COLLOQUIUM MATHEMATICUM

VOL. XLVIII 1984 FASC. 1

ON CONTINUITY OF GROUP HOMOMORPHISMS

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We shall give another solution to an old problem of Mazur [6] recently solved by Labuda and Mauldin [5]. Our solution (Theorem below) is less elementary but it yields a more general result. The solution is based on the following

LEMMA. *Let G and H be Abelian Hausdorff topological groups with G metrizable and complete, and let $f\colon G \to H$ be a homomorphism. If for every Borel function $x\colon [0, 1] \to G$ with separable range*(1) *the superposition fx is m-measurable, where m denotes the Lebesgue measure, then f is continuous.*

Proof. As easily seen, we may assume that G is separable. In view of [3], Theorem 3.8, it is enough to show that f is universally measurable, i.e., given a finite Borel measure μ on G, f is μ-measurable. We assume, without loss of generality, that μ is nonatomic and normalized. Then, according to an isomorphism theorem due to Marczewski ([7], 4.1 (ii)), there exists a Borel isomorphism $x\colon [0, 1] \to G$ such that $\mu(x(B)) = m(B)$ for every Borel set $B \subset [0, 1]$. In particular, $x(A)$ is μ-measurable for every m-measurable set $A \subset [0, 1]$. Hence, in view of the formula $f^{-1}(V) = x(fx)^{-1}(V)$, we see that f is μ-measurable.

THEOREM. *Let G and H be Abelian Hausdorff topological groups with G metrizable, complete, connected and locally arcwise connected, and let $f\colon G \to H$ be a homomorphism. If for every continuous function $x\colon [0, 1] \to G$ the superposition fx is m-measurable, then f is continuous.*

Proof. In view of the Lemma, it is enough to show that, given a Borel function $x\colon [0, 1] \to G$ with separable range, fx is m-measurable. To this end, applying Lusin's theorem, take compact sets $K_n \subset [0, 1]$ such that $x|K_n$ are continuous and $m([0, 1]\backslash K_n) \to 0$. Now, by [4], § 50, I, Theorem 5 (2), there exist continuous functions $x_n\colon [0, 1] \to G$ such that $x_n|K_n = x|K_n$.

(1) In fact, as G is metrizable, the separability condition is redundant ([2], Theorem 1).

(2) This result is formulated in [4] for the case of the Cantor set in [0, 1] but its proof carries over to the general case we need here.

By assumption, fx_n is m-measurable. It follows that $fx|K_n$ is m-measurable on K_n. As

$$m\left([0, 1]\setminus \bigcup_{n=1}^{\infty} K_n\right) = 0,$$

we see that fx is m-measurable on $[0, 1]$.

Remarks. 1. In the special case where G is a Banach space and H is a topological vector space, the Theorem is due to Labuda and Mauldin [5]. Their proof still works for G as in our setting. Namely, one has only to apply the extension theorem we have used instead of Dugundji's theorem.

2. The connectedness assumption in the Theorem cannot be totally dispensed with. To see this, take for G the Cantor group $\{0, 1\}^{\aleph_0}$. Then, G being totally disconnected, every continuous function x: $[0, 1] \to G$ is constant. On the other hand, G admits discontinuous characters, i.e., homomorphisms into the circle group. Indeed, put

$$G_0 = \{(\varepsilon_i) \in G\colon\ \varepsilon_i = 0 \text{ eventually}\}$$

and

$$f_0\big((\varepsilon_i)\big) = \prod_{i=1}^{\infty} (-1)^{\varepsilon_i} \quad \text{for } (\varepsilon_i) \in G_0,$$

and extend f_0 to a character f of G.

3. There exist Abelian Hausdorff topological groups which are metrizable, compact and connected and yet not locally arcwise connected. Indeed, the p-adic solenoid is such ([1], Chapitre III, § 7, Exercice 6; cf. also [4], § 50, I, Theorem 1).

REFERENCES

[1] N. Bourbaki, *Topologie générale*, Paris 1960.

[2] Z. Frolík, *A measurable map with analytic domain and metrizable range is quotient*, Bulletin of the American Mathematical Society 76 (1970), p. 1112-1117.

[3] W. H. Graves, *Universal Lusin measurability and subfamily summable families in abelian topological groups*, Proceedings of the American Mathematical Society 73 (1979), p. 45-50.

[4] K. Kuratowski, *Topology II*, Warszawa 1968.

[5] I. Labuda and R. D. Mauldin, *Problem 24 of "The Scottish Book" concerning additive functionals*, this fascicle, p. 89-91.

[6] S. Mazur, *Problem 24*, *The Scottish Book*, p. 95, edited by R. O. Mauldin, Boston 1981.

[7] E. Szpilrajn (Marczewski), *O zbiorach i funkcjach bezwzględnie mierzalnych* [*On absolutely measurable sets and functions*], Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, Section III, 1937, p. 39-68 [Polish, French summary].

Reçu par la Rédaction le 20. 12. 1980