A CONNECTION BETWEEN SPECTRAL RADIUS AND TRACE

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Wimmer [3] has shown that for a complex \((k \times k)\)-matrix \(A\)

\[
\lim_{n \to \infty} \sqrt[n]{|\text{Tr} A^n|} = \rho(A),
\]

where \(\text{Tr} A\) is the usual trace of \(A\) and \(\rho(A)\) is the spectral radius of an operator on \(C^k\) corresponding to \(A\).

It turns out that formula (1) remains true for much more complicated objects. Let \(M\) be a von Neumann algebra of operators on a Hilbert space \(H\), let \(M^+\) be a set of non-negative elements in \(M\), and \(\varphi\) a trace defined in \(M^+\). Let \(M_1\) denote the ideal in \(M\) generated by the set of all \(T\) in \(M^+\) with \(\varphi(T) < \infty\). Then the trace \(\varphi\) has a unique extension (also denoted by \(\varphi\)) to a linear form on \(M_1\). Observe that the left-hand and right-hand sides of equality (1) make sense for arbitrary \(T \in M_1\). But not always we can get the equality. The aim of this paper is to show that if \(M\) is a purely atomic von Neumann algebra, e.g. if \(M\) is generated by its minimal projections, then (1) holds for every \(T \in M_1\). We prove also that (1) characterizes these algebras.

The method presented in the proof is a generalization of Wimmer's idea. I am indebted to Dr. T. Pytlik for his help and suggestions.

Let \(M\) be a von Neumann algebra and let \(\varphi\) and \(M_1\) be as above. The following proposition is well known ([2], I, § 6, Theorem 8):

**Proposition.** The inequality \(|\varphi(ST)| \leq \|S\|\varphi(|T|)\) holds for every \(S \in M\) and \(T \in M_1\).

**Corollary 1.** We have

\[
\lim_{n \to \infty} \sqrt[n]{|\varphi(T^n)|} \leq \varphi(T) \quad \text{for every } T \in M_1.
\]

**Theorem.** Let \(M\) be a semifinite von Neumann algebra and let \(\varphi\) be a normal, faithful and semifinite trace in \(M\). If \(M\) is purely atomic, then

\[
\varphi(T) = \lim_{n \to \infty} \sqrt[n]{|\varphi(T^n)|} \quad \text{for every } T \in M_1.
\]
If $M$ is not purely atomic, then there exists a $T \in M_1$ such that $\varphi(T^n) = 0$ for $n = 1, 2, \ldots$, but $\varrho(T) \neq 0$.

So formula (2) gives a characterization of purely atomic von Neumann algebras among semifinite ones.

Proof. We may consider $M$ as an algebra of operators on a Hilbert space $H$. If $M$ is purely atomic, then there exists a maximal family $E_i$, $i \in I$, of mutually orthogonal projections in the center of the algebra $M$ and $\sum E_i$ is the identity on $H$. Each algebra $ME_i$ is a factor of type 1, and so it is isomorphic to the full algebra $L(H_i)$ of operators on some Hilbert space $H_i$ (cf. [2], p. 121). Let $T_i$ denote the image of $TE_i$ in $L(H_i)$. The trace $\varphi|_{ME_i}$ is proportional to the usual trace $\varphi_i$ in $L(H_i)$ (cf. [2], I, § 6, Theorem 3, Corollary) and for $T \in M_1$ we have

$$\varphi(T) = \sum_{i \in I} \varphi(T E_i) = \sum_{i \in I} \sigma_i \varphi_i(T_i).$$

Let $T \in M_1$. By Corollary 1, only the inequality

$$\varrho(T) \leq \lim_{n \to \infty} \sqrt[n]{|\varphi(T^n)|}$$

must be shown. It is trivial when $\varrho(T) = 0$, and we may assume $\varrho(T) > 0$. Consider the function

$$f(z) = \sum_{n=1}^\infty \varphi(T^n) z^n$$

which is holomorphic on the circle with radius

$$R = \frac{1}{\lim_{n \to \infty} \sqrt[n]{|\varphi(T^n)|}}.$$

To prove that $R \leq 1/\varrho(T)$ we show that there exist singularities of the function $f(z)$ arbitrarily close to a circle of radius $1/\varrho(T)$. We have

$$f(z) = \sum_{n=1}^\infty \varphi(T^n) z^n = \sum_{n=1}^\infty \sum_{i \in I} \sigma_i \varphi_i(T_i^n) z^n = \sum_{n=1}^\infty \sum_{i \in I} \sum_k \lambda_{k,i}^n z^n,$$

where $\lambda_{k,i}$, $k = 1, 2, \ldots$, is the sequence (may be finite) of all non-zero eigenvalues of $T_i$.

The series on the right-hand side is absolutely summable for $|z| < R_\alpha$, where

$$R_\alpha^{-1} = \lim_{n \to \infty} \sqrt[n]{\sum_{i \in I} \sum_k |\lambda_{k,i}^n|} \leq \varrho(|T|) = \varrho(T),$$
so for \(|z| < 1/e(T)\) we may change the order of summation. Therefore

\[
f(z) = \sum_{i \leq I} \sigma_i \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \lambdai_{k,t} z^n = \sum_{i \leq I} \sigma_i \sum_{k=1}^{\infty} \lambdai_{k,t} z^n.
\]

Thus \(1/\lambdai_{k,t}\) are singularities of \(f(z)\), and since

\[
e(T) = \sup_{i \leq I} e(T_i) = \sup_{i \leq I} \sup_{k} |\lambdai_{k,t}|,
\]

we get the desired statement.

Now suppose \(M\) is not purely atomic. Then it is a direct product of a purely atomic von Neumann algebra and a non-trivial von Neumann algebra \(M_e\) without minimal projections.

Consider a restriction of \(\varphi\) to \(M_e\). Since the trace \(\varphi\) is semifinite, in \(M_e\) there exists a projection \(P\) with a finite trace (we may assume \(\varphi(P) = 1\)), and since \(M_e\) has no minimal projection, there exists a sequence of projections \(P_{k,l}\) for \(k = 1, 2, \ldots\) and \(l = 1, 2, \ldots, 2^k\) with the following properties:

1. \(P_{k,1}, \ldots, P_{k,2^k}\) are mutually orthogonal for \(k = 1, 2, \ldots\);
2. \(P_{1,1} + P_{1,2} = P\) and \(P_{k,2l-1} + P_{k,2l} = P_{k-1,l}\) for \(k = 2, 3, \ldots\) and \(l = 1, 2, \ldots, 2^{k-1}\);
3. \(\varphi(P_{k,l}) = 1/2^k\) for \(k = 1, 2, \ldots\) and \(l = 1, 2, \ldots, 2^k\).

The sequence \(P_{k,l}\) generates an abelian von Neumann algebra which is isomorphic to a von Neumann algebra \(L^\infty(Z, \mu)\), where \(\mu\) is a non-atomic probability measure on a compact set \(Z\) (cf. [2], p. 116-118). Since \((Z, \mu)\), as a measure space, is isomorphic to the interval \([0, 1]\) with Lebesgue measure, we have an isometric embedding of \(L^\infty(0, 1)\) into \(M\), which preserves the trace (on \(L^\infty(0, 1)\) the trace is \(\int\)). If \(T\) in \(M_1\) is the operator corresponding to the function \(e^{\lambda\mu}\), then \(\varphi(T^n) = 0\) for \(n = 1, 2, \ldots\), but \(\varphi(T) \neq 0\).

Remarks. 1. Formula (2) remains true for all elements \(T \in M\) such that \(T^k \in M\) for an integer \(k\).

2. Let \(G\) be a locally compact unimodular group and let \(\text{VN}(G)\) denote the von Neumann algebra of \(G\). Every compact group has a purely atomic von Neumann algebra. But there exist non-compact (non-abelian) groups with that property. An example of such a group due to Fell has been given by Baggett ([1], p. 142).

Corollary 2. Let \(G\) be a locally compact unimodular group. The formula

\[
\lim_n \frac{\|f^{(n)}\|_2}{\|f\|_2} = \lim_n \frac{\|f^{(n)}(e)\|}{\|f(e)\|},
\]

where \(f^{(n)} = f * f * \ldots * f\) (\(n\) times) and \(e\) is the identity in \(G\), holds for every \(f \in L^2(G) \cap \text{VN}(G)\) if and only if \(\text{VN}(G)\) is purely atomic.
In particular,

**Corollary 3.** For a compact group $G$ and $f \in L^1(G) \cap L^2(G)$ we have

$$\lim_{n \to \infty} \sqrt[n]{\|f^{(n)}\|_1} = \lim_{n \to \infty} \sqrt[n]{\|f^{(n)}\|_2} = \lim_{n \to \infty} \sqrt[n]{|f^{(n)}(e)|}.$$ 

**References**


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