}$$

On the basis of (6.4) and of (6.17) we get the following system of equations:

$$z_2 z_4 - z_1^2 \overset{*}{=} -1, \quad z_1 z_4 - z_2 z_3 \overset{*}{=} 0, \quad z_1 z_3 - z_4^2 \overset{*}{=} 0. \tag{6.18}$$

Solving this system we obtain the solutions

$$z_1 \overset{*}{=} \pm 1, \quad z_2 \overset{*}{=} c, \quad z_3 \overset{*}{=} 0, \quad z_4 \overset{*}{=} 0, \tag{6.19}$$

where c is an arbitrary real number.

Let us take the transformation of the coordinate system defined by the matrix

$$\|A^{\lambda}_{\lambda'}\| = \begin{Vmatrix} \varepsilon & 0 \\ \gamma & \delta \end{Vmatrix}, \quad \varepsilon = \pm 1,\ \delta \neq 0. \tag{6.20}$$

This transformation preserves the canonical form (6.17) of the tensor $b_{\lambda\mu}$ [13]. Substituting in the matrix (6.20) the values

$$\varepsilon = \operatorname{sgn} z_1, \quad \gamma = -\frac{\varepsilon z_2}{3}, \tag{6.21}$$

we get in the new coordinate system the following values for the components of tensor $t^{\nu}_{\lambda\mu}$ taken from the allowable set $\mathfrak{M}_{00}$:

$$z'_1 = 1, \quad z'_2 = 0, \quad z'_3 = 0, \quad z'_4 = 0. \tag{6.22}$$

So we have got the third transitive domain $\mathfrak{M}^3_{00}$ for the tensor $t^{\nu}_{\lambda\mu}$ taken from the allowable set $\mathfrak{M}_{00}$.

Thus we have proved the following

THEOREM 9. *For the tensor $t^{\nu}_{\lambda\mu}$ from $\mathfrak{M}_{00}$ for which the tensor $b_{\lambda\mu}$, determined by* (6.3), *belongs to the transitive domain $\mathfrak{N}_3$ we have the transitive domain determined by the conditions*

$$\mathfrak{M}^3_{00}:\begin{cases} z_1z_4-z_1^2\leqslant 0, \quad z_1z_3-z_4^2\leqslant 0, \\ (z_2z_4-z_1^2)^2+(z_1z_3-z_4^2)^2>0, \\ 4(z_2z_4-z_1^2)(z_1z_3-z_4^2)=(z_1z_4-z_2z_3)^2, \end{cases}$$

where z_i $(i=1,2,3,4)$ are the essential components of the tensor $t^{\nu}_{\lambda\mu}$ in $\mathfrak{M}_{00}$. An arbitrary constant function is a scalar concomitant in this domain.

Analogically to Theorem 9 we can prove the next two theorems, the proofs of which can be found in [12].

THEOREM 10. *For the tensor $t^{\nu}_{\lambda\mu}$ from $\mathfrak{M}_{00}$ for which the tensor $b_{\lambda\mu}$, defined by* (6.3), *belongs to the transitive domain $\mathfrak{N}_5$ we have the transitive domain determined by the conditions*

$$\mathfrak{M}^4_{00}:\quad\begin{cases} z_2z_1-z_1^2<0, \quad z_1z_3-z_4^2<0, \\ 4(z_2z_4-z_1^2)(z_1z_3-z_4^2)>(z_1z_4-z_2z_3)^2, \end{cases}$$

where z_i $(i=1,2,3,4)$ are the essential components of the tensor $t^{\nu}_{\lambda\mu}$ in $\mathfrak{M}_{00}$. An arbitrary constant function is a scalar concomitant in this domain.

THEOREM 11. *For the tensor $t^{\nu}_{\lambda\mu}$ from $\mathfrak{M}_{00}$ for which the tensor $b_{\lambda\mu}$, defined by* (6.3), *belongs to the transitive domain $\mathfrak{N}_6$ we have the transitive domain determined by the conditions*

$$\mathfrak{M}^5_{00}:\ 4(z_2z_4-z_1^2)(z_1z_3-z_4^2)<(z_1z_4-z_2z_3)^2,$$

where z_i $(i=1,2,3,4)$ are the essential components of the tensor $t^{\nu}_{\lambda\mu}$ in $\mathfrak{M}_{00}$. An arbitrary constant function is a scalar concomitant in this domain.

7. Remarks concerning the density concomitants of a tensor $t^{\nu}_{\lambda\mu}$ in a 2-dimensional space. On the basis of [1] it is known that the general density concomitant of object (1.1) is the product of a particular non-vanishing density concomitant and a general scalar concomitant of this object. So in those allowable sets of the tensor $t^{\nu}_{\lambda\mu}$ in which we have the non-vanishing density obtained from the tensor $t^{\nu}_{\lambda\mu}$ we can determine the general density concomitant.

In the allowable set $\mathfrak{M}_2$ the determinant of the components of $\overset{1}{v}_{\lambda}$ and $\overset{2}{v}_{\lambda}$ is the particular non-vanishing density concomitant,

$$\mathfrak{v}\overset{\mathrm{df}}{=}\det\|\overset{\varrho}{v}_{\alpha}\| \qquad (\alpha,\varrho=1,2). \tag{7.1}$$

It is known that $\mathfrak{v}$ is an ordinary density of weight 1, and $|\mathfrak{w}|$ is a Weyl density of the same weight.

Thus we have the following

THEOREM 12. *The most general density concomitant of weight* $(-p)$ *of the tensor* $t^{\nu}_{\lambda\mu}$ *in the allowable set* $\mathfrak{M}_2$ *is of the form*

$$f(t^{\nu}_{\lambda\mu}) = \varepsilon|\mathfrak{v}|^p\varphi(\overset{1}{\underset{11}{\omega}}, \overset{2}{\underset{11}{\omega}}, \overset{1}{\underset{22}{\omega}}, \overset{2}{\underset{22}{\omega}}),$$

where

$$\varepsilon = \begin{cases} 1 & \textit{for W-density}, \\ \operatorname{sgn}\mathfrak{v} & \textit{for G-density}, \end{cases}$$

and φ *is an arbitrary function of* 4 *variables.*

In the allowable set $\mathfrak{M}^1_{\varkappa 1}$ the object $\mathfrak{g}$ defined by (4.7) is the particular non-vanishing W-density concomitant.

Thus we have the following

THEOREM 13. *The most general W-density concomitant of weight* $(-p)$ *of the tensor* $t^{\nu}_{\lambda\mu}$ *in the allowable set* $\mathfrak{M}^1_{\varkappa 1}$ *is of the form*

$$f(t^{\nu}_{\lambda\mu}) = |\mathfrak{g}|^{p/2}\varphi\left(\varkappa, \operatorname{sgn}\mathfrak{g}, \frac{\mathfrak{f}\operatorname{sgn}\mathfrak{g}}{\mathfrak{g}^2}, \frac{|\mathfrak{w}|\operatorname{sgn}\mathfrak{g}}{|\mathfrak{g}|^{5/2}}\right),$$

where φ *is an arbitrary function.*

In the allowable set $\mathfrak{M}^{21}_{\varkappa 1}$ the determinant of the components of the tensor $a_{\lambda\mu}$, defined by (4.23), is the particular non-vanishin W-density concomitant.

THEOREM 14. *The most general W-density concomitant of weight* $(-p)$ *of the tensor* $t^{\nu}_{\lambda\mu}$ *in the allowable set* $\mathfrak{M}^{21}_{\varkappa 1}$ *is of the form*

$$f(t^{\nu}_{\lambda\mu}) = |\det[a_{\lambda\mu}]|^{p/2}\varphi(\varkappa, \tau),$$

where φ *is an arbitrary function.*

In the transitive domains $\mathfrak{M}^4_{00}$ and $\mathfrak{M}^5_{00}$ the determinant of the components of the tensor $b_{\lambda\mu}$, defined by (6.3), is the particular non-vanishing W-density concomitant.

Thus we have the following

THEOREM 15. *The most general W-density concomitant of weight* $(-p)$ *of the tensor* $t^{\nu}_{\lambda\mu}$ *in the transitive domains* $\mathfrak{M}^4_{00}$ *and* $\mathfrak{M}^5_{00}$ *is of the form*

$$f(t^{\nu}_{\lambda\mu}) = C\cdot|\det[b_{\lambda\mu}]|^{p/2},$$

where C *is an arbitrary real number.*

We shall deal with the problem of determination of all density concomitants of the tensor $t^{\nu}_{\lambda\mu}$ in the next paper.

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